THE EXTENDED ALGEBRA OF THE MINIMAL MODELS

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ABSTRACT. The minimal models $\mathcal{M}(p',p)$ with $p' > 2$ have a unique (non-trivial) simple current of conformal dimension $h = \frac{1}{4}(p'-2)(p-2)$. The representation theory of the extended algebra defined by this simple current is investigated in detail. All highest weight representations are proved to be irreducible: There are thus no singular vectors in the extended theory. This has interesting structural consequences. In particular, it leads to a recursive method for computing the various terms appearing in the operator product expansion of the simple current with itself. The simplest extended models are analysed in detail and the question of equivalence of conformal field theories is carefully examined.

1. INTRODUCTION

This article is the second in a series in which we analyse well-known conformal field theories from the viewpoint of an extended symmetry algebra defined by a simple current. Here, we present the minimal model case. This series was initiated by [1], in which we constructed and studied the extended algebras of the $\mathsf{SU}(2)$ Wess-Zumino-Witten models. Whilst these two articles are formally independent, we will see that the main results obtained in [1] have formal analogues in the study of the minimal model extended algebras. The construction of the latter is in fact easier in many respects, because the algebra is only graded by the conformal dimension, and not additionally by the $\mathfrak{sl}(2)$-weight. However, this structural reduction makes the analysis of the minimal model extended algebras somewhat more difficult than the $\mathsf{SU}(2)$ case.

1.1. **Motivation.** The reformulation of the minimal models from the point of view of the extended algebra generated by their simple current fits within the general program of trying to understand and/or derive fermionic character formulae (see for example [2]) by intrinsic conformal field theoretical methods. Such fermionic formulae reflect the description of the space of states in terms of quasi-particles subject to some restriction rules. Within the framework developed here, the quasi-particles are represented by the modes of the simple current. The first objective towards constructing the fermionic formula is to derive a complete set of constraints on strings of these simple current modes so as to obtain a complete description of the space of states. The character will then be the generating functions of these states.

The best studied minimal models are those with $p' = 2$. Their basis of states has been derived in [3] (through an analysis inspired in part by that of the parafermionic models in [4]). For fixed p, the basis is formulated in terms of the Virasoro modes and it is controlled by partitions with difference 2 at distance k, where $p = 2k+3$. However, these models do not fit within our framework

May 23, 2007.

as they possess no simple current. More precisely, the simple current of the $\mathcal{M}(2,p)$ models reduces to the identity field.

The next simplest cases are the $\mathcal{M}(3,p)$ models. Here, there exists a non-trivial simple current, namely $\phi_{2,1}$. These minimal models have been reformulated from the point of view of the extended algebra generated by $\phi_{2,1}$ in [5], and this reformulation has led to the proposal of a basis of states expressed solely in terms of the $\phi_{2,1}$ modes[1] (see also related results in [7]). Although some justifications for the linear independence and spanning property of the states were discussed, no formal proofs were presented, although the resulting fermionic characters were derived, reproducing known expressions (given for instance in [8]).

There are two natural ways to extend these results to $p' > 3$. One can either pursue the construction of a basis in terms of $\phi_{2,1}$ — noting that for $p' > 3$, $\phi_{2,1}$ is no longer a simple current — or look for an extended formulation in terms of the new model's simple current. The description of the basis of states for the $\mathcal{M}(p',p)$ models with $p' > 3$ in terms of the $\phi_{2,1}$ modes is plagued with serious technical difficulties rooted in the fact that the operator product expansion of $\phi_{2,1}$ with itself has more than one channel. This construction is pursued in [9]. Here we follow the second approach, which amounts to giving the role of basic generator, played by $\phi_{2,1}$ in the $\mathcal{M}(3,p)$ extended algebra, to the corresponding simple current of the general $\mathcal{M}(p',p)$ model, $p' \geqslant 3$. This simple current is $\phi_{p'-1,1}$ (Section 1.3).

In [5], the $\mathcal{M}(3,p)$ basis of states is formulated in terms of p-dependent constraints on the strings of modes of $\phi_{2,1}$ at distance 1 and 2. However, the precise mechanism by which these constraints arise was not clearly isolated[2]. Do these conditions on the basis of states follow solely from the operator product expansion of $\phi_{1,2}$ with itself, by considering a sufficiently large number of terms (as exploratory computations seem to indicate)? Do the vacuum and simple current singular vectors play any role with regard to these constraints? Moreover, in [5] it was implicitly assumed that there are no singular vectors in the extended algebra, so that no subtractions were needed to construct the character. However, no arguments supporting this assumption were presented there, apart from the fact that ignoring such potential singular vectors did indeed reproduce the correct characters. Finally, all issues relating to linear independence of the proposed basis could be treated rather informally because the fermionic characters, with which the state generating function could be compared, were already known in precisely the same form.

For $p' > 3$, the situation is completely different. Although fermionic expressions are known for all irreducible minimal models [10], they are not obviously of the type that would suit the combinatorial descriptions of the bases found for $p' = 3$. To make progress then, we require definite conclusions concerning the representation theory of the algebra defined by the simple current modes. It is therefore mandatory (both for the $p' = 3$ case and its generalisation) to consolidate

[1] Note that another proposed basis has been presented in [6], without much rationale. This alternative basis involves both the Virasoro and $\phi_{2,1}$ modes and it unifies, in a sense, the $p' = 2$ and $p' = 3$ bases.

[2] See in particular the discussion presented in the two paragraphs before [5, Eqn. (3.16)].

the foundation and reanalyse carefully the mathematical formalism implied by this simple current approach. This is the subject of the present paper. Basis issues will be reported in a sequel [11].

We mention that reformulating theories through a quasi-particle approach has many physical advantages and applications. As some of these have already been discussed in [1], we will not address them further here.

1.2. **Outline.** The article is organised as follows. Some preliminary results are first collected in the following subsection, where we also fix the notation. The extended algebra itself is defined in Section 2. In short, this algebra is defined by the operator product expansion of the simple current $\phi_{p'-1,1}$ with itself. We first examine the commutativity and associativity of the defining operator product expansion, and a neat (rigorous) argument is presented for the necessity of introducing an $\mathcal{S}$-type operator in this expansion, anticommuting with the simple current modes when $4h_{p'-1,1}$ is odd. Whilst the necessity of such an operator is not seen at the level of the full (non-chiral) theory (which explains why it was not observed until recently), we show that its presence is nevertheless essential to a consistent treatment of the representation theory.

It is then a simple matter to extend the Virasoro algebra to the algebra defined by the simple current modes. An important consequence of the dimension of $\phi_{p'-1,1}$ not necessarily being integral or half-integral is that the defining relations of the algebra of modes must be formulated in terms of generalised commutation relations involving infinite sums. The representation theory of this *extended algebra* is then developed in Section 3. We emphasise here the central role played by the monodromy charge. Symmetry properties of the generalised commutation relations are also studied, and we identify for later use those generalised commutation relations, amongst the many that we generate, which are actually independent.

Various examples of extended minimal models are presented in Section 4. In the context of extended algebras, we clarify the meaning of equivalence of two conformal field theories. We do this by introducing a sequence of (partially) extended models whose infinite limit is the genuine extension under study. These "incomplete" versions are in essence defined by truncating the depth at which the operator product expansion of $\phi_{p'-1,1}$ with itself is probed.

The central result of the paper is then presented in Section 5. There it is argued that in the extended algebra, all modules are free of singular vectors. This is first supported by a number of explicit computations within the simplest minimal models. Then, a generic argument is presented which demonstrates that in the vacuum module of the extended theory, there are no singular vectors. In other words, the vacuum Verma module is already irreducible. Finally, an appeal to a (claimed) general property of the Virasoro vacuum singular vector (the one at grade $(p'-1)(p-1)$) is used to lift this conclusion to the other (extended) highest weight modules appearing in the theory. The claimed general property in this case is that the corresponding null field completely controls the spectrum of the minimal model as well as *all* of its singular vectors.

Restated differently, our main result states that when taking into account sufficiently many of the terms in the operator product expansion of $\phi_{p'-1,1}$ with itself, the vacuum and the simple current principal singular vectors identically vanish (and we claim that this then implies the vanishing of the other singular vectors). This means that the corresponding Virasoro singular vectors are coded as identities in the model's defining operator product expansion. But once this is established, it can be turned around and used to deduce recurrence relations for the various terms of this operator product expansion. Such relations are derived in Section 6. These appear to be rather powerful technical tools, as the displayed illustrative computations exemplify.

1.3. **Notation and Preliminaries.** The minimal models may be defined as those two-dimensional conformal field theories whose state space is constructed out of a finite number of irreducible representations of the Virasoro algebra $\mathfrak{Vir}$. This is the infinite dimensional Lie algebra spanned by the modes L_n ($n \in \mathbb{Z}$) and C, subject to the commutation relations

$$\left[L_m, L_n\right] = (m-n) L_{m+n} + \binom{m+1}{3} \delta_{m+n,0} \frac{C}{2} \qquad \text{and} \qquad \left[L_m, C\right] = 0. \tag{1.1}$$

The L_n are the modes of the energy-momentum field

$$T(z) = \sum_n L_n z^{-n-2}, \tag{1.2}$$

whose operator product expansion is

$$T(z) T(w) = \frac{C/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \tag{1.3}$$

The generator C is central, and acts on the irreducible representations comprising the theory as multiplication by c, the central charge.

Up to a finite ambiguity in the modular invariant, the minimal models are classified by two coprime integers $p > p' > 1$ (in fact, by their ratio). We will generally denote a minimal model by $\mathcal{M}(p', p)$, presuming that the modular invariant is diagonal unless otherwise specified (we will actually only be concerned with the structure of the theory at the chiral level). The central charge of $\mathcal{M}(p', p)$ is given by

$$c = 1 - \frac{6(p-p')^2}{pp'}, \tag{1.4}$$

and the irreducible $\mathfrak{Vir}$-modules which appear (the spectrum) have highest weight states of conformal dimension

$$h_{r,s} = \frac{(pr-p's)^2 - (p-p')^2}{4pp'}, \tag{1.5}$$

where $1 \leqslant r \leqslant p'-1$ and $1 \leqslant s \leqslant p-1$. We note the symmetry $h_{p'-r,p-s} = h_{r,s}$ which implies a redundancy in the spectrum. We will generally denote a primary field of conformal dimension $h_{r,s}$ by $\phi_{r,s}(z)$, and the corresponding highest weight state by $\left|\phi_{r,s}\right\rangle$. The aforementioned symmetry is then expressed by the field identifications $\phi_{r,s}(z) = \phi_{p'-r,p-s}(z)$. In particular, the identity field is

$\phi_{1,1}(z) = \phi_{p'-1,p-1}(z)$, and the corresponding highest weight state is the vacuum, which we denote by $\left|0\right\rangle \equiv \left|\phi_{1,1}\right\rangle$.

The fusion rules for the (diagonal) minimal models are known explicitly [12, 13], and read

$$\phi_{r,s} \times \phi_{r',s'} = \sideset{}{'}\sum_{m=1+|r-r'|}^{\min\{r+r'-1,2p'-r-r'-1\}} \sideset{}{'}\sum_{n=1+|s-s'|}^{\min\{s+s'-1,2p-s-s'-1\}} \phi_{m,n}, \tag{1.6}$$

where the primed summation indicates that m and n increment in twos. We are interested in so-called *simple currents*, fields whose fusion with any primary field gives back a single primary field. We see that $\phi_{r',s'}$ is therefore a simple current precisely when

$$1 + \left|r - r'\right| = \min\left\{r + r' - 1, 2p' - r - r' - 1\right\} \qquad \text{for all } 1 \leqslant r \leqslant p' - 1, \tag{1.7}$$

$$\text{and} \qquad 1 + \left|s - s'\right| = \min\left\{s + s' - 1, 2p - s - s' - 1\right\} \qquad \text{for all } 1 \leqslant s \leqslant p - 1. \tag{1.8}$$

We consider Equation (1.8) first. When $s + s' \leqslant p$, this becomes $|s - s'| = s + s' - 2$, so squaring gives

$$(s-1)\left(s'-1\right) = 0 \qquad \text{for all } 1 \leqslant s \leqslant p - s'. \tag{1.9}$$

Thus $s' = 1$ or $s' = p - 1$ (forcing $s = 1$). Similarly, when $s + s' \geqslant p$, $|s - s'| = 2p - s - s' - 2$, so

$$(s-p+1)\left(s'-p+1\right) = 0 \qquad \text{for all } p - s' \leqslant s \leqslant p - 1. \tag{1.10}$$

Again, $s' = p - 1$ or $s' = 1$ (forcing $s = p - 1$). The analysis of Equation (1.7) is identical, so we find that the only simple currents are

$$\phi_{1,1} = \phi_{p'-1,p-1} \qquad \text{and} \qquad \phi_{1,p-1} = \phi_{p'-1,1}. \tag{1.11}$$

For $p' = 2$, these coincide, so there are no non-trivial simple currents. For $p' > 2$, there is a unique non-trivial simple current $\phi_{p'-1,1}$, which we will generally denote by ϕ for brevity. Its conformal dimension is given by

$$h \equiv h_{p'-1,1} = \frac{(p'-2)(p-2)}{4} \tag{1.12}$$

and its fusion rules take the simple form

$$\phi \times \phi_{r,s} = \phi_{p'-r,s}. \tag{1.13}$$

Note that $\phi \times \phi = \phi_{1,1}$.

We wish to extend the symmetry algebra of the minimal model $\mathcal{M}(p', p)$ by adjoining the modes ϕ_n of the simple current to the Virasoro modes L_m. Because the conformal dimension h of the simple current is not integral in general, we do not expect that this extended algebra will be a Lie algebra, but will instead be defined by generalised commutation relations. Mathematically, we therefore seek a (graded) associative algebra $\mathfrak{A}_{p',p}$ generated by the ϕ_n, in which $\mathfrak{Vir}$, or rather its universal enveloping algebra $\mathcal{U}\left(\mathfrak{Vir}_{p',p}\right)$ appears as a (graded) subalgebra. Here, the subscript

"p', p" indicates that we identify C with $c\,\mathrm{id}$ in this universal enveloping algebra (although we will often drop this subscript for brevity in what follows).

Before turning to the construction of this algebra, let us note that a part of the structure is already available to us. As $\phi(w)$ is a primary field, its operator product expansion with the energy-momentum tensor is

$$T(z)\,\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \ldots \tag{1.14}$$

Assuming that these two fields are mutually bosonic, this implies the familiar commutation rule

$$\left[L_m, \phi_n\right] = \left(m(h-1) - n\right)\phi_{m+n}. \tag{1.15}$$

This (mutually) bosonic behaviour can in fact be derived, for example from the analogue of the Jacobi identity (see [1, Eq. 2.5]).

We can also extend the canonical antilinear antiautomorphism $L_m \to L_{-m}$ (defining the adjoint on representations) to the simple current modes. The grading by conformal dimension restricts this extended adjoint to have the form $\phi_n \to \varepsilon\phi_{-n}$, where $|\varepsilon| = 1$. Requiring this to be an antiautomorphism with respect to Equation (1.15) gives no further constraints on ε, so we may choose $\varepsilon = 1$ for simplicity. In other words, we choose the adjoint of the extended theory to be given by

$$L_m^\dagger = L_{-m} \qquad \text{and} \qquad \phi_n^\dagger = \phi_{-n}. \tag{1.16}$$

We will verify later (Section 2.2) that this defines an antilinear automorphism of the *full* extended algebra $\mathfrak{A}_{p',p}$, defined by generalised commutation relations.

2. Constructing the Extended Algebra

2.1. **Commutativity and Associativity.** We will suppose from now on that $p' \geqslant 3$, so that there always exists a unique non-trivial simple current ϕ of order 2 and conformal dimension $h = \frac{1}{4}(p'-2)(p-2)$. Since $\phi \times \phi = \phi_{1,1}$, the corresponding operator product expansion must be of the form

$$\phi(z)\,\phi(w) = \sum_{j=0}^{\infty} A^{(j)}(w)\,(z-w)^{j-2h}, \tag{2.1}$$

where $A^{(0)}(w)$ should be the identity field. Observe that the exponent $2h$ need not be integral, signifying that we should take some care when using Equation (2.1) in formal manipulations.

One of the requirements of the fields of a conformal field theory is that they should satisfy some sort of *mutual locality* principle, meaning that the operator product expansion of two fields should be independent (up to a statistical phase) of the order in which the fields appear[3]. Applying this principle to the operator product expansion (2.1) however leads to a slight ambiguity when $2h \notin \mathbb{Z}$:

[3] We have already applied this principle in deriving Equation (1.15) from Equation (1.14) — there the fields were mutually *bosonic*, meaning that the statistical phase factor is unity.

Comparing the expansions of $\phi(z)\phi(w)$ and $\phi(w)\phi(z)$ is not possible unless we first agree on how to compare $(z-w)^{j-2h}$ with $(w-z)^{j-2h}$.

This issue may be bypassed by reformulating the mutual locality principle as the following commutativity requirement:

$$(z-w)^{2h}\phi(z)\phi(w) = \phi(w)\phi(z)(w-z)^{2h}. \tag{2.2}$$

Inserting the expansion (2.1) and its $z \leftrightarrow w$ analogue into this equation now leads to integral powers of $z-w$ and $w-z$, settling the ambiguity mentioned above. We mention that in considering the mutual locality of two different fields (such as the construction in [1]), we need only require that the corresponding equality in Equation (2.2) hold up to a phase factor. However, it is easy to check that when the two fields are identical (the case of interest here), this phase factor is necessarily unity.

Another requirement of the operator product expansion of fields is that it defines an associative operation. Assuming this requirement, the triple product $\phi(x)\phi(z)\phi(w)$ is unambiguously defined. We now derive an interesting conclusion by combining this requirement with the commutativity condition, Equation (2.2), and the generic operator product expansion (2.1).

Applying commutativity twice, we may write

$$(x-w)^{2h}(z-w)^{2h}\phi(x)\phi(z)\phi(w) = \phi(w)\phi(x)\phi(z)(w-x)^{2h}(w-z)^{2h}. \tag{2.3}$$

Expanding $\phi(x)\phi(z)$ as in Equation (2.1), multiplying both sides by $(x-z)^{2h-\gamma}$ (for some arbitrary $\gamma \in \mathbb{Z}$), and contour-integrating x around z gives

$$\sum_{j=0}^{\gamma-1} \binom{2h}{\gamma-j-1} \left[A^{(j)}(z)\phi(w)(z-w)^{4h+j-\gamma+1} - (-1)^{j-\gamma+1}\phi(w)A^{(j)}(z)(w-z)^{4h+j-\gamma+1}\right] = 0. \tag{2.4}$$

As $4h \in \mathbb{Z}$, this may be simplified to

$$\sum_{j=0}^{\gamma-1} \binom{2h}{\gamma-j-1} (z-w)^{4h+j-\gamma+1} \left[A^{(j)}(z)\phi(w) - (-1)^{4h}\phi(w)A^{(j)}(z)\right] = 0, \tag{2.5}$$

from which we may conclude that

$$A^{(j)}(z)\phi(w) = (-1)^{4h}\phi(w)A^{(j)}(z), \tag{2.6}$$

for each j, by analysing $\gamma = 1,2,3,\ldots$ consecutively. This proves that the fields $A^{(j)}(z)$ appearing in the operator product expansion (2.1) commute with $\phi(w)$ when $2h \in \mathbb{Z}$ (the fields are mutually bosonic), but anticommute with $\phi(w)$ when $2h \notin \mathbb{Z}$ (the fields are mutually fermionic).

Recall now that $A^{(0)}(z)$ was supposed to be the identity field. This clearly contradicts the above commutativity conclusion when $2h \notin \mathbb{Z}$. Indeed, the $A^{(j)}(z)$ with $j > 0$ should be (Virasoro) descendants of the identity field, hence should be expressible in terms of normally-ordered products of $T(z)$ and its derivatives. But these are also mutually bosonic with respect to $\phi(w)$ (Section 1.3), so we face a similar contradiction. This contradiction can only be satisfactorily resolved if we assume that each $A^{(j)}(z)$ contains an operator $\mathcal{S}$ which satisfies

$$\mathcal{S}T(z) = T(z)\mathcal{S}, \qquad \text{but} \qquad \mathcal{S}\phi(w) = (-1)^{4h}\phi(w)\mathcal{S}. \tag{2.7}$$

Such an operator $\mathcal{S}$ was first introduced in [5] for the $\mathcal{M}(3,p)$ models (though with a less direct justification).

It is worth noting that associativity does *not* require the introduction of $\mathcal{S}$-type operators in the corresponding full non-chiral theories (with the diagonal modular invariant) [14]. Essentially, the antiholomorphic component will contribute an additional factor of $(-1)^{4h}$ to Equation (2.6), removing the contradiction that necessitated the appearance of $\mathcal{S}$. Whilst it may then be argued that this operator is in some sense "unphysical", the fact remains that the discipline of conformal field theory rests heavily upon its chiral foundations, and many of its applications (especially in mathematics, but also physically when boundaries are concerned) require a consistent chiral formulation. We will illustrate this explicitly in Section 5.1, by demonstrating that the properties of $\mathcal{S}$ are *crucial* in showing that the $\mathcal{M}(3,5)$ singular vectors vanish identically.

We will find it convenient to explicitly factor this operator $\mathcal{S}$ out from each of the $A^{(j)}(z)$, redefining the latter so that the operator product expansion is

$$\phi(z)\phi(w) = \mathcal{S}\sum_{j=0}^{\infty} A^{(j)}(w)(z-w)^{j-2h}. \tag{2.8}$$

This expansion replaces Equation (2.1), to which we shall not refer again. Thus, for example, $A^{(0)}(w)$ is now genuinely the identity field. We note that $\mathcal{S}^2$ commutes with $\phi(z)$, hence with its modes, and is therefore a multiple of the identity in any irreducible module of the extended algebra (which will be constructed shortly). We also note that $\mathcal{S}$ must leave the vacuum $\left|0\right\rangle$ invariant, so that it does not interfere with the state-field correspondence.

2.2. **Algebraic Structure.** We turn now to the derivation of the algebra defined by the modes of $\phi(z)$. At a formal level, this uses a standard trick [15] involving the evaluation of

$$R_{m,n}(\gamma) = \oint_0 \oint_w \phi(z)\phi(w) z^{m-h+\gamma-1} w^{n+h-1} (z-w)^{2h-\gamma} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \qquad (\gamma \in \mathbb{Z}) \tag{2.9}$$

in two distinct ways. We can expand the operator product directly, using Equation (2.8), or we can break the z-contour around w into the difference of two contours about the origin, one with $|z| > |w|$ and the other with $|z| < |w|$. The final result is a generalised commutation relation, parametrised by γ, m and n:

$$\sum_{\ell=0}^{\infty} \binom{\ell-2h+\gamma-1}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} - (-1)^{\gamma}\phi_{n+2h-\gamma-\ell}\phi_{m-2h+\gamma+\ell}\right] = \mathcal{S}\sum_{j=0}^{\gamma-1} \binom{m-h+\gamma-1}{\gamma-1-j} A^{(j)}_{m+n}, \tag{2.10}$$

where $A^{(j)}_{m+n}$ denotes the modes of the fields $A^{(j)}(w)$ appearing in the operator product expansion (2.8). Note that γ determines how many terms of this operator product expansion contribute to the corresponding generalised commutation relation.

We observe that if $2h \in \mathbb{Z}$, the generalised commutation relations with $\gamma = 2h$ reduce to commutation or anticommutation relations:

$$\phi_m \phi_n - (-1)^{2h} \phi_n \phi_m = \mathcal{S} \sum_{j=0}^{2h-1} \binom{m+h-1}{2h-1-j} A^{(j)}_{m+n}. \tag{2.11}$$

In contrast, when $2h \notin \mathbb{Z}$, every generalised commutation relation has infinitely many terms on the left hand side.

It remains to determine the fields $A^{(j)}(w)$, at least for small j. As in [1], we invert Equation (2.8),

$$\mathcal{S} A^{(j)}(w) = \oint_w \phi(z)\, \phi(w)\, (z-w)^{2h-j-1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \tag{2.12}$$

let both sides act on the vacuum, and send w to 0. The result determines the corresponding states as

$$\left|A^{(j)}\right\rangle = \mathcal{S}\left|A^{(j)}\right\rangle = \phi_{h-j}\phi_{-h}\left|0\right\rangle = \phi_{h-j}\left|\phi\right\rangle. \tag{2.13}$$

[The fields $A^{(j)}(w)$ may be expressed purely in terms of $T(w)$ and its derivatives, hence $\left|A^{(j)}\right\rangle$ is a linear combination of the grade j Virasoro descendants of the vacuum, and so $\mathcal{S}\left|A^{(j)}\right\rangle = \left|A^{(j)}\right\rangle$.]

There is no such descendant at grade 1, so $\left|A^{(1)}\right\rangle = 0$, hence $A^{(1)}(w) = 0$. There is a unique Virasoro descendant at grade 2, so $\left|A^{(2)}\right\rangle$ must be proportional to $L_{-2}\left|0\right\rangle$. This constant of proportionality may be evaluated by using Equations (1.15) and (1.1) to derive

$$\left\langle 0\middle|L_2\middle|A^{(2)}\right\rangle = \left\langle 0\middle|L_2 \phi_{h-2}\phi_{-h}\middle|0\right\rangle = h \qquad \text{and} \qquad \left\langle 0\middle|L_2 L_{-2}\middle|0\right\rangle = \frac{c}{2}. \tag{2.14}$$

It follows that that $A^{(2)}(w) = \frac{2h}{c} T(w)$. Similarly, $A^{(3)}(w) = \frac{h}{c}\partial T(w)$.

At grade 4, there are two possible Virasoro descendants, so we have $\left|A^{(j)}\right\rangle = \alpha L_{-4}\left|0\right\rangle + \beta L_{-2}^2\left|0\right\rangle$, for some unknowns α and β. Applying $\left\langle 0\right|L_4$ and $\left\langle 0\right|L_2^2$ to both sides gives two linear equations to solve, which may be solved to give

$$A^{(4)}(w) = \frac{3h(c-2h+4)}{2c(5c+22)} \partial^2 T(w) + \frac{2h(5h+1)}{c(5c+22)} :T(w)\,T(w):. \tag{2.15}$$

Of course these expressions are all well-known [16].

We remark that when $c = \frac{-22}{5}$, this last computation breaks down because the vacuum $\mathfrak{Vir}$-module has a singular vector at grade 4. As far as the inner-product is concerned, $L_{-4}\left|0\right\rangle$ and $L_{-2}^2\left|0\right\rangle$ are not linearly independent. We would therefore need to compute $\left|A^{(4)}\right\rangle$ separately for this special value of the central charge (choosing which state we regard as independent). Of course, this central charge corresponds to the $\mathcal{M}(2,5)$ model, which is not of interest here, as it has no non-trivial simple current. However, singular vectors will eventually appear (for example, at grade 6 for $\mathcal{M}(3,4)$) and therefore complicate the calculations.

It should be clear then that a knowledge of the singular vectors of the Virasoro vacuum module is essential to continuing these derivations. Whilst this knowledge is not beyond reach [17], the implementation rapidly becomes cumbersome at higher grades. It is therefore worth observing that it is possible to derive recurrence relations for the $A^{(j)}(w)$. We will present such derivations as an application of our extended algebra formalism in Section 6.1.

For later reference, we display a few of the generalised commutation relations. When applying these relations, it is often useful to explicitly indicate the value of γ employed. As in [1], we will use "$\stackrel{\gamma}{=}$" to indicate an equality obtained using a generalised commutation relation with parameter γ. Clearly when $\gamma \leqslant 0$, the right-hand-side of the generalised commutation relations vanish, giving

$$\sum_{\ell=0}^{\infty} \binom{\ell-2h+\gamma-1}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} - (-1)^{\gamma} \phi_{n+2h-\gamma-\ell}\phi_{m-2h+\gamma+\ell} \right] \stackrel{\gamma \leqslant 0}{=} 0. \tag{2.16}$$

Similarly, $A^{(0)}(w) = 1$ implies that the $\gamma = 1$ generalised conjugation relation is

$$\sum_{\ell=0}^{\infty} \binom{\ell-2h}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} + \phi_{n+2h-1-\ell}\phi_{m-2h+1+\ell} \right] \stackrel{1}{=} \delta_{m+n,0}\mathcal{S}. \tag{2.17}$$

Continuing, we have:

$$\sum_{\ell=0}^{\infty} \binom{\ell-2h+1}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} - \phi_{n+2h-2-\ell}\phi_{m-2h+2+\ell} \right] \stackrel{2}{=} -(n+h-1)\,\delta_{m+n,0}\mathcal{S}, \tag{2.18}$$

$$\sum_{\ell=0}^{\infty} \binom{\ell-2h+2}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} + \phi_{n+2h-3-\ell}\phi_{m-2h+3+\ell} \right] \stackrel{3}{=} \left[\binom{n+h-1}{2} \delta_{m+n,0} + \frac{2h}{c} L_{m+n} \right] \mathcal{S}, \tag{2.19}$$

$$\sum_{\ell=0}^{\infty} \binom{\ell-2h+3}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} - \phi_{n+2h-4-\ell}\phi_{m-2h+4+\ell} \right] \stackrel{4}{=} \left[-\binom{n+h-1}{3} \delta_{m+n,0} + \frac{h}{c}(m-n-2h+4) L_{m+n} \right] \mathcal{S}, \tag{2.20}$$

and

$$\begin{aligned} \sum_{\ell=0}^{\infty} \binom{\ell-2h+4}{\ell} \left[\phi_{m-\ell}\phi_{n+\ell} + \phi_{n+2h-5-\ell}\phi_{m-2h+5+\ell} \right] \stackrel{5}{=} \Bigg[&\binom{n+h-1}{4} \delta_{m+n,0} \\ &+ \left[\frac{3h(c-2h+4)}{c(5c+22)} \binom{m+n+3}{2} - \frac{h}{c}(m-h+4)(n+h-1) \right] L_{m+n} \\ &+ \frac{2h(5h+1)}{c(5c+22)} \sum_{r\in\mathbb{Z}} :L_r L_{m+n-r}: \Bigg] \mathcal{S}. \end{aligned} \tag{2.21}$$

We now define[4] the extended symmetry algebra $\mathfrak{A}_{p',p}$ of the minimal model $\mathcal{M}(p',p)$ (with $p'>2$) to be the graded (by conformal dimension) associative algebra generated by the modes ϕ_n, subject to the set of generalised commutation relations, Equation (2.10), and equipped with the adjoint $\phi_n^\dagger = \phi_{-n}$. It is easy to check that $R_{m,n}(\gamma)^\dagger = R_{-n,-m}(\gamma)$ (using the left-hand-side of Equation (2.10)), hence that this adjoint defines a genuine antilinear antiautomorphism of $\mathfrak{A}_{p',p}$. We summarise this result by noting that this adjoint makes our extended symmetry algebra into a graded *-algebra (the grading being by the conformal weight and the *-algebra meaning simply that the adjoint satisfies the usual properties with respect to the algebra operations). Note that the generalised commutation relation with $\gamma=1$ requires $\mathcal{S}$ to be self-adjoint.

3. Representation Theory

3.1. **Monodromy Charge.** Consider now the Virasoro highest weight state $\left|\phi_{r,s}\right\rangle$ corresponding to the primary field $\phi_{r,s}(z)$. The simple current fusion rules (1.13) imply that

$$\phi(z)\,\phi_{r,s}(w) = \frac{\eta_{r,s}\phi_{p'-r,s}(w)}{(z-w)^{\theta_{r,s}}} + \ldots, \tag{3.1}$$

where $\eta_{r,s}$ is a constant (or possibly an operator like $\mathcal{S}$ if required by associativity). The leading exponent $\theta_{r,s}$ is then given by

$$\theta_{r,s} = h + h_{r,s} - h_{p'-r,s} = 1 - rs + \frac{1}{2}\left[p(r-1) + p'(s-1)\right] \in \frac{1}{2}\mathbb{Z}. \tag{3.2}$$

As ϕ is a simple current, the omitted terms in the operator product expansion (3.1) all have $(z-w)$-exponents of the form $j-\theta_{r,s}$, where $j\in\mathbb{Z}_+$.

The common value (modulo 1) of these exponents tells us how to expand $\phi(z)$ into modes, when acting on $\left|\phi_{r,s}\right\rangle$. We see this by noting that

$$\phi(z)\left|\phi_{r,s}\right\rangle = \lim_{w\to 0}\phi(z)\,\phi_{r,s}(w)\left|0\right\rangle = \eta_{r,s} z^{-\theta_{r,s}}\left|\phi_{p'-r,s}\right\rangle + \ldots \tag{3.3}$$

implies that $\phi(z)$ must be expanded in powers of z equal to $-\theta_{r,s}$ modulo 1:

$$\phi(z)\left|\phi_{r,s}\right\rangle = \sum_{n\in\mathbb{Z}+\theta_{r,s}-h} \phi_n z^{-n-h}\left|\phi_{r,s}\right\rangle. \tag{3.4}$$

We interpret this as investing each highest weight state with an associated charge, $\theta_{r,s}$, whose value modulo 1 dictates which modes of the simple current field may act upon this state. Modes corresponding to the wrong charge do not have a well-defined action on this state. We shall refer to $\theta_{r,s}$ as the $\mathfrak{u}(1)$-charge of $\left|\phi_{r,s}\right\rangle$, and we shall call its value modulo 1 the *monodromy charge* (following [18]).

[4] This definition is in fact not quite complete because we have thus far avoided specifying the values that the index in ϕ_n can take. As with all fields exhibiting non-bosonic statistics, these values depend upon the state on which the mode acts. This will be specified when discussing the representation theory of the symmetry algebra in Section 3.

Whilst the monodromy charge controls which indices on the modes ϕ_n are allowed when acting on a highest weight state, the $\mathfrak{u}(1)$-charge tells us for which indices this action necessarily gives 0. By writing $\phi_n \left| \phi_{r,s} \right\rangle$ as a contour integral involving the operator product expansion (3.1), it is easy to see when the integrand becomes regular (hence when the integral vanishes). The result is

$$\phi_n \left| \phi_{r,s} \right\rangle = 0 \qquad \text{for all } n > \theta_{r,s} - h, \tag{3.5}$$

and of course this is non-vanishing when $n = \theta_{r,s} - h$. In other words, the $\mathfrak{u}(1)$-charge specifies the *first descendant* of the state $\left| \phi_{r,s} \right\rangle$ with respect to the extended algebra $\mathfrak{A}_{p',p}$.

The $\mathfrak{u}(1)$-charge is non-negative on the Virasoro highest weight states $\left| \phi_{r,s} \right\rangle$. To prove this, we note that as $\gcd\{p', p\} = 1$, either p or p' must be odd. Without loss of generality, we suppose it is p (otherwise swap p and p', and r and s in what follows). Then, for each $s = 1, \ldots, p-1$, there is a unique $r \in \mathbb{R}$ such that $\theta_{r,s} = 0$. From Equation (3.2), this value is

$$r = 1 + \frac{(p'-2)(s-1)}{2s-p}. \tag{3.6}$$

Suppose that this unique r lies between 1 and $p'-1$ (non-inclusive). Then

$$0 < \frac{s-1}{2s-p} < 1, \qquad \text{as } p' \geqslant 3. \tag{3.7}$$

If $s > p/2$, this requires $s > p-1$. Similarly, if $s < p/2$, this requires $s < 1$, so both conclusions fall outside the allowed range for s. It therefore follows that for $s = 1, \ldots, p-1$, $\theta_{r,s} \neq 0$ for $1 < r < p'-1$. Now, $\theta_{1,s} = \theta_{p'-1,p-s} = \frac{1}{2}(p'-2)(s-1) > 0$ for all $s \neq 1$, so we conclude that $\theta_{r,s} \geqslant 0$ with equality if and only if $\phi_{r,s}$ is the identity.

We can generalise the notion of monodromy charge to descendant states in the same fashion. It is not difficult to infer (as in [1]) from Equations (1.15) and (2.10) that the monodromy charge of a state is left invariant (modulo 1) by the application of a Virasoro mode L_m, but is changed by $2h$ (again modulo 1) by the application of a mode ϕ_n.

3.2. $\mathfrak{A}_{p',p}$-Verma Modules.

We define an $\mathfrak{A}_{p',p}$-highest weight state to be a state $\left| \psi \right\rangle$ satisfying

$$\phi_n \left| \psi \right\rangle = L_m \left| \psi \right\rangle = 0 \qquad \text{for all } m,n > 0. \tag{3.8}$$

We include annihilation under the Virasoro modes of positive index to ensure that an $\mathfrak{A}_{p',p}$-highest weight state is necessarily[5] a Virasoro-highest weight state. An $\mathfrak{A}_{p',p}$-Verma module is then the module generated from such a highest weight state by the action of the (allowed) ϕ_n, modulo the algebra relations (the generalised commutation relations). We will denote the $\mathfrak{A}_{p',p}$-Verma module generated from the highest weight state $\left| \phi_{r,s} \right\rangle$ by $\mathcal{V}_{r,s}^{p',p}$.

[5] The Virasoro highest weight condition follows easily from $\phi_n \left| \psi \right\rangle = 0$ when h is sufficiently small (use the generalised commutation relation with $\gamma = 4$ for example). However, it is not clear if this continues to hold true for all h.

Consider therefore an arbitrary $\mathfrak{A}_{p',p}$-highest weight state $\left|\psi\right\rangle$. Being a $\mathfrak{Vir}$-highest weight state, it has (Section 3.1) a definite $\mathfrak{u}(1)$-charge θ. Its first descendant is then given by Equation (3.5) as

$$\phi_{\theta-h}\left|\psi\right\rangle \neq 0. \tag{3.9}$$

If $\theta > h$, this contradicts the definition of highest weight state given in Equation (3.8), so we conclude that $\mathfrak{A}_{p',p}$-highest weight states must necessarily have $\mathfrak{u}(1)$-charge $\theta \leqslant h$.

Now recall that the Virasoro highest weight states $\left|\phi_{r,s}\right\rangle$ appearing in the minimal models have $\theta_{r,s} \geqslant 0$ and $\theta_{r,s} \in \frac{1}{2}\mathbb{Z}$. It is easy to derive the identity

$$\theta_{r,s} + \theta_{p'-r,s} = \frac{1}{2}\left(p'-2\right)\left(p-2\right) = 2h \tag{3.10}$$

from Equation (3.2), from which we deduce that $\theta_{r,s} \leqslant h$ if and only if $\theta_{p'-r,s} \geqslant h$. In other words, $\left|\phi_{r,s}\right\rangle$ is an $\mathfrak{A}_{p',p}$-highest weight state precisely when $\left|\phi_{p'-r,s}\right\rangle$ is not (unless $\theta_{r,s} = h$, a special case that we shall discuss shortly).

It follows that these $\mathfrak{Vir}$-highest weight states of charge greater than h must occur as descendants (with respect to the extended algebra) of the $\mathfrak{A}_{p',p}$-highest weight states, whose charges are not more than h. We specify this relationship precisely by considering the (first) descendant state $\phi_{\theta_{r,s}-h}\left|\phi_{r,s}\right\rangle$ (where $\theta_{r,s} \leqslant h$). The conformal dimension of this descendant is $h_{r,s} - \theta_{r,s} + h = h_{p'-r,s}$, by Equation (3.2), suggesting that

$$\phi_{\theta_{r,s}-h}\left|\phi_{r,s}\right\rangle = \left|\phi_{p'-r,s}\right\rangle, \tag{3.11}$$

the $\mathfrak{Vir}$-highest weight state. This can be confirmed by acting with the positive Virasoro modes ($n > 0$) and using Equation (3.5):

$$L_n\phi_{\theta_{r,s}-h}\left|\phi_{r,s}\right\rangle = \left[L_n, \phi_{\theta_{r,s}-h}\right]\left|\phi_{r,s}\right\rangle = \left(n\left(h-1\right) - \theta_{r,s} + h\right)\phi_{n+\theta_{r,s}-h}\left|\phi_{r,s}\right\rangle = 0. \tag{3.12}$$

We remark that it is possible for the highest weight state $\left|\phi_{r,s}\right\rangle$ to have $\theta_{r,s} = h$. In fact, this occurs if and only if $r = p'/2$ or $s = p/2$ (hence cannot occur if p and p' are both odd). We have then two $\mathfrak{Vir}$-highest weight states, $\left|\phi_{r,s}\right\rangle$ and $\phi_0\left|\phi_{r,s}\right\rangle$, of the same conformal dimension. These are both $\mathfrak{A}_{p',p}$-highest weight states, according to the above definition. We now ask whether these two highest weight states are linearly independent. A relevant observation to this question is that as $\phi_0\left|\phi_{r,s}\right\rangle$ is a $\mathfrak{Vir}$-highest weight state,

$$\left\langle\phi_{r,s}\middle|\phi_0^2\middle|\phi_{r,s}\right\rangle = 1, \qquad \text{so} \qquad \phi_0^2\left|\phi_{r,s}\right\rangle = \left|\phi_{r,s}\right\rangle, \tag{3.13}$$

assuming $\left|\phi_{r,s}\right\rangle$ and $\phi_0\left|\phi_{r,s}\right\rangle$ to be either proportional or orthogonal.

If we take these two highest weight states to be proportional, then Equation (3.13) limits the proportionality constant to ± 1. We therefore have two possible $\mathfrak{A}_{p',p}$-Verma modules which are identical as $\mathfrak{Vir}$-Verma modules, but which are distinguished by the eigenvalue of ϕ_0 on their highest weight states. If we instead take these two highest weight states to be orthogonal, then the linear combinations

$$\left|\phi_{r,s}\right\rangle \pm \phi_0\left|\phi_{r,s}\right\rangle \tag{3.14}$$

are the highest weight states of (distinct) $\mathfrak{A}_{p',p}$-Verma modules. Again, these are identical as $\mathfrak{Vir}$-Verma modules, but can distinguished by the eigenvalue of ϕ_0 on their highest weight states. It follows that the choice of whether $\phi_0\left|\phi_{r,s}\right\rangle$ is proportional or orthogonal to $\left|\phi_{r,s}\right\rangle$ is of no essential importance.

The above discussion suggests that we should augment the definition of an $\mathfrak{A}_{p',p}$-highest weight state to include being an eigenstate of ϕ_0 (assuming that ϕ_0 is allowed to act upon it). The eigenvalue, when defined, would then be ± 1 for states of $\mathfrak{u}(1)$-charge h, and 0 otherwise. Whilst this is in full accord with general Lie-algebraic principles[6], considering eigenstates of ϕ_0 leads to a certain inelegance in the formalism. In particular, we are forced to relinquish our picture of the ϕ_n as being *intertwiners* between $\mathfrak{Vir}$-Verma modules, that is that the action of each ϕ_m takes us from one $\mathfrak{Vir}$-Verma module to another whilst the action of a subsequent ϕ_n brings us back again.

To restore this intertwining picture, we shall adopt the following convention concerning modules whose highest weight states have $\mathfrak{u}(1)$-charge h: We declare that $\left|\phi_{r,s}\right\rangle$ is an $\mathfrak{A}_{p',p}$-highest weight state and that $\phi_0\left|\phi_{r,s}\right\rangle$ is its orthogonal descendant. In this way, ϕ_0 (as well as the other ϕ_n) act as genuine intertwiners between the $\mathfrak{Vir}$-Verma modules generated from $\left|\phi_{r,s}\right\rangle$ and $\phi_0\left|\phi_{r,s}\right\rangle$. The $\mathfrak{A}_{p',p}$-Verma module[7] generated by $\left|\phi_{r,s}\right\rangle$ then contains two $\mathfrak{Vir}$-highest weight states (given above), just as the $\mathfrak{A}_{p',p}$-Verma modules with $\mathfrak{u}(1)$-charge not equal to h do.

The picture which now emerges is that $\mathfrak{A}_{p',p}$-Verma modules have a highest weight state $\left|\phi_{r,s}\right\rangle$ which is a $\mathfrak{Vir}$-highest weight state with $\theta_{r,s} \leqslant h$, and the first descendant $\phi_{\theta_{r,s}-h}\left|\phi_{r,s}\right\rangle$ is another $\mathfrak{Vir}$-highest weight state $\left|\phi_{p'-r,s}\right\rangle$ which has $\theta_{p'-r,s} \geqslant h$. In fact, this exhausts the set of (independent) Virasoro highest weight states (which are not themselves Virasoro descendants) in any $\mathfrak{A}_{p',p}$-Verma module. The proof of this statement is identical to that of the analogous statement in [1, Prop. 5.2], so we omit it here.

An $\mathfrak{A}_{p',p}$-Verma module therefore decomposes into two $\mathfrak{Vir}$-modules. From a mathematical perspective, what we have shown is that the *injection* of graded *-algebras,

$$\mathcal{U}\left(\mathfrak{Vir}_{p',p}\right) \longrightarrow \mathfrak{A}_{p',p}, \tag{3.15}$$

describing the extension of the symmetry algebra, leads to a *surjection* of the corresponding Verma modules:

$$\widetilde{\mathcal{V}}_{r,s}^{p',p} \oplus \widetilde{\mathcal{V}}_{p'-r,s}^{p',p} \longrightarrow \mathcal{V}_{r,s}^{p',p}. \tag{3.16}$$

[6]Here we mean that ϕ_0 commutes with L_0, C and $\mathcal{S}$ whenever it belongs to $\mathfrak{A}_{p',p}$, hence may be consistently included in a "Cartan subalgebra" (maximal abelian subalgebra) of $\mathfrak{A}_{p',p}$.

[7]This module however has the curious property of decomposing into *two* different $\mathfrak{A}_{p',p}$-Verma modules, headed by the vectors (3.14). This is indicative of the fact that this module is not, strictly speaking, a Verma module, because its highest weight state is not an eigenstate of the *maximal* abelian subalgebra of $\mathfrak{A}_{p',p}$: We have deliberately chosen a highest weight state which is not an eigenvector of ϕ_0. Nevertheless, we will refer to this module as a Verma module (for regularity in exposition), with this slight subtlety understood implicitly.

Here, $\widetilde{\mathcal{V}}_{r,s}^{p',p}$ denotes the $\mathfrak{Vir}_{p',p}$-Verma module whose highest weight state is $\left|\phi_{r,s}\right\rangle$. We mention that this surjection is a homomorphism of $\mathfrak{Vir}$-modules (strictly speaking, $\mathfrak{Vir} \oplus \mathfrak{Vir}$-modules), meaning nothing more than that the action of the Virasoro modes is the same on both sides. Furthermore, because the adjoints of these *-algebras completely determine the sesquilinear forms on the Verma modules up to normalisation, it follows that we can choose (and indeed have chosen) this normalisation so that our surjection is *isometric* (norm-preserving).

3.3. $\mathcal{S}$**-eigenvalues.** Recall from Section 2.1 that $\mathcal{S}$ commutes with all the ϕ_n up to a factor of $(-1)^{4h}$. Its action on an $\mathfrak{A}_{p',p}$-Verma module is therefore completely determined by its eigenvalue on the highest weight state. Using the generalised commutation relations, Equation (2.10), we will compute (some of) these eigenvalues, under the assumption that every $\mathfrak{Vir}$-highest weight state has norm 1. We will describe an algorithm for computing *all* these eigenvalues (recursively) in Section 6.2.

On a $\mathfrak{A}_{p',p}$-highest weight state of $\mathfrak{u}(1)$-charge 0 (which is necessarily the vacuum $\left|0\right\rangle$), the generalised commutation relation with $\gamma = 1$ gives

$$\left\langle 0\middle|\mathcal{S}\middle|0\right\rangle \overset{1}{=} \left\langle 0\middle|\phi_h\phi_{-h}\middle|0\right\rangle = \left\langle \phi\middle|\phi\right\rangle = 1 \quad \Rightarrow \quad \mathcal{S}\left|0\right\rangle = \left|0\right\rangle. \tag{3.17}$$

This is of course a consequence of the normalisation we assumed at the end of Section 2.1. However, if $\left|\psi\right\rangle$ is a $\mathfrak{A}_{p',p}$-highest weight state of charge $\frac{1}{2}$, then we derive

$$\left\langle \psi\middle|\mathcal{S}\middle|\psi\right\rangle \overset{1}{=} \left\langle \psi\middle|2\phi_{h-1/2}\phi_{1/2-h}\middle|\psi\right\rangle = 2 \quad \Rightarrow \quad \mathcal{S}\left|\psi\right\rangle = 2\left|\psi\right\rangle, \tag{3.18}$$

as $\phi_{1/2-h}\left|\psi\right\rangle$ is a (normalised) Virasoro highest weight state. Interestingly, charge 1 $\mathfrak{A}_{p',p}$-highest weight states $\left|\psi\right\rangle$ require $\gamma = 3$:

$$1 = \left\langle \psi\middle|\phi_{h-1}\phi_{1-h}\middle|\psi\right\rangle \overset{3}{=} \left\langle \psi\middle|\frac{2h}{c}L_0\mathcal{S}\middle|\psi\right\rangle \quad \Rightarrow \quad \mathcal{S}\left|\psi\right\rangle = \frac{c}{2hh_{\psi}}\left|\psi\right\rangle, \tag{3.19}$$

where h_{ψ} is the conformal dimension of $\left|\psi\right\rangle$, as does the charge $\frac{3}{2}$ case:

$$1 = \left\langle \psi\middle|\phi_{h-3/2}\phi_{3/2-h}\middle|\psi\right\rangle \overset{3}{=} \left\langle \psi\middle|\frac{1}{2}\left(\frac{-1}{8}+\frac{2h}{c}L_0\right)\mathcal{S}\middle|\psi\right\rangle \quad \Rightarrow \quad \mathcal{S}\left|\psi\right\rangle = \frac{16c}{16hh_{\psi}-c}\left|\psi\right\rangle. \tag{3.20}$$

Similarly, the charge 2 case requires $\gamma = 5$:

$$\begin{aligned} 1 = \left\langle \psi\middle|\phi_{h-2}\phi_{2-h}\middle|\psi\right\rangle &\overset{5}{=} \left\langle \psi\right| \left[\frac{-h\left(c+18h+8\right)}{c\left(5c+22\right)}L_0 + \frac{2h\left(5h+1\right)}{c\left(5c+22\right)}\left(L_0^2+2L_0\right)\right]\mathcal{S}\left|\psi\right\rangle \\ \Rightarrow \quad \mathcal{S}\left|\psi\right\rangle &= \frac{c\left(5c+22\right)}{hh_{\psi}\left[2\left(5h+1\right)h_{\psi}-\left(c-2h+4\right)\right]}\left|\psi\right\rangle, \end{aligned} \tag{3.21}$$

and so on.

We remark that we could have instead chosen the norms of the $\mathfrak{Vir}$-highest weight states (or included constant factors in Equation (3.11) appropriately) so as to make the $\mathcal{S}$-eigenvalues above equal to 1. However, we would still have to compute these norms in any genuine calculation, so this does not represent a simplification of the formalism. We observe that to compute the $\mathcal{S}$-eigenvalue of a $\mathfrak{A}_{p',p}$-highest weight state of charge θ, it is necessary to use a $\gamma = 2\theta + 1$ generalised commutation relation when $\theta \in \mathbb{Z}$, but we can get away with a $\gamma = 2\theta$ relation when $\theta \notin \mathbb{Z}$. This pattern regarding the required orders of generalised commutation relations is quite common when computing with extended algebras, and we shall see why in Section 3.4.

Consider now a theory with $h \in \mathbb{Z}$, $\mathcal{M}(3,10)$ or $\mathcal{M}(5,6)$ for example. According to the above discussion, computing the eigenvalue of $\mathcal{S}$ on an $\mathfrak{A}_{p',p}$-module of charge $\theta = h$ requires us to employ a generalised commutation relation with $\gamma = 2h + 1$ (and no smaller). This is interesting as this relation receives contributions from every singular term of the operator product expansion (2.8), as well as from the first regular term. In other words, it is not possible to compute this $\mathcal{S}$-eigenvalue using only the information contained in the singular terms. We will shortly see further computations where it will prove necessary to use generalised commutation relations which include contributions from further regular terms of the operator product expansion.

It is also interesting to consider the $\mathcal{S}$-eigenvalue if the $\mathfrak{u}(1)$-charge of the $\mathfrak{A}_{p',p}$-highest weight state were negative. We easily find that in this case

$$\phi_{h-\theta}\phi_{\theta-h}\left|\psi\right\rangle \overset{0}{=} 0, \quad \text{but} \quad \phi_{h-\theta}\phi_{\theta-h}\left|\psi\right\rangle \overset{1}{=} \mathcal{S}\left|\psi\right\rangle. \tag{3.22}$$

It follows that $\mathcal{S}$ must vanish identically on any $\mathfrak{A}_{p',p}$-highest weight states of negative charge, hence on the corresponding Verma modules. (We have not bracketed these calculations with $\left\langle\psi\right|$ because we will need this conclusion to apply to *singular* highest weight states in Theorem 5.1.) Note that such a bracketing shows that the first descendant $\phi_{\theta-h}\left|\psi\right\rangle$ of such an highest weight state must be singular (though this is independent of whether the highest weight state itself is singular). In fact, an easy induction argument shows that every descendant of a highest weight state of negative charge is singular, so the corresponding irreducible module will be trivial.

3.4. **Symmetries of the Generalised Commutation Relations.** It may seem that the algebra $\mathfrak{A}_{p',p}$ is determined by an enormous number of generalised commutation relations. However, there is in fact a huge amount of redundancy present in these equations, which we shall now reveal. This redundancy is exposed by two easily derived symmetries of the expressions $R_{m,n}(\gamma)$ (see Equation (2.9)), whose evaluation defines the generalised commutation relations. Using the form of $R_{m,n}(\gamma)$ given on the right-hand-side of Equation (2.10), and the binomial identity

$$\binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1}, \tag{3.23}$$

we easily derive that

$$R_{m,n}(\gamma) - R_{m-1,n+1}(\gamma) = R_{m,n}(\gamma-1). \tag{3.24}$$

Similarly, the left-hand-side of Equation (2.10) makes the following symmetry evident:

$$R_{m,n}(\gamma) = (-1)^{\gamma-1} R_{n+2h-\gamma,m-2h+\gamma}(\gamma). \tag{3.25}$$

It is important to realise that these symmetries should be viewed as identities in $\mathfrak{A}_{p',p}$ which make sense when applied to a state of definite monodromy charge.

Equation (3.24) shows that the generalised commutation relations of given order γ contain all the information inherent in the generalised commutation relations of order $\gamma-1$ (because the latter can be derived from the former). It follows that any generalised commutation relation of order $\gamma' < \gamma$ may be derived from the generalised commutation relations of order γ. This should not be surprising when we recall that γ merely parametrises how many terms from the operator product expansion (2.8) contribute to the generalised commutation relation.

More interestingly, if we fix the conformal dimension $m+n$ (which is of course conserved), then Equation (3.24) implies that every generalised commutation relation of order γ is equivalent to any other arbitrarily chosen generalised commutation relation of order γ, modulo those of order $\gamma-1$. In turn, this generalised commutation relation of order $\gamma-1$ is equivalent to some arbitrarily chosen generalised commutation relation of order $\gamma-1$, modulo those of order $\gamma-2$, and so on. Since $R_{m,n}(\gamma)=0$ for $\gamma \leqslant 0$, we can conclude that any given generalised commutation relation is equivalent to a (finite) linear combination of "basic" generalised commutation relations, one for each order parameter γ. Furthermore, we can choose these basic relations arbitrarily.

There is one slight proviso to the above argument: We must respect monodromy charge restrictions throughout. Practically, this means that for a given conformal dimension $m+n$, there are in fact two disjoint classes of generalised commutation relations, corresponding to the two possible monodromy charges (integral and half-integral) of the states they are to act upon. This follows from Equation (3.24) by realising that it can not be used to relate $R_{m,n}(\gamma)$ with $R_{m-1/2,n+1/2}(\gamma)$, because these would have to act on states of different monodromy charge. A more precise version of our above conclusion is therefore that by restricting to some fixed conformal dimension $m+n$ and monodromy charge θ, any given generalised commutation relation is equivalent to a (finite) linear combination of "basic" generalised commutation relations, of which there is one (which we can choose arbitrarily) for each order parameter γ.

We can improve on this when γ is *even* by noting the following. If $m-n=2h-\gamma$, Equation (3.25) implies that

$$R_{m,n}(\gamma)=0 \qquad (m-n=2h-\gamma). \tag{3.26}$$

Note that this may or may not be allowed by the monodromy charge of the state on which we are acting. If it is not, we note that if $m-n=2h-\gamma+1$, Equation (3.25) implies that $R_{m-1,n+1}(\gamma) = -R_{m,n}(\gamma)$, and substituting into Equation (3.24) gives

$$R_{m,n}(\gamma)=\frac{1}{2}R_{m,n}(\gamma-1) \qquad (m-n=2h-\gamma+1). \tag{3.27}$$

In either case, Equation (3.24) now implies that when γ is even, any generalised commutation relation with parameter γ may be expressed as a linear combination of generalised commutation relations with parameter $\gamma-1$. In other words, the set of generalised commutation relations with even parameter γ contains the same information as the set with parameter $\gamma - 1$. From the point of view of a set of basic generalised commutation relations (chosen arbitrarily), this means that we can drop those corresponding to γ even.

This explains the property observed in Section 3.3 that computing the eigenvalue of $\mathcal{S}$ on a highest weight state of charge θ requires a generalised commutation relation of order $\gamma = 2\theta + 1$ when $\theta \in \mathbb{Z}$, but only $\gamma = 2\theta$ when $\theta \notin \mathbb{Z}$. Essentially, $\gamma = 2\theta + 1$ is the correct order, but when $\theta \notin \mathbb{Z}$, $2\theta + 1$ is even, hence it is convenient to use instead the (equivalent) generalised commutation relations of order 2θ.

4. Examples and Equivalences

In this section, we investigate the structure of the extended algebras $\mathfrak{A}_{p',p}$ with several examples. These examples are chosen so that one can identify these extended chiral algebras with those of other familiar theories. In other words, we provide simple (but detailed) instances of an apparent equivalence between conformal field theories. This is followed by a careful study of what such an equivalence means at the level of chiral algebras. We show that there are some subtleties to be addressed here, which we pin down by reconsidering what we mean by an extension of an algebra. We then illustrate our conclusions with an interesting extended example (relegated to Appendix A) in which a more involved conformal field theory equivalence is derived.

4.1. **Examples.**

$\underline{\mathcal{M}(3,4)}$. The simplest minimal model exhibiting a simple current is that corresponding to the Ising model, whose central charge is $c = \frac{1}{2}$. The simple current $\phi = \phi_{1,3} = \phi_{2,1}$ has conformal dimension $h = \frac{1}{2}$, and it is well-known that the extended theory describes a free fermion. This may be seen explicitly by substituting the first few $A^{(j)}(w)$ (derived in Section 2.2) into Equation (2.8), with $h = c = \frac{1}{2}$:

$$\phi(z)\,\phi(w) = \mathcal{S}\left[\frac{1}{z-w} + 2T(w)(z-w) + \partial T(w)(z-w)^2 + \ldots\right]. \tag{4.1}$$

The presence of the $\mathcal{S}$ in this equation (and those that follow) is not an essential complication, as it commutes with the ϕ_n by Equation (2.7), hence is a multiple of the identity on each $\mathfrak{A}_{3,4}$-Verma module. We may therefore treat it as a scaling factor, which we could set equal to the identity by suitably choosing the norms of the constituent $\mathfrak{Vir}$-highest weight states (Section 3.3). Alternatively (and equivalently), we may redefine the fermionic field as

$$\psi(z) = \mathcal{S}^{-1/2}\phi(z). \tag{4.2}$$

We note that the operator product expansion (4.1) implies that

$$T(w) = \frac{1}{2\mathcal{S}} : \partial\phi(w)\,\phi(w) : \; = \frac{1}{2} : \partial\psi(w)\,\psi(w) : , \tag{4.3}$$

as befits a fermionic theory. Finally, we recover the familiar anticommutation relation from the generalised commutation relation with $\gamma = 1$:

$$\phi_m \phi_n + \phi_n \phi_m \overset{1}{=} \delta_{m+n,0} \mathcal{S} \qquad \Longleftrightarrow \qquad \left\{ \psi_m, \psi_n \right\} \overset{1}{=} \delta_{m+n,0}. \tag{4.4}$$

$\underline{\mathcal{M}(3,5)}$. Now $h = \frac{3}{4}$ and $c = \frac{-3}{5}$, and we have the simplest example of a *graded parafermionic* theory, $\widehat{\mathfrak{osp}}(1\,|\,2)_1 / \widehat{\mathfrak{u}}(1)$ [19,20]. This follows from the operator product expansion

$$\phi(z)\,\phi(w) = \mathcal{S}\left[\frac{1}{(z-w)^{3/2}} - \frac{5}{2} T(w)\,(z-w)^{1/2} - \ldots\right], \tag{4.5}$$

and the identifications (with the notation used in [19])

$$\phi(z) \longleftrightarrow \psi_{1/2}(z) \qquad \text{and} \qquad T(z) \longleftrightarrow \frac{-2}{5} \mathcal{O}^{(1/2)}(z). \tag{4.6}$$

We remark that associativity (Section 2.1) forces the operator product expansion of this graded parafermion $\psi_{1/2}(z)$ to involve a non-trivial operator analogous to $\mathcal{S}$ (at least when the $\widehat{\mathfrak{osp}}(1\,|\,2)$-level is equal to 1). This was overlooked in the original treatments (but corrected in [5]).

One may wonder why we do not just redefine the simple current field, as we did with $\mathcal{M}(3,4)$, to remove $\mathcal{S}$ from $\phi(z)\,\phi(w)$. Taking $\mathcal{S}^{1/2}$ such that $\mathcal{S}^{1/2}\phi = \mathfrak{i}\phi\mathcal{S}^{1/2}$, we can define $\psi(z) = e^{-\mathfrak{i}\pi/4}\mathcal{S}^{-1/2}\phi(z)$ to achieve this goal. However, we then face the problem that $\psi(z)$ satisfies the same commutativity relations as $\phi(z)$, so associativity (Section 2.1) demands that $\psi(z)\,\psi(w)$ involve an $\mathcal{S}$-type operator, contradicting the fact that it was constructed so as not to. The problem here is that it is in fact not possible to make the above redefinition because $\mathcal{S}$ does *not* have such a square root[8] when anticommuting with ϕ.

$\underline{\mathcal{M}(3,8)}$ *and* $\underline{\mathcal{M}(4,5)}$. When $h = \frac{3}{2}$, the extended theory defines a *superconformal* field theory. We can see this explicitly by taking the generalised commutation relation

$$\phi_m \phi_n + \phi_n \phi_m \overset{3}{=} \mathcal{S}\left[\binom{m+\frac{1}{2}}{2} \delta_{m+n,0} + \frac{3}{c} L_{m+n}\right], \tag{4.8}$$

and defining

$$G_n = \sqrt{\frac{2c}{3\mathcal{S}}}\phi_n \tag{4.9}$$

(note that $\mathcal{S}$ and ϕ_n commute) to get the familiar relations

$$\left\{G_m, G_n\right\} \overset{3}{=} 2L_{m+n} + \left(m^2 - \tfrac{1}{4}\right) \delta_{m+n,0} \frac{c}{3}. \tag{4.10}$$

[8]Of course $\mathcal{S}$ has *many* square roots, being a self-adjoint operator on a complex (pre-)Hilbert space. What we claim is that none of them commute with the simple current field up to a constant multiplier (necessary for the above redefinition to work). Any prospective square root, $\mathcal{T}$ say, for which $\mathcal{T}\phi = \lambda\phi\mathcal{T}$, must satisfy $\lambda^2 = -1$, hence

$$\mathcal{T}|0\rangle = \mathcal{T}\phi_h\phi_{-h}|0\rangle = \lambda^2\mathcal{S}\mathcal{T}|0\rangle = \lambda^2\mathcal{T}|0\rangle = -\mathcal{T}|0\rangle \qquad (\text{as } \phi_h\phi_{-h}|0\rangle \overset{1}{=} |0\rangle). \tag{4.7}$$

Obviously $\mathcal{T}|0\rangle$ cannot vanish as $\mathcal{S}|0\rangle = |0\rangle$, so our claim if proved: No such square root $\mathcal{T}$ can exist. Conversely, it is easy to construct such a square root when $\mathcal{S}$ commutes with ϕ, because $\mathcal{S}$ is just a multiple of the identity on each extended Verma module.

The central charges of $\mathcal{M}(3,8)$ and $\mathcal{M}(4,5)$ ($\frac{-21}{4}$ and $\frac{7}{10}$) identify their extensions with the superconformal field theories $\mathcal{SM}(2,8)$ and $\mathcal{SM}(3,5)$ respectively.

4.2. **Equivalences and Algebra Isomorphisms.** We have just seen that the algebraic structure of the extensions of several minimal models can be identified with that of other well-known conformal field theories. Specifically, we have based this (loose) identification on the explicit form of a *finite* number of terms of the operator product expansion $\phi(z)\phi(w)$ (or on the terms of a particular generalised commutation relation). However, this only guarantees that the generalised commutation relations defining the extended chiral algebra have analogues under our identification when γ is sufficiently small, hence not for arbitrary values of γ. As might be expected, this means that it is strictly speaking *incorrect* to declare that these extended minimal models are precisely the respective well-known models that we have loosely identified them with, because their chiral algebras may not be isomorphic. We are therefore led to a consideration of precisely what we mean when we say that two conformal field theories are equivalent and equally, what we mean by saying that one is an extension of another.

Before discussing these considerations, let us remark that this concern is indeed valid, because such mismatches between chiral algebras have tangible consequences at the level of representation theory. This is best illustrated with the example of the extended algebra of the $h=\frac{3}{2}$ models $\mathcal{M}(3,8)$ and $\mathcal{M}(4,5)$, considered in Section 4.1. There, we identified these extended algebras with those of certain *superconformal* minimal models, exhibiting a super-Virasoro algebra ($\mathfrak{sVir}$) symmetry. Specifically, extending $\mathcal{M}(3,8)$ gave (roughly speaking) $\mathcal{SM}(2,8)$, and extending $\mathcal{M}(4,5)$ gave $\mathcal{SM}(3,5)$.

The easiest way of seeing that the corresponding chiral algebras are *not* isomorphic is to note that the $\mathfrak{A}_{3,8}$ and $\mathfrak{A}_{4,5}$-Verma modules are irreducible (as we shall prove in Section 5.2), whereas it is well-known that those of the corresponding super-Virasoro algebras are not (they contain non-trivial singular vectors). As the definitions of highest weight state and Verma module for these two chiral algebras are compatible[9], the notion of irreducibility should be preserved by any algebra isomorphism. We are therefore forced to conclude that $\mathfrak{A}_{3,8}$ and $\mathfrak{A}_{4,5}$ are *not* actually isomorphic to (the universal enveloping algebra of) the corresponding super-Virasoro algebras, $\mathfrak{sVir}_{2,8}$ and $\mathfrak{sVir}_{3,5}$, respectively.

The identification made in Section 4.1 (which we have just questioned) was limited to Equation (4.8), the generalised commutation relations with $\gamma = 3$. The results of Section 3.4 demonstrate that these generalised commutation relations imply those with $\gamma < 3$, but not necessarily those with $\gamma \geqslant 5$. Our non-isomorphism result therefore requires that there exist further identities satisfied in $\mathfrak{A}_{3,8}$ and $\mathfrak{A}_{4,5}$, which have no counterpart in the super-Virasoro algebra (and are responsible for the irreducibility of the Verma modules of the extended theory).

[9] We refer to [1, Sec. 5.4] for an example where they are not.

For example, we could consider the generalised commutation relation with $\gamma = 4$, which receives a contribution from the first regular term in the operator product expansion $\phi(z)\phi(w)$. At the level of fields, we have (using Equation (4.9))

$$:\phi(w)\phi(w): = A^{(4)}(w) = \frac{3\mathcal{S}}{2c}\partial T(w) \qquad \Rightarrow \qquad :G(w)G(w): = \partial T(w). \tag{4.11}$$

The latter relation is of course standard for superconformal field theories, and reflects the ($\gamma = 3$) anticommutation relation

$$\left\{G_{-3/2}, G_{-3/2}\right\} = 2L_{-3}. \tag{4.12}$$

(This illustrates the fact that when γ is even, the generalised commutation relations of order γ may be obtained from those of order $\gamma - 1$.) The generalised commutation relations with $\gamma = 5$ however imply the identity

$$:\partial G(w)G(w): = \frac{17}{5c+22} :T(w)T(w): + \frac{3(c+1)}{2(5c+22)}\partial^2 T(w), \tag{4.13}$$

which is not a generic identity in superconformal field theory. Indeed, we will see in Section 5.1 that such a relation cannot be satisfied identically, because the $\gamma = 5$ generalised commutation relations are sufficient to prove the vanishing of the first singular vector in the extended algebra vacuum Verma module, but this vector does not vanish identically in the super-Virasoro vacuum Verma module.

This example is typical of the general situation: Comparing a finite number of terms in the defining operator product expansions (or generalised commutation relations) is not generally enough to demonstrate an isomorphism of chiral algebras (even if all singular terms are included). To make this point precise, we introduce a sequence of extended chiral algebras $\mathfrak{A}^{(\gamma)}_{p',p}$, $\gamma \in \mathbb{Z}$, in which only the generalised commutation relations of order γ are imposed. The results of Section 3.4 allow us to depict this sequence as follows:

$$\mathfrak{F}_{p',p} \to \cdots \to \mathfrak{A}^{(-1)}_{p',p} \xrightarrow{\cong} \mathfrak{A}^{(0)}_{p',p} \to \mathfrak{A}^{(1)}_{p',p} \xrightarrow{\cong} \mathfrak{A}^{(2)}_{p',p} \to \mathfrak{A}^{(3)}_{p',p} \xrightarrow{\cong} \mathfrak{A}^{(4)}_{p',p} \to \mathfrak{A}^{(5)}_{p',p} \to \cdots \to \mathfrak{A}_{p',p}. \tag{4.14}$$

Here, $\mathfrak{F}_{p',p}$ denotes the free algebra generated by the ϕ_n modes, and the arrow from $\mathfrak{A}^{(\gamma-1)}_{p',p}$ to $\mathfrak{A}^{(\gamma)}_{p',p}$ represents the quotient map obtained by factoring out the (two-sided) ideal generated by the generalised commutation relations of order γ. The "fully" extended algebra $\mathfrak{A}_{p',p}$ is then the *direct limit* of these "partially" extended algebras:

$$\mathfrak{A}_{p',p} = \varinjlim \mathfrak{A}^{(\gamma)}_{p',p}. \tag{4.15}$$

In this formalism, the precise version of the correspondence between $\mathfrak{A}_{3,8}$, $\mathfrak{A}_{4,5}$ and the appropriate super-Virasoro algebras is as follows[10]:

$$\mathfrak{A}^{(3)}_{3,8} \cong \mathcal{U}\big(\mathfrak{sVir}_{2,8}\big) \ncong \mathfrak{A}^{(5)}_{3,8} \qquad \text{and} \qquad \mathfrak{A}^{(3)}_{4,5} \cong \mathcal{U}\big(\mathfrak{sVir}_{3,5}\big) \ncong \mathfrak{A}^{(5)}_{4,5}. \tag{4.17}$$

[10] We remark that the algebra isomorphisms constructed in [1, Sec. 4] must also be understood in this sense. Specifically, if we denote the extended chiral algebra of the level-k $\mathsf{SU}(2)$ Wess-Zumino-Witten model by $\widehat{\mathfrak{A}}_k$, then the precise

The compatibility of the respective definitions of a highest weight state imply that these isomorphic chiral algebras have isomorphic Verma modules, and more importantly, isomorphic irreducible highest weight modules. In general then, a simple current defines an infinite family of extended chiral algebras, parametrised by the order (γ) of the defining generalised commutation relations. In this sense, $\mathcal{U}\left(\mathfrak{sVir}_{2,8}\right)$ is *an* extension of $\mathcal{U}\left(\mathfrak{Vir}_{3,8}\right)$, but not a "maximal" extension (in the obvious sense).

Physically however, it is not the Verma modules which are fundamental, but the corresponding irreducible modules. We might expect that different extensions (defined by different γ) would have isomorphic irreducible representations, and this is indeed true under one important proviso. This follows from the fact that it is immaterial to the construction of the irreducible modules from the Verma modules whether the singular vectors vanish identically or are quotiented out. The proviso follows from the fact that the algebraic structure of the extensions must be sufficient to be able to compute whether a given vector is singular or not. The general, but fundamental, requirement of actually being able to compute with a given symmetry algebra leads to quite non-trivial bounds on γ. For example, we have seen in Section 3.3 that computing $\mathcal{S}$-eigenvalues requires $\gamma > 2h$.

A full analysis of this proviso will not be required here, because we only consider extensions in which we know *a priori* that computation is possible. The upshot is then that the isomorphisms and non-isomorphisms of (4.17) still imply the corresponding isomorphisms between the irreducible $\mathfrak{A}_{3,8}$ and $\mathfrak{A}_{4,5}$-modules and the irreducible super-Virasoro modules (though not between the corresponding Verma modules). These isomorphisms therefore detail *explicitly* the conformal field theory equivalences

$$\mathcal{M}(3,8) \equiv \mathcal{SM}(2,8) \qquad \text{and} \qquad \mathcal{M}(4,5) \equiv \mathcal{SM}(3,5), \tag{4.18}$$

and imply the corresponding character identities.

We conclude this discussion by mentioning the other equivalences noted in Section 4.1, involving the theories describing the free fermion and the (level 1) graded parafermion. If we denote their chiral algebras by $\mathfrak{F}$ and $\mathfrak{G}$ respectively, then the algebra isomorphisms described there take the form

$$\mathfrak{A}^{(3)}_{3,4} \cong \mathfrak{F} \qquad \text{and} \qquad \mathfrak{A}^{(3)}_{3,5} \cong \mathfrak{G}. \tag{4.19}$$

Here, we define $\mathfrak{F}$ as the associative algebra defined by the anticommutation relations of Equation (4.4) and the relation

$$L_n = \frac{-1}{2} \sum_m \left(m + \frac{1}{2}\right) :\phi_m \phi_{n-m}: , \tag{4.20}$$

isomorphisms proved there are

$$\widehat{\mathfrak{A}}^{(3)}_2 \cong \mathfrak{A}^{(3)}_{3,4} \otimes \mathfrak{A}^{(3)}_{3,4} \otimes \mathfrak{A}^{(3)}_{3,4} \qquad \text{and} \qquad \widehat{\mathfrak{A}}^{(2)}_4 \cong \mathcal{U}\left(\widehat{\mathfrak{sl}}(3)_1\right) \ncong \widehat{\mathfrak{A}}_4. \tag{4.16}$$

We expect that the first of these isomorphisms can be extended to all $\gamma \geqslant 3$ (hence for the fully extended algebras). The presence of non-trivial $\widehat{\mathfrak{sl}}(3)_1$-singular vectors implies that such an extension is not possible in the second case (hence the non-isomorphism indicated).

corresponding to Equation (4.3) (and necessary for conformal symmetry). Similarly, we define $\mathfrak{G}$ as the associative algebra defined by the generalised commutation relations given in [19]. We will see in Section 5.1 that for $\mathfrak{A}_{3,4}$, $\gamma = 3$ is sufficient to prove that all singular vectors vanish. Therefore, even if there were further non-trivial algebraic relations (corresponding to $\gamma \geqslant 5$ for example), these relations would have to act trivially on every representation (irreducible *or* Verma).

5. Singular Vectors

Recall from Section 3.2 that there is an isometric surjection (3.16) mapping the (direct sum of) the two constituent $\mathfrak{Vir}$-Verma modules onto the corresponding $\mathfrak{A}_{p',p}$-Verma module. Furthermore, this is a homomorphism of $\mathfrak{Vir}$-modules. The natural question to ask then is which vectors get mapped to zero. In other words, what is the kernel of this homomorphism. As our homomorphism is norm-preserving, we immediately see that this kernel can only consist of null vectors. Clarifying our original question slightly, we are led to ask whether *all* the $\mathfrak{Vir}$-null vectors are mapped to zero, that is, is the kernel precisely the set of null vectors? A negative answer to this question necessarily requires that there exist non-trivial singular vectors in the $\mathfrak{A}_{p',p}$-Verma module.

Now, the irreducible $\mathfrak{Vir}$-modules comprising the minimal models are well-known to be quotients of the corresponding Verma modules by a submodule generated by two[11] singular vectors, which we refer to as the *principal* singular vectors. The explicit form of these singular vectors is only known in special cases [17], but their grades follow easily from Kac's determinant formula. Specifically, the $\mathfrak{Vir}$-Verma module $\widetilde{\mathcal{V}}_{r,s}^{p',p}$ has principal singular vectors at grades rs and $(p'-r)(p-s)$.

We will restrict ourselves to considering the images of the principal singular vectors in the vacuum $\mathfrak{A}_{p',p}$-Verma module $\mathcal{V}_{1,1}^{p',p}$. There are three to consider, at grades $p'-1$ and $p-1$, corresponding to the simple current $\mathfrak{Vir}$-module, and at grade $(p'-1)(p-1)$, corresponding to the vacuum $\mathfrak{Vir}$-module. We will first investigate the explicit form of these singular vectors in the simplest minimal models, before lifting our conclusions to the general case. For clarity, we will usually restrict attention to the principal singular vector of lowest grade.

5.1. **Examples.**

$\underline{\mathcal{M}(3,4)}$. We recall the well-known fact that the irreducible modules (Fock spaces) of the extended algebra are freely generated, that is, no null vectors are encountered in their construction. Indeed,

[11] In the vacuum Verma module $\widetilde{\mathcal{V}}_{1,1}^{p',p}$, the first of these is $L_{-1}|0\rangle$, which is usually taken to vanish identically. Physically, this reflects the requirement that the (chiral) vacuum be invariant under the (chiral) global conformal transformations generated by L_1, L_0 and L_{-1} — that is, under the (chiral) conformal group. Mathematically, this may be derived as a simple consequence of the state-field correspondence: $L_{-1}|0\rangle$ corresponds to the derivative of the identity field.

the first (principal) singular vector of the extended vacuum module is

$$\begin{aligned}\left(L_{-2}-\frac{3}{4}L_{-1}^2\right)\left|\phi\right\rangle &= \left(L_{-2}-\frac{3}{4}L_{-1}^2\right)\phi_{-1/2}\left|0\right\rangle = \phi_{-1/2}L_{-2}\left|0\right\rangle \\ &\overset{3}{=} \frac{1}{2}\phi_{-1/2}\phi_{-3/2}\phi_{-1/2}\left|0\right\rangle,\end{aligned} \tag{5.1}$$

which clearly vanishes due to the anticommutation relations (4.4). Similarly, the second principal singular vector is

$$\begin{aligned}\left(L_{-3}-4L_{-2}L_{-1}+\frac{4}{3}L_{-1}^3\right)\phi_{-1/2}\left|0\right\rangle &= \left(L_{-3}\phi_{-1/2}-4L_{-2}\phi_{-3/2}+8\phi_{-7/2}\right)\left|0\right\rangle \\ &= \left(\phi_{-1/2}L_{-3}-4\phi_{-3/2}L_{-2}\right)\left|0\right\rangle \\ &\overset{3}{=} \left(\frac{1}{4}\phi_{-1/2}\phi_{-5/2}\phi_{-1/2}-2\phi_{-3/2}\phi_{-3/2}\phi_{-1/2}\right)\left|0\right\rangle \\ &= 0.\end{aligned} \tag{5.2}$$

Finally, a direct computation using the $\gamma = 3$ generalised commutation relation shows that the last singular vector

$$\left(L_{-6}+\frac{22}{9}L_{-4}L_{-2}-\frac{31}{36}L_{-3}^2-\frac{16}{27}L_{-2}^3\right)\left|0\right\rangle \tag{5.3}$$

may be expressed as a linear combination of the vectors $\phi_{-11/2}\phi_{-1/2}\left|0\right\rangle$, $\phi_{-9/2}\phi_{-3/2}\left|0\right\rangle$, and $\phi_{-7/2}\phi_{-5/2}\left|0\right\rangle$, whose coefficients exactly cancel.

To summarise, the singular vectors of the extended Verma module $\mathcal{V}_{1,1}^{3,4}$ all vanish identically. This Verma module is thus irreducible. The same can easily be verified for the remaining $\mathfrak{A}_{3,4}$-Verma module $\mathcal{V}_{1,2}^{3,4}$, whose highest weight state has conformal dimension $\frac{1}{16}$.

<u>$\mathcal{M}(3,5)$</u>. It is not hard to show that the first (principal) singular vector may again be shown to vanish identically:

$$\begin{aligned}\left(L_{-2}-\frac{3}{5}L_{-1}^2\right)\phi_{-3/4}\left|0\right\rangle &\overset{3}{=} \left(\frac{-4}{5}\phi_{-7/4}\phi_{-1/4}\mathcal{S}^{-1}\phi_{-3/4}-\frac{6}{5}\phi_{-11/4}\phi_{3/4}\mathcal{S}^{-1}\phi_{-3/4}-\frac{6}{5}\phi_{-11/4}\right)\left|0\right\rangle \\ &= \left(\frac{4}{5}\phi_{-7/4}\phi_{-1/4}\phi_{-3/4}+\frac{6}{5}\phi_{-11/4}\phi_{3/4}\phi_{-3/4}-\frac{6}{5}\phi_{-11/4}\right)\left|0\right\rangle \\ &= \left(\frac{6}{5}\phi_{-11/4}-\frac{6}{5}\phi_{-11/4}\right)\left|0\right\rangle = 0.\end{aligned} \tag{5.4}$$

In the third equality here, we have made use of the simple relations

$$\phi_{h-1}\phi_{-h}\left|0\right\rangle \overset{1}{=} 0 \qquad \text{and} \qquad \phi_{h}\phi_{-h}\left|0\right\rangle \overset{1}{=} \left|0\right\rangle, \tag{5.5}$$

which apply quite generally and which will be used without comment in the future. We remark that to obtain this vanishing result, we must remember that $\mathcal{S}$ anticommutes with the ϕ_n and leaves $\left|0\right\rangle$ invariant. It should be clear that if there were no $\mathcal{S}$-operator, then the final result above would be

$\frac{-12}{5}\phi_{-11/4}\left|0\right\rangle$ rather than zero, implying that

$$\left(L_{-2}+\frac{3}{5}L_{-1}^{2}\right)\left|\phi\right\rangle=0 \tag{5.6}$$

(which contradicts the fact that this linear combination is not even null).

The second principal singular vector also vanishes identically, although the demonstration of this is deferred to Appendix B. This computation captures the essential complications of such a verification for the second primary singular vector. The vanishing of the Virasoro vacuum singular vector (at grade 8) may be demonstrated similarly, but this and the corresponding computations for the other $\mathfrak{A}_{3,5}$-Verma module will be omitted.

$\underline{\mathcal{M}(3,7)}$. We have $h=\frac{5}{4}$ and $c=\frac{-25}{7}$. The first (principal) singular vector is

$$\begin{aligned}\left(L_{-2}-\frac{3}{7}L_{-1}^{2}\right)\phi_{-5/4}\left|0\right\rangle &=\left(\phi_{-5/4}L_{-2}+\frac{3}{4}\phi_{-13/4}-\frac{6}{7}\phi_{-13/4}\right)\left|0\right\rangle\\ &\overset{3}{=}\left(\frac{-10}{7}\phi_{-5/4}\phi_{-3/4}\phi_{-5/4}-\frac{3}{28}\phi_{-13/4}\right)\left|0\right\rangle\\ &\overset{3}{=}\left(\frac{-10}{7}\left[\frac{7}{20}L_{-2}-\frac{3}{8}\phi_{-13/4}\phi_{5/4}\right]\phi_{-5/4}-\frac{3}{28}\phi_{-13/4}\right)\left|0\right\rangle\\ &=\frac{-1}{2}\left(L_{-2}-\frac{3}{7}L_{-1}^{2}\right)\phi_{-5/4}\left|0\right\rangle,\end{aligned} \tag{5.7}$$

and so it again vanishes. This calculation typifies the general procedure: The Virasoro modes are commuted to the right until they act on the vacuum, then the $\gamma=3$ generalised commutation relation is used to write them in terms of ϕ-modes. We then apply a suitable generalised commutation relation to the two leftmost ϕ-modes in order to re-express them in terms of Virasoro modes, finally getting back a multiple of the singular vector.

$\underline{\mathcal{M}(3,8)}$. We consider the first singular vector of $\mathcal{M}(3,8)$ ($c=\frac{-21}{4}$), and apply the procedure outlined above. Commuting and using $\gamma=3$ gives

$$\left(L_{-2}-\frac{3}{8}L_{-1}^{2}\right)\left|\phi\right\rangle\overset{3}{=}\left(\frac{-7}{4}\phi_{-3/2}\phi_{-1/2}-\frac{1}{8}L_{-1}^{2}\right)\left|\phi\right\rangle. \tag{5.8}$$

We now apply $\gamma=5$ to $\phi_{-3/2}\phi_{-1/2}$ and tidy up, getting

$$\left(L_{-2}-\frac{3}{8}L_{-1}^{2}\right)\left|\phi\right\rangle\overset{5}{=}-6\left(L_{-2}-\frac{3}{8}L_{-1}^{2}\right)\left|\phi\right\rangle, \tag{5.9}$$

hence the (by now) expected vanishing.

$\underline{\mathcal{M}(4,5)}$. If we try to apply this procedure to the $\mathcal{M}(4,5)$ vector $\left(L_{-2}-\frac{3}{8}L_{-1}^{2}\right)\left|\phi\right\rangle$ ($c=\frac{7}{10}$), we find that we end up with precisely this vector again, rather than a non-trivial multiple of it:

$$\left(L_{-2}-\frac{3}{8}L_{-1}^{2}\right)\left|\phi\right\rangle\overset{3}{=}\left(\frac{7}{30}\phi_{-3/2}\phi_{-1/2}-\frac{1}{8}L_{-1}^{2}\right)\left|\phi\right\rangle\overset{5}{=}\left(L_{-2}-\frac{3}{8}L_{-1}^{2}\right)\left|\phi\right\rangle. \tag{5.10}$$

Of course, this vector is not null in the $\mathcal{M}(4,5)$ model, so it should not be surprising that it does not vanish identically. The first singular vector is in fact

$$\left(L_{-3}-\frac{4}{3}L_{-2}L_{-1}+\frac{4}{15}L_{-1}^{3}\right)\left|\phi\right\rangle, \tag{5.11}$$

and our procedure with $\gamma = 3$, 5, and then 5 again, proves that it indeed vanishes identically.

5.2. **The General Case.** Based on these computations, it seems reasonable to conjecture that all the Virasoro singular vectors vanish identically when mapped into the appropriate $\mathfrak{A}_{p',p}$-Verma module. A proof is presented below for the vacuum module. Unfortunately, the same argument cannot be extended to all $\mathfrak{A}_{p',p}$-Verma modules. In particular, it breaks down completely when applied to modules of charge h. However, after presenting this proof, we will discuss why the vanishing of the singular vectors in the $\mathfrak{A}_{p',p}$-vacuum Verma module in fact implies the corresponding result for the other Verma modules. We mention in passing that the method established in Section 5.1 to demonstrate the vanishing of the singular vectors requires using the generalised commutation relations with $\gamma = 4h-1$ (or, equivalently, $4h-2$ when $4h$ is odd). These clearly receive contributions from many regular terms in the operator product expansion (2.8).

Theorem 5.1. *The vacuum $\mathfrak{A}_{p',p}$-Verma module $\mathcal{V}_{1,1}^{p';p}$ of the $\mathcal{M}(p',p)$ model with $p > p' > 2$ has no singular vectors, hence is irreducible.*

In [1, Thm. 5.3], we have established the analogous result for the $\mathsf{SU}(2)$ Wess-Zumino-Witten models (and all modules) by direct, though highly non-trivial, computation. Such a direct proof is not possible in the present context as we do not have as detailed a knowledge of the explicit form of the principal Virasoro singular vectors. The method of proof we give below is therefore indirect, being essentially a proof by contradiction. We remark that this contradiction argument is in fact quite general and should serve as a prototype for future generalisations.

Before giving the formal proof of Theorem 5.1, let us outline the simple idea behind it. The vacuum $\mathfrak{A}_{p',p}$-Verma module consists of two Virasoro modules (of unknown character). If one of these modules contained a non-vanishing singular vector, then it would have to possess a non-vanishing singular $\mathfrak{A}_{p',p}$-descendant *in the other Virasoro module*. By comparing the depth (conformal dimension) of this supposedly singular descendant with the depths of the known singular vectors of this other module, we derive a contradiction, showing that there could not have been such a non-vanishing singular vector to start with. We illustrate this idea schematically in Figure 1.

Proof. We argue by contradiction. Let us begin with the first primitive singular vector $\left|\chi_1\right\rangle$ of the $\mathfrak{Vir}$-Verma module $\widetilde{\mathcal{V}}_{p'-1,1}^{p',p}$ at grade $p'-1$. In $\mathcal{V}_{1,1}^{p';p}$, $\left|\chi_1\right\rangle$ is a $\mathfrak{Vir}$-highest weight state of conformal dimension $p'-1+h$. As there can be no null vectors of conformal dimension lower than this in $\mathcal{V}_{1,1}^{p';p}$, it follows that $\left|\chi_1\right\rangle$ must in fact be an $\mathfrak{A}_{p',p}$-highest weight state. Suppose that the singular vector $\left|\chi_1\right\rangle$ does *not* vanish identically (although its norm does of course). It must then have a $\mathfrak{u}(1)$-charge θ (not exceeding h). In addition, $\left|\chi_1\right\rangle$ is an eigenstate of $\mathcal{S}$ with eigenvalue $(-1)^{4h} \neq 0$,

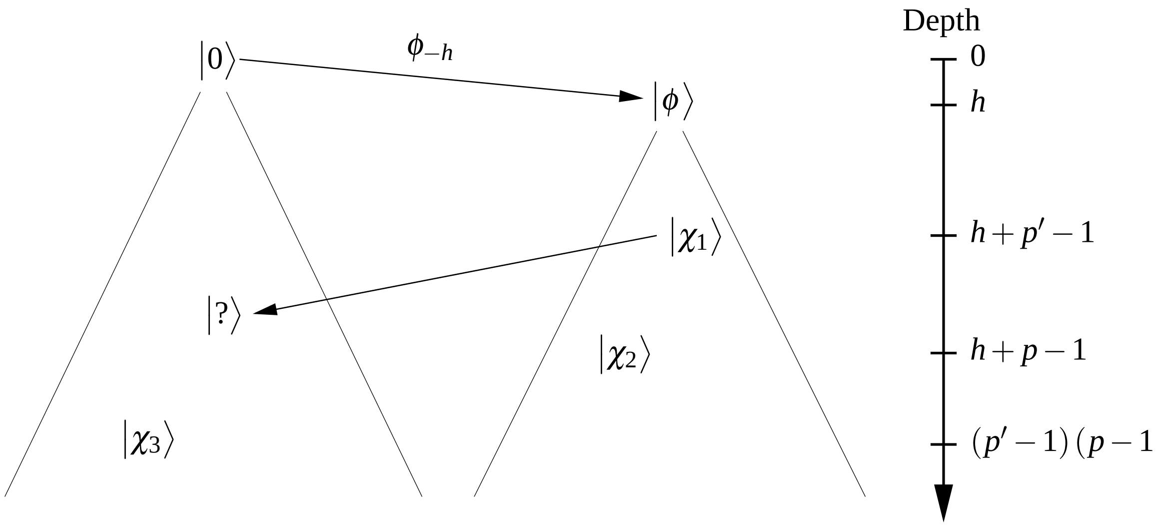

FIGURE 1. A pictorial representation of the vacuum $\mathfrak{A}_{p',p}$-Verma module $\mathcal{V}_{1,1}^{p',p}$, illustrating the mechanism behind the proof of Theorem 5.1. The states $|\chi_i\rangle$, $i = 1,2,3$, are the principal Virasoro singular vectors, and $|?\rangle$ is the purported singular descendant whose non-existence gives the contradiction.

from which we conclude that θ cannot be negative (Section 3.3). As the first descendant of $|\chi_1\rangle$ is obtained by acting with $\phi_{\theta-h}$, it has conformal dimension $p'-1+2h-\theta$. This descendant is non-zero by definition, and its norm obviously vanishes:

$$\langle\chi_1|\phi_{h-\theta}\phi_{\theta-h}|\chi_1\rangle \propto \langle\chi_1|\chi_1\rangle = 0. \tag{5.12}$$

On the other hand, $\phi_{\theta-h}|\chi_1\rangle$ is a state in $\widetilde{\mathcal{V}}_{1,1}^{p',p}$ which is not a descendant of $L_{-1}|0\rangle$ (as this has been taken to vanish identically). It is therefore either the Virasoro vacuum singular vector, or a (Virasoro) descendant thereof. The desired contradiction is now obtained (as $p \geqslant 4$) because the first singular element $|\chi_3\rangle$ of $\widetilde{\mathcal{V}}_{1,1}^{p',p}$ (hence the first state with zero norm that has not yet been proven to vanish identically) has conformal dimension $(p'-1)(p-1)$, and

$$p'-1+2h-\theta \geqslant \big(p'-1\big)(p-1) \qquad \Rightarrow \qquad p \leqslant 2-\frac{2\theta}{p'} \leqslant 2. \tag{5.13}$$

The singular vector $|\chi_1\rangle$ therefore vanishes identically. Clearly its Virasoro descendants will then also vanish identically.

The next primitive singular vector is $|\chi_2\rangle \in \widetilde{\mathcal{V}}_{p'-1,1}^{p',p}$ at grade $p-1$. Its conformal dimension is then $p-1+h$. Since we have just shown that there are no non-vanishing null vectors of lower conformal dimension in $\mathcal{V}_{1,1}^{p',p}$, $|\chi_2\rangle$ must also be an $\mathfrak{A}_{p',p}$-highest weight state of charge $0 \leqslant \theta \leqslant h$. If non-vanishing, it would necessarily have a non-vanishing descendant of conformal dimension $p-1+2h-\theta$, which is smaller than $(p'-1)(p-1)$ unless $p' \leqslant 2-\frac{2\theta}{p} \leqslant 2$, another contradiction (as $p' \geqslant 3$). So, $|\chi_2\rangle$ and its Virasoro descendants also vanish identically.

Finally, the primitive singular vector $\left|\chi_3\right\rangle \in \widetilde{\mathcal{V}}^{p',p}_{1,1}$ at grade $(p'-1)(p-1)$ should have, if non-vanishing itself, a non-vanishing singular descendant in $\widetilde{\mathcal{V}}^{p',p}_{p'-1,1}$. But we have seen that all the singular vectors in $\widetilde{\mathcal{V}}^{p',p}_{p'-1,1}$ vanish identically, hence there is no such descendant. This last contradiction proves the result. ∎

Abstractly, this result may be restated as follows: Every Virasoro singular vector, hence every null vector, in $\widetilde{\mathcal{V}}^{p',p}_{1,1}$ and $\widetilde{\mathcal{V}}^{p',p}_{p'-1,1}$ is mapped to zero by the surjection (3.16). Thus, we have completely characterised the kernel of this map. This is summarised in the following corollary.

Corollary 5.2. *When $r = s = 1$, the kernel of the surjection (3.16) is precisely the submodule of Virasoro null vectors, hence*

$$\widetilde{\mathcal{L}}^{p',p}_{1,1} \oplus \widetilde{\mathcal{L}}^{p',p}_{p'-1,1} \cong \mathcal{V}^{p',p}_{1,1} \cong \mathcal{L}^{p',p}_{1,1}, \tag{5.14}$$

where $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ denote the irreducible $\mathfrak{Vir}_{p',p}$ *and $\mathfrak{A}_{p',p}$-highest weight modules (respectively).*

We consider now the argument behind the proof of Theorem 5.1, as it applies to the other $\mathfrak{A}_{p',p}$-modules of the theory. The problem with this argument is that it is easy to see that it cannot work for all modules. The most extreme example illustrating this is an $\mathfrak{A}_{p',p}$-module whose highest weight state has charge h. Such modules are composed of two identical $\mathfrak{Vir}$-modules, connected by the mode ϕ_0. Evidently, a singular vector in one of these $\mathfrak{Vir}$-modules has a counterpart in the other at the same grade, so establishing our conclusion the same way would amount to justifying that the charge of the singular vector is greater than h (contradicting it being an $\mathfrak{A}_{p',p}$-highest weight state).

Instead, we recall that in a rational conformal field theory the vacuum singular vector is supposed to determine which highest weight modules can be consistently added to the theory. In other words, it controls the spectrum of the theory. Specifically, a non-trivial singular vector $\left|\chi\right\rangle$ of the vacuum module corresponds to a null field $\chi(z)$ under the state-field correspondence. The zero-mode χ_0 of the null field then selects the allowed highest weight states through the simple requirement that

$$\chi_0\left|\psi\right\rangle = 0. \tag{5.15}$$

It is worth emphasising that this is a consistency requirement that all conformal field theories must satisfy. It has been checked explicitly for the $\mathcal{M}(2,p)$ models in [3], where it is claimed that the result for general minimal models was proven in [21].

This begs the question: Why do the singular vectors of the other highest weight modules of the theory not further restrict the spectrum? The answer appears to be: Because these other singular vectors may all be obtained from the vacuum singular vector. More precisely, we have:

Claim. *If $\left|\chi\right\rangle$ is the $\mathcal{M}(p',p)$ vacuum singular vector at grade $(p'-1)(p-1)$, and $\left|\zeta_{r,s}\right\rangle$ is the* first *principal singular vector of the Verma module headed by $\left|\psi_{r,s}\right\rangle$ (with r and s chosen so that $rs < (p'-r)(p-s)$), then*

$$\left|\zeta_{r,s}\right\rangle = \chi_{-rs}\left|\psi_{r,s}\right\rangle, \tag{5.16}$$

up to an arbitrary normalisation. The second principal singular vector is then a linear combination of $\chi_{-(p'-r)(p-s)}\left|\psi_{r,s}\right\rangle$ *and Virasoro descendants of* $\left|\zeta_{r,s}\right\rangle$.

Obviously, $\chi_{-rs}\left|\psi_{r,s}\right\rangle$ is null, hence proportional to $\left|\zeta_{r,s}\right\rangle$, so our Claim is just that this proportionality constant is non-zero. Note that $\chi_n\left|\psi_{r,s}\right\rangle$ vanishes identically for all $n < rs$.

This is not hard to verify in simple cases (for instance, $\mathcal{M}(2,5)$ and $\mathcal{M}(3,4)$). Indeed, it follows easily from [3, Thm. 3.6] that it is true for all $\mathcal{M}(2,p)$ models. However, we are not aware of proofs of this Claim for general minimal models, even though we are certain of its truth. We shall therefore assume that it is valid in what follows.

The power of this Claim should be evident. By proving that the vacuum singular vector identically vanishes in the extended theory (itself a consequence of the vanishing of the singular vectors of the simple current module), we obtain the vanishing of the null field $\chi(z)$, and hence the vanishing of each mode χ_n. If the singular vectors of every (allowed) Virasoro module are induced by these modes acting on the relevant highest weight states, then they must also vanish identically in the extended theory. The implication of Corollary 5.2 is therefore the following strengthening:

Corollary 5.3. *Assuming the above Claim,*

$$\widetilde{\mathcal{L}}_{r,s}^{p',p} \oplus \widetilde{\mathcal{L}}_{p'-r,s}^{p',p} \cong \mathcal{V}_{r,s}^{p',p} \cong \mathcal{L}_{r,s}^{p',p}, \tag{5.17}$$

for all $1 \leqslant r \leqslant p'-1$, $1 \leqslant s \leqslant p-1$.

In other words, every such $\mathfrak{A}_{p',p}$-Verma module is irreducible, and decomposes under $\mathfrak{Vir}_{p',p}$ into the direct sum of two irreducible modules. This is our main result.

Let us conclude this section with a final remark. From the point of view of the algebra $\mathfrak{A}_{p',p}$, it is not natural to leave the central role in determining the spectrum to the Virasoro vacuum singular vector. It would be more natural to have a picture in which the two principal singular vectors of the simple current module would themselves control the whole spectrum (after all, it is the operator product expansion of the simple current with itself that is supposed to contain all the information of the theory). We will present computations supporting this expectation in the next section.

6. Some Simple Applications

The absence of the singular vectors in the $\mathfrak{A}_{p',p}$-modules which has just been demonstrated has immediate structural consequences. In particular, the first principal singular vector of ϕ vanishing identically implies powerful recurrence relations for the fields $A^{(j)}(w)$ appearing in the operator product expansion (2.8) of $\phi(z)$ with itself. These relations provide a fundamental calculational tool for computing within these extended algebras. We illustrate this by showing how they may be used to efficiently determine the $\mathcal{S}$-eigenvalues (proceeding directly as in Section 3.3 rapidly becomes cumbersome when the monodromy charges are large). As fundamental computational tools, it is important to determine the applicability of these recursion relations. We will see that on occasion a given recursion relation may fail to determine a small (finite) number of the $A^{(j)}(w)$, and isolate this phenomenon precisely through an amusing exercise in number theory. We conclude the section with a brief discussion on the role of the singular vectors of the extended vacuum module in restricting the spectrum of the theory.

6.1. **Recursion Relations for the $\left|A^{(j)}\right\rangle$.** We illustrate the derivation of these recursion relations for the $\mathcal{M}(3,p)$ models, for which $h = \frac{1}{4}(p-2)$. The first principal singular vector of the simple current module is

$$\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)\left|\phi\right\rangle, \tag{6.1}$$

so its vanishing in $\mathcal{V}_{2,1}^{3,p}$ and the usual commutation relations give

$$\begin{aligned} 0 &= \phi_{h-j+2}\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)\left|\phi\right\rangle \\ &= \left[\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)\phi_{h-j+2} - \frac{3(2h+1-j)}{2h+1}L_{-1}\phi_{h-j+1} + \frac{j(8h+1-3j)}{2(2h+1)}\phi_{h-j}\right]\left|\phi\right\rangle. \end{aligned} \tag{6.2}$$

Recalling Equation (2.13), we can recast this result in the form

$$\frac{j(8h+1-3j)}{2(2h+1)}\left|A^{(j)}\right\rangle = \frac{3(2h+1-j)}{2h+1}L_{-1}\left|A^{(j-1)}\right\rangle - \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)\left|A^{(j-2)}\right\rangle. \tag{6.3}$$

The prefactor on the left hand side vanishes (thus $\left|A^{(j)}\right\rangle$ is not determined) when $j=0$ or $3j = 8h+1$. But the latter can only occur if $2p-3 = 8h+1 \in 3\mathbb{Z}$, hence $p \in 3\mathbb{Z}$. As $p' = 3$ and p must be coprime, we therefore find that for all $j > 0$,

$$\left|A^{(j)}\right\rangle = \frac{6(2h+1-j)}{j(8h+1-3j)}L_{-1}\left|A^{(j-1)}\right\rangle + \frac{3}{j(8h+1-3j)}L_{-1}^2\left|A^{(j-2)}\right\rangle - \frac{2(2h+1)}{j(8h+1-3j)}L_{-2}\left|A^{(j-2)}\right\rangle. \tag{6.4}$$

With the initial conditions $\left|A^{(0)}\right\rangle = \left|0\right\rangle$ and $\left|A^{(-1)}\right\rangle = 0$, this recursively determines all the terms of the operator product expansion, Equation (2.8), for the $\mathcal{M}(3,p)$ models. Noting that for these models $c = -2h(8h-5)/(2h+1)$, this relation effortlessly reproduces the results of Section 2.2 (for these models) and more:

$$\begin{aligned} \left|A^{(1)}\right\rangle &= 0, \\ \left|A^{(2)}\right\rangle &= -\tfrac{2h+1}{8h-5}L_{-2}\left|0\right\rangle, \\ \left|A^{(3)}\right\rangle &= -\tfrac{1}{2}\tfrac{2h+1}{8h-5}L_{-3}\left|0\right\rangle, \\ \left|A^{(4)}\right\rangle &= \tfrac{2h+1}{(8h-5)(8h-11)}\left[-3(h-1)L_{-4} + \tfrac{1}{2}(2h+1)L_{-2}^2\right], \\ \left|A^{(5)}\right\rangle &= \tfrac{2h+1}{(8h-5)(8h-11)}\left[-2(h-1)L_{-5} + \tfrac{1}{2}(2h+1)L_{-3}L_{-2}\right], \end{aligned} \tag{6.5}$$

$$\left|A^{(6)}\right\rangle = \tfrac{2h+1}{(8h-5)(8h-11)(8h-17)}\Big[-\left(12h^2-38h+23\right)L_{-6}+(2h+1)(3h-5)L_{-4}L_{-2} + (2h+1)(h-2)L_{-3}^2-\tfrac{1}{6}(2h+1)^2L_{-2}^3\Big],$$

$$\left|A^{(7)}\right\rangle = \tfrac{2h+1}{(8h-5)(8h-11)(8h-17)}\Big[-\tfrac{3}{4}\left(12h^2-38h+23\right)L_{-7}+\tfrac{1}{2}(2h+1)(4h-7)L_{-5}L_{-2} +\tfrac{3}{4}(2h+1)(2h-3)L_{-4}L_{-3}-\tfrac{1}{4}(2h+1)^2L_{-3}L_{-2}^2\Big].$$

One can similarly derive recursion relations for higher p'. For example, the $\mathcal{M}(4,p)$ models have $h=\frac{1}{2}(p-2)$ and singular vector

$$\left(L_{-3}-\frac{2}{h}L_{-2}L_{-1}+\frac{1}{h(h+1)}L_{-1}^3\right)\left|\phi\right\rangle, \tag{6.6}$$

leading to the recursion relation

$$\begin{aligned}\left|A^{(j)}\right\rangle = \Bigg\{&\frac{3(2h+2-j)(2h+1-j)-2(h+1)(3h+1-j)}{j(h-j)(3h+1-j)}L_{-1}\left|A^{(j-1)}\right\rangle \\ &+\frac{3(2h+2-j)}{j(h-j)(3h+1-j)}L_{-1}^2\left|A^{(j-2)}\right\rangle-\frac{2(h+1)(2h+2-j)}{j(h-j)(3h+1-j)}L_{-2}\left|A^{(j-2)}\right\rangle \\ &+\frac{1}{j(h-j)(3h+1-j)}L_{-1}^3\left|A^{(j-3)}\right\rangle-\frac{2(h+1)}{j(h-j)(3h+1-j)}L_{-2}L_{-1}\left|A^{(j-3)}\right\rangle \\ &+\frac{h(h+1)}{j(h-j)(3h+1-j)}L_{-3}\left|A^{(j-3)}\right\rangle\Bigg\},\end{aligned} \tag{6.7}$$

with $\left|A^{(0)}\right\rangle=\left|0\right\rangle$ and $\left|A^{(-1)}\right\rangle=\left|A^{(-2)}\right\rangle=0$. We note that $h=\frac{1}{2}p-1$ implies that $h,3h+1\notin\mathbb{Z}$, so this recursion relation does indeed define $\left|A^{(j)}\right\rangle$ for all $j>0$. Whilst this recursion relation may appear somewhat intimidating, its derivation and utilisation are easy to implement on a computer algebra system.

It is very interesting to compare the $p'=3$ and $p'=4$ recursion relations in the single case where they both apply: $\mathcal{M}(3,4)$. As shown in Table 1, their respective solutions $\left|A^{(j)}\right\rangle$ agree for $j\leqslant 5$, but do *not* agree for $j\geqslant 6$. However, the difference between the two $j=6$ solutions can be checked to be

$$\frac{90}{7007}\left(L_{-6}+\frac{22}{9}L_{-4}L_{-2}-\frac{31}{36}L_{-3}^2-\frac{16}{27}L_{-2}^3\right)\left|0\right\rangle, \tag{6.8}$$

a multiple of the vacuum singular vector (given in Equation (5.3)). Likewise, higher-grade solutions differ by descendants of this singular vector. At first, it seems somewhat astonishing that these recursion relations, derived from the principal singular vectors of the simple current $\mathfrak{Vir}$-module, may be used to compute the singular vector of corresponding vacuum module. On second thoughts though, this is perhaps not entirely unexpected given that the solutions to the recursion relations must be consistent. But more fundamentally, this is quite consistent with the concluding remarks

j	Basis	$p'=3$ solution	$p'=4$ solution
2	(L_{-2})	(2)	(2)
3	(L_{-3})	(1)	(1)
4	(L_{-4},L_{-2}^2)	$(\frac{3}{7},\frac{2}{7})$	$(\frac{3}{7},\frac{2}{7})$
5	$(L_{-5},L_{-3}L_{-2})$	$(\frac{2}{7},\frac{2}{7})$	$(\frac{2}{7},\frac{2}{7})$
6	$(L_{-6},L_{-4}L_{-2},L_{-3}^2,L_{-2}^3)$	$(\frac{2}{13},\frac{2}{13},\frac{6}{91},\frac{4}{273})$	$(\frac{76}{539},\frac{6}{49},\frac{83}{1078},\frac{12}{539})$
7	$(L_{-7},L_{-5}L_{-2},L_{-4}L_{-3},L_{-3}L_{-2}^2)$	$(\frac{3}{26},\frac{10}{91},\frac{6}{91},\frac{2}{91})$	$(\frac{57}{539},\frac{40}{539},\frac{39}{539},\frac{18}{539})$

TABLE 1. Solutions $\lvert A^{(j)}\rangle$ to the $p'=3$ and $p'=4$ recursion relations when $p=4$ and $p=3$, respectively. Both describe the operator product expansion of the Ising model simple current (the free fermion) with itself. The second column gives an ordered basis (when acting on $\lvert 0\rangle$) for the subspace of the vacuum Virasoro module of grade j, and the third and fourth columns give the coefficients of the solutions with respect to this basis.

of Section 5.2: In the extended chiral algebra framework, all information concerning the spectrum should be derivable from the two principal singular vectors of the simple current module.

6.2. **Recursion Relations for $\mathcal{S}$-Eigenvalues.** In Section 3.3, we derived general expressions for the $\mathcal{S}$-eigenvalues of a highest weight state $\lvert\psi\rangle$ of $\mathfrak{u}(1)$-charge $\theta \leqslant 2$. As the charge increases, these expressions become increasingly more difficult to compute, as evidenced by Equations (3.17–3.21). In general, we may write

$$1 = \langle\psi\rvert\phi_{h-\theta}\phi_{\theta-h}\lvert\psi\rangle \overset{2\theta+1}{=} \langle\psi\rvert\mathcal{S}\sum_{j=0}^{2\theta}\binom{\theta}{2\theta-j}A_0^{(j)}\lvert\psi\rangle$$
$$\Rightarrow \qquad \langle\psi\rvert\mathcal{S}\lvert\psi\rangle = \left[\sum_{j=0}^{2\theta}\binom{\theta}{2\theta-j}\langle\psi\rvert A_0^{(j)}\lvert\psi\rangle\right]^{-1}. \tag{6.9}$$

The problem lies in computing the terms

$$f_j \equiv \langle\psi\rvert A_0^{(j)}\lvert\psi\rangle \tag{6.10}$$

for large j. But this problem is tailor-made for the recursion relations of Section 6.1. A recursion relation for the $\lvert A^{(j)}\rangle$ induces a similar relation for the corresponding fields $A^{(j)}(w)$, and therefore for their modes. We illustrate this for the $\mathcal{M}(3,p)$ models. Equation (6.4) implies that

$$A_0^{(j)} = -\alpha_j(j-1)A_0^{(j-1)} + \beta_j(j-1)(j-2)A_0^{(j-2)} + \gamma_j\sum_{r\in\mathbb{Z}} :L_rA_{-r}^{(j-2)}: \tag{6.11}$$

(here α_j, β_j and γ_j denote the coefficients of $L_{-1}\left|A^{(j-1)}\right\rangle$, $L_{-1}^2\left|A^{(j-2)}\right\rangle$ and $L_{-2}\left|A^{(j-2)}\right\rangle$ in Equation (6.4), respectively), hence that

$$f_j = -\alpha_j\,(j-1)\,f_{j-1} + \Big[\beta_j\,(j-1)\,(j-2) + \gamma_j\,\big(h_\psi + j - 2\big)\Big]\,f_{j-2}. \tag{6.12}$$

With $f_0 = 1$ and $f_{-1} = 0$, it is now a trivial task to determine the $\mathcal{S}$-eigenvalues.

For example, when $p = 17$ ($h = \frac{15}{4}$), the highest weight states of charge $0, \frac{1}{2}, 1, \ldots, \frac{15}{2} = 2h$ are determined by Equation (6.12) to have respective eigenvalues

$$1, 2, \tfrac{50}{11}, \tfrac{25}{2}, \tfrac{95}{2}, 380, -4940, 12350, -12350, 4940, -380, -\tfrac{95}{2}, -\tfrac{25}{2}, -\tfrac{50}{11}, -2, \text{ and } -1. \tag{6.13}$$

It is a strong consistency test of our formalism that these computations respect the symmetry implied by Equations (2.7) and (3.11), which relates the $\mathcal{S}$-eigenvalues of the states of charge θ and $2h-\theta$. Indeed, this symmetry is by no means apparent from the recursion relations.

Observe that Equation (6.12) may also be used to formally obtain $\mathcal{S}$-eigenvalues for higher charge states, for example a charge 8 highest weight state of $\mathcal{M}(3,17)$ would have eigenvalue $\frac{-17}{31}$ according to our recursion formula. Given that $\mathcal{S}$-eigenvalues of states of negative charge must vanish (Section 3.3), this appears to be at odds with the $\mathcal{S}$-eigenvalue symmetry mentioned above. But such states cannot actually be present in the theory, as they do not respect the singular vectors in the simple current module (as we will see in Section 6.4). Given that the above recursion relations were originally derived from these singular vectors, it is not surprising that the "non-physical" solutions to these relations do not satisfy the $\mathcal{S}$-eigenvalue symmetry we might otherwise expect.

6.3. **General Applicability.** It should be clear that the method employed to derive the recursion relations for the $\left|A^{(j)}\right\rangle$ may be applied quite generally, the only obstacle being the determination of whether the common denominator of the coefficients ever vanishes (for $j \in \mathbb{Z}_+$). It is tempting to conjecture that this denominator never vanishes, as in Equations (6.4) and (6.7), hence that for every $\mathcal{M}(p',p)$ (with $p > p' > 2$), these recursion relations determine $\left|A^{(j)}\right\rangle$ for all positive j. However, it turns out that for $p' = 6$, the common denominator for the coefficients of the recursion relation obtained from the level 5 singular vector is (up to a multiplicative constant involving h)

$$j\,(h-j)\,(h-1-3j)\,(2h+1-j)\,(10h+8-3j)\,, \tag{6.14}$$

which vanishes for $j = h = p-2 \in \mathbb{Z}$ and $j = 2h+1 = 2p-3 \in \mathbb{Z}$. It follows that for these two values of j (as well as $j = 0$), the $p' = 6$ recursion relation does not determine $\left|A^{(j)}\right\rangle$.

Remarkably, it seems that this property that the common denominator factors nicely is a general feature of $\mathcal{M}(p',p)$ models. Based on explicit calculations for $p' \leqslant 10$, we conjecture that the common denominator of the coefficients of the recursion relation derived from the grade $p'-1$ principal singular vector is (up to an h-dependent multiplicative constant which can be normalised away) given by

$$\prod_{k=1}^{p'-1} \left[k\,(k-1)\,p - p'\,(j+k-1)\right]. \tag{6.15}$$

The factor with $k=1$ gives the expected vanishing when $j=0$. The other factors give zero (for some j) when p' divides $k\,(k-1)\,p$ (abbreviated $p' \mid k\,(k-1)\,p$), hence when $p' \mid k\,(k-1)$ (for some $1<k<p'$). In Appendix C (Proposition C.1), we show that such k only exist when p' is not a prime power. Thus for $p' = 6, 10, 12, 15, \ldots$, the corresponding recursion relation will fail to determine $\left|A^{(j)}\right\rangle$ for at least two (positive) values for j (the precise number is determined in Proposition C.2).

One can analogously derive similar recursion relations from descendant singular vectors. We shall call these "descendant recursion relations" to distinguish them from the "principal recursion relations" discussed above. It is easy to show that these descendant recursion relations have coefficients whose common denominator is that of the principal recursion relation (conjectured to be (6.15)), multiplied by other j-dependent factors which depend on the particular descendant used in the derivation. Hence these descendant recursion relations will also fail to determine the $\left|A^{(j)}\right\rangle$ for the same set of j as the principal recursion relation (and may fail for other j as well).

However, there are two independent principal singular vectors in the $\mathfrak{Vir}$-Verma module $\widetilde{\mathcal{V}}_{p'-1,1}^{p',p}$, so we should be able to derive two independent principal recurrence relations (we have already compared the solutions of these for $\mathcal{M}(3,4)$ in Table 1). At the level of the common denominator of the coefficients of these relations, swapping between the principal singular vectors merely amounts to swapping p' and p. Assuming the conjecture (6.15), we may therefore ask if there are $\left|A^{(j)}\right\rangle$ (with $j>0$) which are not determined by either principal recurrence relation. Numerical studies suggest[12] that the answer is "no": The $\left|A^{(j)}\right\rangle$, and hence the operator product expansion of $\phi(z)$ with itself, may always be computed from these two principal recurrence relations. However, we have not been able to construct a proof of this claim.

Finally, we mention an intriguing reformulation of our conjectured expression (6.15) for the common denominator of the principal recurrence relation derived from the principal singular vector at grade $p'-1$. We can express this singular vector in the form

$$\left|\chi\right\rangle = \sum_{\lambda \in \mathcal{P}_{p'-1}} a_\lambda\, L_{-\lambda_1} L_{-\lambda_2} \cdots L_{-\lambda_{\ell(\lambda)}} \left|\phi\right\rangle, \tag{6.16}$$

where $\mathcal{P}_n$ is the set of partitions $\lambda = \left(\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\ell(\lambda)}\right)$ of n, and the a_λ are unknown (rational) coefficients. We can follow the derivation of the corresponding recursion relation, only keeping track of the coefficient of the term in which all Virasoro modes are removed by commutation with the ϕ-mode. This coefficient defines the common denominator of the recursion relation, so our

[12] We used MAPLE to search for such j, and found that no such j exist when $p', p \leqslant 1000$.

conjecture for this implies that

$$\sum_{\lambda \in \mathcal{P}_{p'-1}} a_\lambda \prod_{k=1}^{\ell(\lambda)} \left[(\lambda_k + 1) h + \sum_{i=k+1}^{\ell(\lambda)} \lambda_i - j \right] = \prod_{k=1}^{p'-1} \left[k(k-1) p - p' (j + k - 1) \right], \tag{6.17}$$

where $h = \frac{1}{4}(p'-2)(p-2)$ (and the a_λ are appropriately normalised).

We have not tried to prove this formula, but it has been checked up to $p' = 10$ (for all p). It seems amazing to us that the coefficients of the singular vector should satisfy such a nice relation — indeed, for $p' < 6$, this relation completely determines the coefficients a_λ (up to an overall normalisation). One can of course substitute in explicit expressions [17] for the singular vector coefficients, leading to bemusing identities of considerable complexity. We will not attempt to analyse these here, but it would be very interesting to try to understand these observations at a deeper level.

6.4. **Singular Vectors and the Spectrum.** As remarked at the end of Section 5.2, it is well-established (though difficult to prove) that the chiral spectrum of a rational conformal field theory is to a large extent controlled by the singular vectors of the vacuum module (with respect to the chiral symmetry algebra). For example, $\mathcal{M}(3,4)$ has a principal singular vector $|\chi\rangle$ in its vacuum $\mathfrak{Vir}$-module $\widetilde{\mathcal{V}}_{1,1}^{3,4}$ at grade 6. The corresponding field $\chi(z)$ is therefore null, hence its modes χ_n must map *any* state of the theory into a singular state. By computing the action of χ_0 on an arbitrary highest weight state, one finds that the result is singular (in fact vanishing) if and only if the dimension of the highest weight state is 0, $\frac{1}{16}$ or $\frac{1}{2}$.

In this way, the principal vacuum singular vector determines the chiral spectrum of the theory. We ask the obvious question of whether this property is preserved in the extended theories: Do the principal singular vectors of the simple current module also determine the spectrum of the (extended) theory? We have seen in Section 6.1 that (at least for the Ising model) these singular vectors already “know about” the vacuum singular vector, so we expect that the answer to our question is “yes”.

This may indeed be demonstrated in simple cases. We suppose that $|\psi\rangle$ is a (Virasoro) highest weight state of monodromy charge θ and conformal dimension Δ. In an $\mathcal{M}(3,p)$ model, the singular vector (6.1) defines a null field $\chi(z)$ (vanishing in the extended theory) whose modes have the form

$$\chi_n = \sum_{r \in \mathbb{Z}} :L_r \phi_{n-r}: - \frac{3(h+n)(h+n+1)}{2(2h+1)} \phi_n \equiv 0. \tag{6.18}$$

We may therefore calculate

$$\begin{aligned} 0 = \chi_{\theta-h} |\psi\rangle &= \left(\phi_{\theta-h} L_0 + \phi_{\theta-h+1} L_{-1} - \frac{3\theta(\theta+1)}{2(2h+1)} \phi_{\theta-h} \right) |\psi\rangle \\ &= \left[\Delta + \theta - \frac{3\theta(\theta+1)}{2(2h+1)} \right] \phi_{\theta-h} |\psi\rangle. \end{aligned} \tag{6.19}$$

As $\phi_{\theta-h}\left|\psi\right\rangle \neq 0$, this gives a simple relation between the charge and dimension of any highest weight state in an $\mathcal{M}(3,p)$ model:

$$\Delta = \frac{\theta(3\theta-4h+1)}{2(2h+1)}. \tag{6.20}$$

Clearly, this does not completely determine the spectrum of the theory[13]. What is needed is a second relation between the charge and dimension, and this is provided by the second principal singular vector. As an example, in the Ising model ($h=\frac{1}{2}$), this second relation is easily verified to be

$$(2\theta-1)\left(2\Delta - \frac{\theta(2\theta+1)}{3}\right) = 0. \tag{6.21}$$

Solving the two spectrum-constraining relations then gives $\theta = 0, \frac{1}{2}, 1$ and $\Delta = 0, \frac{1}{16}, \frac{1}{2}$ (respectively), as expected.

Of course, this analysis can be repeated for other models and other singular vectors, with similar results. We restrict ourselves to a final remark. It is possible to show explicitly (for example with $\mathcal{M}(5,6)$) that the constraints derived from the principal singular vectors select all fields from the Kac table, and *not* just those which contribute to the D-type modular invariant.

Acknowledgements

We would like to thank Matthias Gaberdiel and Antun Milas for interesting correspondence during the preparation of this article, and Patrick Jacob for useful discussions. We also benefited greatly from the reviewer's comments regarding the modules corresponding to fixed points of the simple current, and thank them for their detailed observations. This research is supported by NSERC.

Appendix A. The Decomposition of $\mathcal{M}(3,10)$

In this appendix (which is a continuation of Section 4), we illustrate fully the process of deriving a conformal field theory equivalence by constructing an isomorphism of chiral algebras. Uniquely amongst the minimal models, $\mathcal{M}(3,10)$ has a simple current of conformal dimension $h=2$. The central charge is $c = \frac{-44}{5}$, so the operator product expansion takes the form

$$\phi(z)\phi(w) = \mathcal{S}\left[\frac{1}{(z-w)^4} - \frac{5}{11}\frac{T(w)}{(z-w)^2} - \frac{5}{22}\frac{\partial T(w)}{z-w} + \dots\right]. \tag{A.1}$$

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[13]Interestingly, the $\mathcal{M}(3,p)$ spectrum is completely determined by this constraint *and* by requiring that the charge be an integer or half-integer between 0 and h. However, this complete determination is a peculiarity of the $\mathcal{M}(3,p)$ models. In general, some of these charges will also be forbidden.

Again, $\mathcal{S}$ commutes with the ϕ_n, and its eigenvalues on the five $\mathfrak{A}_{3,10}$-highest weight states $\left|\phi_{1,s}\right\rangle$ ($s = 1, \ldots, 5$) are given by Equations (3.17–3.21) as

$$\begin{gathered}\mathcal{S}\left|0\right\rangle = \left|0\right\rangle, \qquad \mathcal{S}\left|\phi_{1,2}\right\rangle = 2\left|\phi_{1,2}\right\rangle, \qquad \mathcal{S}\left|\phi_{1,3}\right\rangle = \tfrac{11}{2}\left|\phi_{1,3}\right\rangle, \\ \mathcal{S}\left|\phi_{1,4}\right\rangle = 44\left|\phi_{1,4}\right\rangle, \qquad \text{and} \qquad \mathcal{S}\left|\phi_{1,5}\right\rangle = -110\left|\phi_{1,5}\right\rangle.\end{gathered} \tag{A.2}$$

We consider the linear combination $aT(z) + b\phi(z)$, $a, b \in \mathbb{C}$. This is a field of conformal dimension 2 whose operator product expansion with itself is easily read off from the operator product expansions (1.3), (1.14) and (A.1). It can be verified that this linear combination again defines a Virasoro field (that is, the *four* singular terms of the operator product expansion have the form of those in (1.3)) if and only if

$$a = 1, \quad b = 0 \qquad \text{or} \qquad a = \frac{1}{2}, \quad b = \pm \mathrm{i}\sqrt{\frac{11}{10\mathcal{S}}}. \tag{A.3}$$

We define

$$T^{\pm}(z) = \frac{1}{2}T(z) \pm \mathrm{i}\sqrt{\frac{11}{10\mathcal{S}}}\phi(z), \tag{A.4}$$

and note that these Virasoro fields $T^{\pm}(z)$ correspond to a central charge of $c_{\pm} = \frac{-22}{5}$ (which in turn corresponds to the minimal model $\mathcal{M}(2,5)$). Furthermore,

$$T^{+}(z)\,T^{-}(w) = \,:T^{+}(w)\,T^{-}(w):\, + \ldots \tag{A.5}$$

has no singular terms, as is readily checked, so the corresponding Virasoro algebras are independent. We have therefore constructed an isomorphism of algebras:

$$\mathfrak{A}_{3,10}^{(4)} \cong \mathcal{U}\big(\mathfrak{Vir}_{2,5}\big) \otimes \mathcal{U}\big(\mathfrak{Vir}_{2,5}\big). \tag{A.6}$$

The superscript $^{(4)}$ reminds us that this algebra is defined by the generalised commutation relations with $\gamma = 4$ (only the singular terms of the operator product expansions have been used in the construction of this isomorphism).

It is important to note that the Virasoro modes

$$L_n^{\pm} = \frac{1}{2}L_n \pm \mathrm{i}\sqrt{\frac{11}{10\mathcal{S}}}\phi_n \tag{A.7}$$

corresponding to the fields $T^{\pm}(z)$ are only defined to act on states of integral monodromy charge (so $n \in \mathbb{Z}$). The representation theory corresponding to the isomorphism (A.6) is therefore restricted to the $\mathfrak{A}_{3,10}$-modules headed by $\left|0\right\rangle$, $\left|\phi_{1,3}\right\rangle$ and $\left|\phi_{1,5}\right\rangle$, which build the *D*-type modular invariant of $\mathcal{M}(3,10)$. It will not be possible to relate the modules headed by $\left|\phi_{1,2}\right\rangle$ and $\left|\phi_{1,4}\right\rangle$ to any module of $\mathfrak{Vir}_{2,5}^{\otimes 2}$. This situation is analogous to that of [1, Sec. 4.2], in which the equivalence of the *D*-type $\mathsf{SU}(2)_4$ and *A*-type (diagonal) $\mathsf{SU}(3)_1$ Wess-Zumino-Witten models was constructed.

We can identify the nature of the highest weight states under the $\mathfrak{Vir}_{2,5}^{\otimes 2}$-action by computing

$$L_0^{\pm}\left|0\right\rangle = 0 \qquad \text{and} \qquad L_0^{\pm}\left|\phi_{1,3}\right\rangle = \frac{1}{2}L_0\left|\phi_{1,3}\right\rangle = \frac{-1}{5}\left|\phi_{1,3}\right\rangle, \tag{A.8}$$

using Equation (1.5). As expected, $\left|\phi_{1,7}\right\rangle$ and $\left|\phi\right\rangle = \left|\phi_{1,9}\right\rangle$ are not even highest weight states under this action:

$$L_1^{\pm}\left|\phi_{1,7}\right\rangle = \pm\frac{\mathfrak{i}}{\sqrt{5}}\phi_1\phi_{-1}\left|\phi_{1,3}\right\rangle = \pm\frac{\mathfrak{i}}{\sqrt{5}}\left|\phi_{1,3}\right\rangle \neq 0, \tag{A.9}$$

$$L_2^{\pm}\left|\phi\right\rangle = \pm\mathfrak{i}\sqrt{\frac{11}{10}}\phi_2\phi_{-2}\left|0\right\rangle = \pm\mathfrak{i}\sqrt{\frac{11}{10}}\left|0\right\rangle \neq 0. \tag{A.10}$$

The analysis for $\left|\phi_{1,5}\right\rangle$ requires a little more delicacy. Let $\left|\widetilde{\phi}_{1,5}\right\rangle = \phi_0\left|\phi_{1,5}\right\rangle$, so that

$$L_0^{\pm}\left|\phi_{1,5}\right\rangle = \left(\frac{1}{2}L_0 \pm \frac{\mathfrak{i}\sqrt{-1}}{10}\phi_0\right)\left|\phi_{1,5}\right\rangle = \frac{-1}{10}\left|\phi_{1,5}\right\rangle \pm \frac{\mathfrak{i}\sqrt{-1}}{10}\left|\widetilde{\phi}_{1,5}\right\rangle. \tag{A.11}$$

By similarly computing $L_0^{\pm}\left|\widetilde{\phi}_{1,5}\right\rangle$, we see that the $\mathfrak{Vir}_{2,5}^{\otimes 2}$-highest weight states are actually $\left|\phi_{1,5}\right\rangle + \left|\widetilde{\phi}_{1,5}\right\rangle$ and $\left|\phi_{1,5}\right\rangle - \left|\widetilde{\phi}_{1,5}\right\rangle$. Their conformal dimensions are 0 and $\frac{-1}{5}$, but it is not possible to say which state has which dimension without choosing whether $\sqrt{-1}$ should be $\mathfrak{i}$ or $-\mathfrak{i}$.

To summarise, the isomorphism (A.6) induces the following relationship between the highest weight states of $\mathfrak{A}_{3,10}$ and $\mathfrak{Vir}_{2,5}^{\otimes 2}$:

$$\left|0\right\rangle \longleftrightarrow \left|0\right\rangle^+ \otimes \left|0\right\rangle^-, \quad \left|\phi_{1,3}\right\rangle \longleftrightarrow \left|\tfrac{-1}{5}\right\rangle^+ \otimes \left|\tfrac{-1}{5}\right\rangle^-, \quad \begin{matrix} \left|\phi_{1,5}\right\rangle + \left|\widetilde{\phi}_{1,5}\right\rangle \\ \left|\phi_{1,5}\right\rangle - \left|\widetilde{\phi}_{1,5}\right\rangle \end{matrix} \longleftrightarrow \begin{matrix} \left|0\right\rangle^+ \otimes \left|\tfrac{-1}{5}\right\rangle^- \\ \left|\tfrac{-1}{5}\right\rangle^+ \otimes \left|0\right\rangle^- \end{matrix}. \tag{A.12}$$

Here we have denoted the $\mathfrak{Vir}_{2,5}$-highest weight state of conformal dimension Δ by $\left|\Delta\right\rangle^{\pm}$ (including a label $^{\pm}$ to distinguish the two copies of $\mathfrak{Vir}_{2,5}$). As the respective definitions of highest weight state are compatible with this relationship, this implies the following isomorphisms of irreducible modules (using Corollaries 5.2 and 5.3):

$$\widetilde{\mathcal{L}}_{1,1}^{2,5} \otimes \widetilde{\mathcal{L}}_{1,1}^{2,5} \cong \mathcal{V}_{1,1}^{3,10} \cong \widetilde{\mathcal{L}}_{1,1}^{3,10} \oplus \widetilde{\mathcal{L}}_{1,9}^{3,10}, \tag{A.13}$$

$$\widetilde{\mathcal{L}}_{1,2}^{2,5} \otimes \widetilde{\mathcal{L}}_{1,2}^{2,5} \cong \mathcal{V}_{1,3}^{3,10} \cong \widetilde{\mathcal{L}}_{1,3}^{3,10} \oplus \widetilde{\mathcal{L}}_{1,7}^{3,10}, \tag{A.14}$$

$$\left(\widetilde{\mathcal{L}}_{1,1}^{2,5} \otimes \widetilde{\mathcal{L}}_{1,2}^{2,5}\right) \oplus \left(\widetilde{\mathcal{L}}_{1,2}^{2,5} \otimes \widetilde{\mathcal{L}}_{1,1}^{2,5}\right) \cong \mathcal{V}_{1,5}^{3,10} \cong \widetilde{\mathcal{L}}_{1,5}^{3,10} \oplus \widetilde{\mathcal{L}}_{1,5}^{3,10}.$$

Of course, this last chain of isomorphisms may be more succinctly expressed in the form

$$\widetilde{\mathcal{L}}_{1,1}^{2,5} \otimes \widetilde{\mathcal{L}}_{1,2}^{2,5} \cong \widetilde{\mathcal{L}}_{1,2}^{2,5} \otimes \widetilde{\mathcal{L}}_{1,1}^{2,5} \cong \widetilde{\mathcal{L}}_{1,5}^{3,10}, \tag{A.15}$$

so Equations (A.13–A.15) complete the tensor product multiplication table for the irreducible $\mathfrak{Vir}$-modules making up $\mathcal{M}(2,5)$. This table has been previously derived (though indirectly) in [22, Prop. 7.2.1] using results on the asymptotic growth of the characters of these irreducible $\mathfrak{Vir}$-modules.

We conclude by studying the effect of imposing the $\gamma = 5$ generalised commutation relation on our construction. In particular, we impose the (equivalent) field identification

$$:\phi(w)\phi(w): = \mathcal{S}\left[\frac{5}{22}:T(w)T(w): - \frac{3}{22}\partial^2 T(w)\right]. \tag{A.16}$$

As $:\phi(w)T(w): = :T(w)\phi(w):$, Equation (A.4) gives

$$:T^+(w)T^-(w): = \frac{1}{2}:T(w)T(w): - \frac{3}{20}\partial^2 T(w). \tag{A.17}$$

As the $T^\pm(w)$ generate the modes defining each side of the tensor product $\mathfrak{Vir}_{2,5} \otimes \mathfrak{Vir}_{2,5}$, we might expect that this product vanishes. However, we see that it cannot, as this would require

$$\left[L_{-2}^2 - \frac{3}{5}L_{-4}\right]\left|0\right\rangle = 0, \tag{A.18}$$

and this vector is not even singular in $\mathcal{M}(3,10)$. Nevertheless, we can back-substitute $T = T^+ + T^-$ to show that

$$\frac{1}{2}\left(:T^+(w)T^+(w): - \frac{3}{20}\partial^2 T^+(w)\right) + \frac{1}{2}\left(:T^-(w)T^-(w): - \frac{3}{20}\partial^2 T^-(w)\right) = 0. \tag{A.19}$$

Applying this to the $\mathfrak{Vir}_{2,5}^{\otimes 2}$-vacuum $\left|0\right\rangle^+ \otimes \left|0\right\rangle^-$, we therefore recover the vanishing of the $\mathcal{M}(2,5)$ singular vector (in each copy):

$$\left[\left(L_{-2}^\pm\right)^2 - \frac{3}{5}L_{-4}^\pm\right]\left|0\right\rangle^\pm = 0. \tag{A.20}$$

It follows that $\mathfrak{A}_{3,10}^{(5)}$ and $\mathcal{U}\left(\mathfrak{Vir}_{2,5}\right) \otimes \mathcal{U}\left(\mathfrak{Vir}_{2,5}\right)$ are not isomorphic, because the former contains a defining relation which proves that the primitive $\mathcal{M}(2,5)$ vacuum singular vectors are identically zero (which cannot be proven in $\mathcal{U}\left(\mathfrak{Vir}_{2,5}\right)$).

Appendix B. The Vanishing of an $\mathcal{M}(3,5)$ Singular Vector

This appendix is devoted to outlining (by example) how one can demonstrate that the second principal singular vector of the vacuum Verma module vanishes identically. For $\mathcal{M}(3,5)$, this singular vector is explicitly given (up to normalisation) by

$$\left|\chi\right\rangle \equiv \left(L_{-4} - \frac{55}{57}L_{-3}L_{-1} - \frac{45}{38}L_{-2}^2 + \frac{125}{57}L_{-2}L_{-1}^2 - \frac{125}{342}L_{-1}^4\right)\phi_{-3/4}\left|0\right\rangle. \tag{B.1}$$

We have established in Section 5.1 that the first singular vector $\left|\omega\right\rangle$ vanishes identically, hence so do all its descendants. In particular, its descendants at grade four,

$$\left|\xi\right\rangle \equiv L_{-1}^2\left|\omega\right\rangle = \left(L_{-4} + L_{-3}L_{-1} + \frac{1}{2}L_{-2}L_{-1}^2 - \frac{3}{10}L_{-1}^4\right)\phi_{-3/4}\left|0\right\rangle \tag{B.2}$$

and $$\left|\zeta\right\rangle \equiv L_{-2}\left|\omega\right\rangle = \left(L_{-2}^2 - \frac{3}{5}L_{-2}L_{-1}^2\right)\phi_{-3/4}\left|0\right\rangle, \tag{B.3}$$

must vanish identically. Note that $\left|\chi\right\rangle$ is not (obviously) a linear combination of these descendants.

We compute:

$$\begin{aligned} L_{-4}\left|\phi\right\rangle &= \left(\phi_{-3/4}L_{-4} + \frac{7}{4}\phi_{-19/4}\right)\left|0\right\rangle \\ &\stackrel{5}{=} \phi_{-3/4}\left(\frac{-8}{3}\phi_{-13/4}\phi_{-3/4} - \frac{5}{3}L_{-2}^2\right)\left|0\right\rangle + \frac{7}{96}L_{-1}^4\left|\phi\right\rangle \\ &= \left(\frac{-8}{3}\phi_{-3/4}\phi_{-13/4} - \frac{5}{3}L_{-2}^2 + \frac{25}{12}L_{-2}L_{-1}^2 - \frac{241}{1152}L_{-1}^4\right)\left|\phi\right\rangle. \end{aligned} \tag{B.4}$$

The ϕ-bilinear may be evaluated recursively:

$$\begin{gathered} \left(\phi_{-3/4}\phi_{-13/4} - \frac{5}{2}\phi_{-7/4}\phi_{-9/4} + \frac{15}{8}\phi_{-11/4}\phi_{-5/4} - \frac{15}{384}\phi_{-19/4}\phi_{3/4}\right)\left|\phi\right\rangle \stackrel{-1}{=} 0, \\ \left(\phi_{-7/4}\phi_{-9/4} - \frac{1}{2}\phi_{-11/4}\phi_{-5/4} - \frac{1}{16}\phi_{-19/4}\phi_{3/4}\right)\left|\phi\right\rangle \stackrel{1}{=} 0, \\ \text{and} \qquad \left(\phi_{-11/4}\phi_{-5/4} + \frac{15}{8}\phi_{-19/4}\phi_{3/4}\right)\left|\phi\right\rangle \stackrel{3}{=} \frac{5}{4}L_{-4}\left|\phi\right\rangle \\ \Rightarrow \quad \phi_{-3/4}\phi_{-13/4}\left|\phi\right\rangle = \left(\frac{-25}{32}L_{-4} + \frac{175}{128}\phi_{-19/4}\phi_{3/4}\right)\left|\phi\right\rangle = \left(\frac{-25}{32}L_{-4} + \frac{175}{3072}L_{-1}^4\right)\left|\phi\right\rangle. \end{gathered} \tag{B.5}$$

Substituting back and tidying up, we find that

$$\left|\eta\right\rangle \equiv \left(L_{-4} - \frac{20}{13}L_{-2}^2 + \frac{25}{13}L_{-2}L_{-1}^2 - \frac{1}{3}L_{-1}^4\right)\phi_{-3/4}\left|0\right\rangle = 0. \tag{B.6}$$

Basic linear algebra now shows that $\left|\chi\right\rangle$, $\left|\xi\right\rangle$, $\left|\zeta\right\rangle$ and $\left|\eta\right\rangle$ are linearly dependent. Indeed,

$$\left|\chi\right\rangle = \frac{2725}{1482}\left|\xi\right\rangle - \frac{55}{57}\left|\zeta\right\rangle + \frac{112}{57}\left|\eta\right\rangle, \tag{B.7}$$

which completes the proof that the second principal singular vector of the vacuum $\mathfrak{A}_{3,5}$-Verma module vanishes identically.

Appendix C. Two Number-Theoretic Results

This appendix is devoted to the proof of two simple (related) number-theoretic results, which are used in Section 6.1.

Proposition C.1. *Let $p \geqslant 3$. Then, $p \mid k(k-1)$ for some $1 < k < p$ if and only if p is* not *a prime power ($p \neq q^n$, q prime and $n > 0$).*

Proof. If p is a prime power, $p \mid k(k-1)$ implies $p \mid k$ or $p \mid k-1$ (since $\gcd\{k-1,k\} = 1$). But then, $p \leqslant k$ (as $k \neq 0,1$), so there can be no $1 < k < p$ satisfying $p \mid k(k-1)$.

Suppose then that p is not a prime power, so we may write $p = ab$ where $a, b \in \mathbb{Z}$, $a, b > 1$, and $\gcd\{a, b\} = 1$. We can therefore find $a', b' \in \mathbb{Z}$ such that

$$ab' - ba' = 1. \tag{C.1}$$

Note that this property is invariant under $(a', b') \to (a' + \ell a, b' + \ell b)$, for all $\ell \in \mathbb{Z}$. Hence we may choose ℓ so that

$$0 \leqslant a\left(b' + \ell b\right) = ab' + \ell p \leqslant p - 1. \tag{C.2}$$

Let $k = a(b' + \ell b)$. Then, by Equation (C.1),

$$k - 1 = ab' - 1 + \ell ab = ba' + \ell ab = b\left(a' + \ell a\right), \tag{C.3}$$

hence

$$k(k-1) = a\left(b' + \ell b\right) b\left(a' + \ell a\right) = p\left(b' + \ell b\right)\left(a' + \ell a\right). \tag{C.4}$$

We have therefore constructed k satisfying $0 \leqslant k < p$ and $p \mid k(k-1)$. It remains to show that $k > 1$.

If $k = 0$, $(b' + \ell b) = 0$ (as $a > 1$). But then, substituting b' into Equation (C.1) gives $-b(a' + \ell a) = 1$, contradicting $b > 1$. If $k = 1$, we derive a similar contradiction (from Equation (C.3)), completing the proof. ∎

Proposition C.2. *$p \mid k(k-1)$ for precisely $2^n - 2$ integers k in the range $1 < k < p$, where n is the number of distinct prime divisors of p.*

Proof. The proof of Proposition C.1 shows that every factoring of p into coprime integers $a, b > 1$ produces a unique integer $1 < k < p$ such that $p \mid k(k-1)$. There are of course $2^n - 2$ such factorings, corresponding to the ways of partitioning the distinct prime factors of p into two non-empty subsets (the missing two correspond to choosing either a or b to be 1, corresponding to the empty subset). We need to show that this map from the set of factorings into the set of k satisfying the required conditions is a bijection.

We first show that different factorings give different k. Let $p = ab = \alpha\beta$, $a \neq \alpha$, be two such factorings, producing k and κ respectively. Then, there must exist a prime q dividing one of a and α, but not the other (otherwise a would equal α). Without loss of generality, suppose $q \mid a$ but $q \nmid \alpha$. Then $q \mid p$, so $q \mid \beta$, and

$$k = a\left(b' + \ell b\right) = 0 \pmod q \qquad \text{but} \qquad \kappa = \alpha\left(\beta' + \lambda\beta\right) = \alpha\beta' = 1 \pmod q, \tag{C.5}$$

since $\alpha\beta' - \beta\alpha' = 1$. Therefore, $k \neq \kappa$.

We now show that every k between 1 and p that satisfies $p \mid k(k-1)$ corresponds to such a factoring of p. Given such a k, we set

$$a = \gcd\{k, p\} \qquad \text{and} \qquad b = \gcd\{k-1, p\}. \tag{C.6}$$

Then, $\gcd\{a,b\} = 1$ because $\gcd\{k,k-1\} = 1$, and $ab = p$ since $p \mid k(k-1)$. Finally, $a = 1$ implies $p \mid k-1 > 0$, hence $p \leqslant k-1$, a contradiction as $k < p$. $b = 1$ also contradicts this, so we have the required factoring. We complete the proof by showing that this factoring $p = ab$ does indeed produce our original integer k. Clearly we can trivially satisfy $ab' - ba' = 1$ by choosing

$$b' = \frac{k}{a} = \frac{k}{\gcd\{k,p\}} \qquad \text{and} \qquad a' = \frac{k-1}{b} = \frac{k-1}{\gcd\{k-1,p\}}. \tag{C.7}$$

The integer produced by the proof of Proposition C.1 is thus

$$a\left(b' + \ell b\right) = k + \ell p, \tag{C.8}$$

and clearly $\ell = 0$ as this integer must lie between 1 and p. ■

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