

RELAXED SINGULAR VECTORS, JACK SYMMETRIC FUNCTIONS AND FRACTIONAL LEVEL $\widehat{\mathfrak{sl}}(2)$ MODELS

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ABSTRACT. The fractional level models are (logarithmic) conformal field theories associated with affine Kac-Moody (super)algebras at certain levels $k \in \mathbb{Q}$. They are particularly noteworthy because of several longstanding difficulties that have only recently been resolved. Here, Wakimoto's free field realisation is combined with the theory of Jack symmetric functions to analyse the fractional level $\widehat{\mathfrak{sl}}(2)$ models. The first main results are explicit formulae for the singular vectors of minimal grade in relaxed Wakimoto modules. These are closely related to the minimal grade singular vectors in relaxed (parabolic) Verma modules. Further results include an explicit presentation of Zhu's algebra and an elegant new proof of the classification of simple relaxed highest weight modules over the corresponding vertex operator algebra. These results suggest that generalisations to higher rank fractional level models are now within reach.

1. INTRODUCTION

We begin by briefly summarising some historical details regarding fractional level $\widehat{\mathfrak{sl}}(2)$ models and the application of Jack symmetric functions to conformal field theory. The work presented here forms a part of an ambitious project aimed at elucidating the properties of general fractional level models as fundamental examples of logarithmic conformal field theories. A short digression on the notion of admissibility follows as we use the term in a non-standard manner as compared to much of the literature. This also provides us with an opportunity to fix some notation. Finally, we outline the main results of the research reported here.

1.1. Fractional level models and Jack symmetric functions. The fractional level $\widehat{\mathfrak{sl}}(2)$ models are conformal field theories with a long and notorious history, originally proposed by Kent [1] as non-unitary models whose existence would lead to a uniform coset construction for all the Virasoro minimal models. This proposal received a significant boost from the subsequent announcement of Kac and Wakimoto [2] that $\widehat{\mathfrak{sl}}(2)$ has, for precisely the levels required for this coset construction, a finite set of simple highest weight modules whose characters close under modular transformations. Moreover, they observed that this property persisted for higher rank affine Kac-Moody algebras. However, computations [3–5] of the fusion rules of the purported theories gave confusing and conflicting results. In particular, substituting the modular S-matrix entries into the Verlinde formula resulted in negative fusion multiplicities. Despite a flurry of subsequent work, no resolution was agreed upon; in their discussion, the authors of the textbook [6] suggested that the fractional level models may be “intrinsically sick”.

The first steps towards curing this sickness were made by Gaberdiel [7], whose explicit fusion computations for the $\widehat{\mathfrak{sl}}(2)$ model of level $k = -\frac{4}{3}$ demonstrated that one was forced to consider modules that were not highest weight. Indeed, he found that fusing the highest weight modules of Kac and Wakimoto resulted in infinite numbers of modules whose conformal weights were not bounded below, modules that were not highest weight with respect to any Borel subalgebra (relaxed modules), and modules upon which the Virasoro mode L_0 acted non-semisimply. This showed that the $k = -\frac{4}{3}$ model was not a rational conformal field theory, as had been implicitly assumed in earlier studies, but was, in fact, logarithmic. Subsequent works [8–11] extended these results to $k = -\frac{1}{2}$ and thence to the closely related $\beta\gamma$ ghost systems [11–13]. Moreover, it was also shown [12] that a misunderstanding concerning the role of convergence regions in the modular transformations of Kac and Wakimoto was to blame for the failure of the Verlinde formula. To get a genuine action of the modular group and a working Verlinde formula, one needs to include all the simple relaxed modules and all their twists under spectral flow [14, 15].

We regard these recent advances as demonstrating that the sickness of the $\widehat{\mathfrak{sl}}(2)$ models has been cured and we expect that this cure will be just as effective for higher rank Kac-Moody algebras. Given the logarithmic

nature of the theories, it is now reasonable to expect that the fractional level models will play an important role in understanding logarithmic conformal field theory, much as the non-negative integer level Wess-Zumino-Witten models did for rational conformal field theory. We therefore view the fractional level models as objects that deserve intense study. In particular, one should at least determine the spectrum of simple modules, compute character formulae and verify that the modular properties of these characters lead to non-negative integers upon applying the Verlinde formula. More ambitiously, one would like to understand the fusion ring generated by the simple modules, the structure of the indecomposable modules generated by fusion, and the corresponding three- and four-point correlation functions. From a mathematical perspective, one can ask about homological properties of the spectrum (a category of modules over the corresponding vertex operator algebra) including rigidity and the identification of the projective and injective modules. We think that it is not unreasonable to expect that there are beautiful answers to all these questions, given that they concern structures built from affine Kac-Moody algebras.

While some, though not all, of these questions have been answered for $\widehat{\mathfrak{sl}}(2)$ at levels $k = -\frac{4}{3}$ and $-\frac{1}{2}$, some of the machinery employed is clearly unfeasible for more general levels and ranks. In our opinion, it is likely that the modular story will remain under control within the so-called standard module framework [16, 17]. On the other hand, brute force methods such as the Nahm-Gaberdiel-Kausch algorithm [18, 19] for fusion products are already computationally-prohibitive in all but the simplest cases. Further progress will instead require the development of more general alternatives and free field realisations seem to be the obvious candidates in this respect. Here, we use a free field realisation to address one of the most basic questions of all, that of determining the simple modules of a fractional level theory. We restrict ourselves to the $\widehat{\mathfrak{sl}}(2)$ models as a testing ground, leaving the challenge of higher ranks for future publications.

The standard free field realisation of an affine Kac-Moody algebra is due to Wakimoto [20], for $\widehat{\mathfrak{sl}}(2)$, and Feigin and Frenkel [21] in general. For $\widehat{\mathfrak{sl}}(2)$, the Wakimoto realisation combines a free boson with a pair of bosonic ghosts of central charge $c = 2$. The virtue of free field theories such as these is that their underlying Lie algebras are almost abelian. More precisely, the negative modes (creation operators) all commute among themselves, hence one can invoke symmetric group theory, in particular symmetric polynomials and functions, to analyse certain representation-theoretic questions. For the question of classifying the simple modules of the fractional level $\widehat{\mathfrak{sl}}(2)$ models, it turns out that the key lies in the Jack symmetric functions [22].

The relevance of Jack symmetric functions to conformal field theory goes back to the work of Mimachi and Yamada [23] who realised that they provide elegant expressions for the singular vectors of Verma modules over the Virasoro algebra. Explicit singular vector formulae are useful for many field-theoretic investigations including those of the spectrum of primary fields, the fusion rules and the correlation functions. However, the general singular vector formulae that were then known, for example those of [24, 25], are not particularly tractable for these purposes. On the other hand, the Jack function formulae were derived directly from the Feigin-Fuchs free field realisation of the Virasoro algebra (also known as the Coulomb gas), as developed by Tsuchiya and Kanie [26] and Felder [27], which is far better suited to explicit computation.

Unfortunately, it appears that the power of symmetric function theory was not immediately exploited in conformal field theory studies, perhaps because of an unfamiliarity with Jack symmetric functions (Macdonald's influential textbook [28] did not appear until several years later). However, in more recent times, symmetric function theory has been embraced by the community, particularly as a means to prove the AGT conjecture [29] which relates Liouville conformal field theories to the instanton calculus of Yang-Mills theories (although it now appears that a generalisation of the Jack symmetric functions will be required [30] to prove this conjecture).

The work reported here, using Jack symmetric functions to classify simple modules of the fractional level $\widehat{\mathfrak{sl}}(2)$ models, has its genesis in [31] where this formalism was used to classify, among other things, the simple modules of a family [32] of (logarithmic) conformal field theories called the (p_+, p_-) triplet models. The methods developed for this purpose were also applied to give a far more elegant proof of the singular vector formulae of Mimachi and Yamada. More recently [33], it was shown that these methods lead to an elegant new proof of the classification of the Virasoro minimal model modules. We recall that the original classification proof of Wang [34] combined a

projection formula for the singular vector of the vacuum module, stated by Feigin and Fuchs and eventually proven (fifteen years later) by Astashkevich and Fuchs [35], with some intricate cohomological arguments.

The content of this paper is then that these simplified methods generalise to the fractional level $\widehat{\mathfrak{sl}}(2)$ models. Specifically, we deduce explicit formulae for singular vectors in Wakimoto modules and classify the simple modules. We take this as strong evidence that the symmetric function techniques developed here will further generalise to models over higher rank Kac-Moody algebras and superalgebras. Moreover, it is clear that these techniques can be profitably exploited to investigate other important representation-theoretic questions including the structure theory of relaxed Verma and Wakimoto modules. We hope to report on this in the future.

It should be emphasised that, as with the Virasoro case reported in [33], the $\widehat{\mathfrak{sl}}(2)$ classification result is not new. The highest weight classification is due to Adamović and Milas [36] and, independently, Dong, Li and Mason [37]. However, their proofs mimic that of Wang, relying upon a projection formula for the singular vector of the vacuum module stated (without proof) by Fuchs [38]. From this projection formula, Adamović and Milas also derive a classification result for what we call, following [39], the *relaxed* highest weight modules.¹ It is not clear to us if these projection formulae will generalise easily to higher ranks; the tedium of the proof in the Virasoro case alone warrants, in our opinion, the development of the rather more elegant symmetric function methods. With this in mind, we remark that, to the best of our knowledge, there are no general classification results known for fractional level models of rank greater than 1.

1.2. Basic concepts and notation. Suppose that we have a conformal field theory whose chiral algebra is identified as a Lie (super)algebra \mathfrak{g} , for example, the Virasoro algebra or an affine Kac-Moody (super)algebra. More general chiral algebras can be accommodated within the formalism to follow, but this level of generality will suffice for the purposes of the article. There is always a module of the chiral algebra, called the vacuum module, that carries the structure of a vertex operator algebra \mathcal{V} . The elements of the chiral algebra are organised into fields that generate \mathcal{V} and the (anti)commutation relations of the chiral algebra are equivalent to the operator product expansions of these generating fields. We make the following definition:

Definition. Consider a vertex operator algebra \mathcal{V} corresponding to a Lie (super)algebra \mathfrak{g} as above. If the operator product expansions of the generating fields constitute a complete set of algebraic relations, then \mathcal{V} is said to be *universal*.

The terminology comes from noting that a vertex operator algebra is a quotient of \mathcal{V} if, and only if, it has a set of generating fields that satisfy the same operator product expansions as those of \mathcal{V} .

One way to understand universality is to consider what it means for the vacuum module. When \mathfrak{g} is the Virasoro algebra, the vacuum module of the universal vertex operator algebra is the quotient of the Verma module whose (generating) highest weight vector has conformal weight 0 by the Verma submodule generated by the singular vector of conformal weight 1 (the vacuum must be translation-invariant). For $\mathfrak{g} = \widehat{\mathfrak{sl}}(2)$, one quotients the Verma module generated by the highest weight vector of $\mathfrak{sl}(2)$ -weight 0 by the Verma submodule generated by the singular vector of $\mathfrak{sl}(2)$ -weight -2 (as required by the state-field correspondence). In both cases, it may happen that this universal vacuum module is not simple. But, quotienting by a non-trivial proper submodule amounts to imposing additional relations upon the vertex operator algebra, so the result would no longer qualify as universal.

Suppose now that we have a parametrised family of universal vertex operator algebras. For example, the Virasoro algebra and $\widehat{\mathfrak{sl}}(2)$ each define a one-parameter family of vertex operator algebras, parametrised by the central charge $c \in \mathbb{C}$ and the level $k \in \mathbb{C} \setminus \{-2\}$, respectively. We characterise the non-simple members of this family.

Definition. Suppose that one has a family of universal vertex operator algebras $\{\mathcal{V}_i\}_{i \in I}$, parametrised by some index set I . A given value i of the parameter is said to be *admissible* if the corresponding universal vertex operator algebra \mathcal{V}_i is not simple.

¹The existence of even more general weight modules follows directly from twisting by the spectral flow automorphisms of $\widehat{\mathfrak{sl}}(2)$. Well known in the physics literature, this twisting is an important ingredient in [39,40] and first seems to have been explicitly noted for fractional level $\widehat{\mathfrak{sl}}(2)$ models in [7].

For the Virasoro algebra, the admissible central charges c are then precisely those for which there exist coprime integers $p, p' \in \mathbb{Z}_{\geq 2}$ satisfying $c = 13 - 6(t + t^{-1})$, where $t = \frac{p}{p'}$. These central charges correspond, of course, to the Virasoro minimal models which are commonly denoted by $M(p, p')$. We will also denote the simple Virasoro vertex operator algebra of this (admissible) central charge by $M(p, p')$, for convenience.

For $\widehat{\mathfrak{sl}}(2)$, the structure theory of its Verma modules [41] leads to the following characterisation:

Proposition 1.1. *The admissible levels k of the universal vertex operator algebras of $\widehat{\mathfrak{sl}}(2)$ are precisely those for which there exist coprime integers $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Z}_{\geq 1}$ satisfying $k = -2 + t$, where $t = \frac{u}{v}$.*

Needless to say, the admissible levels of $\widehat{\mathfrak{sl}}(2)$ are precisely those of the fractional level $\widehat{\mathfrak{sl}}(2)$ models (including those of non-negative integer level, for convenience). To emphasise the analogy with the Virasoro minimal models $M(p, p')$, we shall denote by $A_1(u, v)$ both the fractional level $\widehat{\mathfrak{sl}}(2)$ model with $k = -2 + \frac{u}{v}$ and the corresponding simple vertex operator algebra. We regard the $A_1(u, v)$ as the minimal models of $\widehat{\mathfrak{sl}}(2)$. Note that $A_1(k + 2, 1)$ requires $k \in \mathbb{Z}_{\geq 0}$ and is therefore just the level k Wess-Zumino-Witten model on $SU(2)$.

It should be clear now that the focus of our interest is not so much on the universal vertex operator algebras themselves, but rather on their admissible level simple quotients. The point is that these simple quotients will have constrained representation theories, due to their additional defining relations, about which we expect to be able to prove classification theorems. The representation theory of the universal vertex operator algebras is, on the other hand, unconstrained by additional relations so that (almost) every \mathfrak{g} -module is allowed.²

To complete the analogy between Virasoro minimal models and fractional level $\widehat{\mathfrak{sl}}(2)$ models, we consider the modules of the simple quotient vertex operator algebras.

Definition. Suppose that $i \in I$ is admissible for a given family $\{\mathbb{V}_i\}$ of universal vertex operator algebras. Then, any module of the simple quotient of \mathbb{V}_i is said to be an *admissible module* of \mathbb{V}_i .

Note that the admissible modules of the universal Virasoro vertex operator algebra are precisely the central charge $c = 13 - 6(\frac{p}{p'} + \frac{p'}{p})$ modules of the Virasoro minimal model $M(p, p')$. These are the highest weight Virasoro modules corresponding to the entries in the Kac table. The admissible modules of the universal $\widehat{\mathfrak{sl}}(2)$ vertex operator algebras are the subject of this article. As noted above, they were first classified in [36, 37]. We remark that they can also be associated with the entries of a table with similar properties to the Virasoro Kac table, see [15].

Finally, note that the definition of admissible module given above is not the original definition of Kac and Wakimoto [2], who originally defined admissible highest weight modules, for arbitrary affine Kac-Moody algebras, in terms of criteria that guaranteed a character formula, generalising that of Weyl-Kac, and good modular properties. The vertex operator algebra definition given above is certainly more general and, in our opinion, more fundamental. However, the two definitions of admissibility coincide for highest weight $\widehat{\mathfrak{sl}}(2)$ -modules. A generalised version of this coincidence for higher rank affine Kac-Moody algebras has recently appeared in [42].

1.3. Outline. We close with a brief outline of the contents of this paper. Section 2 is a pedagogical introduction to the notion of relaxed highest weight theory, crucial for studying the $A_1(u, v)$ models. The idea actually reduces to a special case of parabolic (also called parahoric or generalised) highest weight theory, but the connection to vertex operator algebras via Zhu's algebra is so important that we feel it warrants separate consideration. This is then followed by a detailed discussion of the relaxed highest weight modules of the three Lie algebras used in the remainder of the paper: The Heisenberg algebra, the (bosonic) $\beta\gamma$ ghost algebra and, of course, $\widehat{\mathfrak{sl}}(2)$.

Section 3 opens with a brief review of the Wakimoto free field realisation of $\widehat{\mathfrak{sl}}(2)$ and the corresponding Wakimoto modules. We pay particular attention to the screening fields and operators as a means to motivate the usage of symmetric function theory. We then discuss the construction of certain singular vectors in highest weight and relaxed highest weight $\widehat{\mathfrak{sl}}(2)$ -modules using Jack symmetric function technology to deduce explicit formulae

²One should only exclude modules, such as the adjoint module, that lead to operator product expansions with essential singularities.

for these (Theorem 3.1). We believe that these formulae are new: Even in the standard highest weight case, the only similar result we are aware of is an old paper of Kato and Yamada [43] where an integral formula is derived and evaluated in special cases using Schur polynomials. For singular vectors of relaxed highest weight modules, the only other formulae we know are of the Malikov-Feigin-Fuchs (complex power) form [39].

One consequence of this singular vector study is that at admissible levels, Wakimoto's construction yields a free field realisation of the universal vertex operator algebra of $\widehat{\mathfrak{sl}}(2)$ rather than of its simple quotient $A_1(u, v)$. This is important as it means that the singular vector of the universal vacuum module is accessible in the Wakimoto realisation, hence it may be exploited to such ends as determining the spectrum of $A_1(u, v)$ -modules. We first obtain an upper bound on the highest weight spectrum (Corollary 4.2) using a surprisingly effortless calculation that combines the form of the vacuum singular vector with the specialisation formula for Jack symmetric functions. This bound turns out to be saturated, but to prove this we must address the more involved relaxed highest weight spectrum. In this case, a few more symmetric function manipulations allow us to identify an explicit presentation for Zhu's algebra (Theorem 4.4) in terms of generators and relations.

It is now easy to classify the simple relaxed highest weight modules of $A_1(u, v)$, reproducing in an elegant fashion the results of [36, 37]. We emphasise that our proofs are, to the best of our knowledge, independent of those which have appeared in the literature. We also deduce that highest weight $A_1(u, v)$ -modules are semisimple, giving a new proof of an old result that (essentially) appeared in [2]. In contrast, $A_1(u, v)$ is shown to admit non-semisimple relaxed highest weight modules with two composition factors and these are characterised using short exact sequences. We moreover conjecture that these modules, together with their simple analogues, exhaust the indecomposable relaxed highest weight $A_1(u, v)$ -modules, remarking that proving this would require, among other things, a more detailed knowledge of the submodule structure of relaxed highest weight $\widehat{\mathfrak{sl}}(2)$ -modules. Finally, we prove that the Virasoro zero mode L_0 acts semisimply on all relaxed highest weight $A_1(u, v)$ -modules, independent of our conjecture on the indecomposable spectrum, and discuss briefly how this is consistent with the expectation that, for $v > 1$, the $A_1(u, v)$ model is a logarithmic conformal field theory.

Appendix A gives a brief, but thorough, introduction to the aspects of symmetric function theory, in particular, those relating to Jack symmetric functions, that are used in the text. This material is all standard and may be found in [28]. We have also included a brief introduction to Zhu's algebra in Appendix B, concentrating on motivating it as an abstract version of the algebra of zero modes acting on the space of "ground states" of a relaxed highest weight module. This appendix is aimed at physicists, in particular, it uses physics conventions for Fourier expansions, but we hope that it will also prove useful to mathematicians.

2. GENERALISING HIGHEST WEIGHT THEORY

In this section, we consider a generalisation of highest weight theory that we will qualify as *relaxed*. Originally introduced for $\widehat{\mathfrak{sl}}(2)$ in order to study a correspondence relating $\widehat{\mathfrak{sl}}(2)$ -modules to those over the $N = 2$ superconformal algebra [39], relaxed highest weight modules have since appeared as necessary constituents of the $SL(2; \mathbb{R})$ Wess-Zumino-Witten model [40], in admissible level fusion rules [7, 11], in relations to logarithmic minimal models [10, 44, 45], in demonstrating the modular invariance of admissible level theories [14, 15], and in the full description of bosonic $\beta\gamma$ ghosts [11, 13]. Necessity aside, we feel that from some points of view, particularly that of Zhu's algebra [46], discussed in Appendix B, it is more natural to consider these relaxed modules instead of the standard highest weight modules that one typically encounters in rational conformal field theory.

2.1. Relaxed highest weight theory. We recall that the formalism of highest weight theory for a Lie algebra \mathfrak{g} is built from a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+. \tag{2.1}$$

This is a vector space direct sum of subalgebras of \mathfrak{g} in which the Cartan subalgebra \mathfrak{h} is abelian and acts semisimply, through the adjoint action, on both \mathfrak{g}_- and \mathfrak{g}_+ . In particular, $[\mathfrak{h}, \mathfrak{g}_\pm] \subseteq \mathfrak{g}_\pm$. Moreover, the subalgebras \mathfrak{g}_- and \mathfrak{g}_+ are assumed to be antiequivalent in that there exists a (linear) order two antiautomorphism, the adjoint \dagger ,

satisfying $\mathfrak{g}_\pm^\dagger = \mathfrak{g}_\mp$. The elements of \mathfrak{h} are supposed to be fixed by \dagger , meaning that each $x \in \mathfrak{h}$ is self-adjoint [47]. This is, however, a little too restrictive; we instead only demand that the adjoint preserves the Cartan subalgebra. Since \mathfrak{h} is abelian, this implies that each $x \in \mathfrak{h}$ is normal: $[x, x^\dagger] = 0$. Note that being a linear antiautomorphism means that $(ax)^\dagger = ax^\dagger$ and $[x, y]^\dagger = [y^\dagger, x^\dagger]$, for all $a \in \mathbb{C}$ and $x, y \in \mathfrak{g}$.

Given such a triangular decomposition, one defines a highest weight vector in a given \mathfrak{g} -module to be a simultaneous eigenvector of the elements of \mathfrak{h} which is annihilated by all the elements of \mathfrak{g}_+ . Any module generated by a single highest weight vector is called a highest weight module. Conspicuous examples include the Verma modules $\mathcal{V}_\lambda = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$, where $\mathcal{U}\mathfrak{g}$ denotes the universal enveloping algebra of \mathfrak{g} , $\mathcal{U}\mathfrak{b}$ that of the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}_+$, $\lambda \in \mathfrak{h}^*$ is the highest weight, and \mathbb{C}_λ is the one-dimensional \mathfrak{b} -module upon which \mathfrak{g}_+ acts as 0 and each $x \in \mathfrak{h}$ acts as $\lambda(x) \in \mathbb{C}$. The highest weight vector generating \mathcal{V}_λ is $\mathbf{1} \otimes_{\mathcal{U}\mathfrak{b}} 1_\lambda$, where $\mathbf{1}$ denotes the unit of $\mathcal{U}\mathfrak{g}$ (and 1_λ that of $\mathbb{C}_\lambda \cong \mathbb{C}$).

The Lie algebras \mathfrak{g} that one typically encounters in conformal field theory have triangular decompositions. However, they also have more structure in that they are graded by the semisimple action of the Virasoro zero mode L_0 so that, if \mathfrak{g}_n denotes the $\text{ad}(L_0)$ -eigenspace of eigenvalue $-n$, then $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$. Moreover, the Cartan subalgebra \mathfrak{h} is generally chosen to include L_0 , which we will always assume is self-adjoint, hence it follows that $\mathfrak{g}_m^\dagger = \mathfrak{g}_{-m}$. Given this structure, the following definition is natural:

Definition. Let \mathfrak{g} be a Lie algebra with triangular decomposition (2.1) (where the Cartan subalgebra may include elements that are not self-adjoint). If there exists $L_0 \in \mathfrak{h}$ such that $\mathfrak{g} = \bigoplus_n \mathfrak{g}_n$, where \mathfrak{g}_n is the eigenspace of $\text{ad}(L_0)$ of eigenvalue $-n$, and L_0 is the zero mode of a subalgebra of \mathfrak{g} isomorphic to the Virasoro algebra, then we will say that \mathfrak{g} is *conformally graded*.

The most obvious example is the Virasoro algebra itself which is clearly conformally graded with $\mathfrak{g}_n = \text{span}\{L_n\}$, for $n \in \mathbb{Z} \setminus \{0\}$, $\mathfrak{g}_0 = \text{span}\{L_0, C\}$ and $\mathfrak{g}_n = \{0\}$ otherwise. This shows that the usual, but somewhat confusing, convention that $[L_0, L_n] = -L_n$ is responsible for \mathfrak{g}_n having $\text{ad}(L_0)$ -eigenvalue $-n$ in the above definition.

Definition. Given a conformally graded Lie algebra \mathfrak{g} , its *relaxed triangular decomposition* is

$$\mathfrak{g} = \mathfrak{g}_< \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_>, \quad (2.2)$$

where $\mathfrak{g}_< = \bigoplus_{n<0} \mathfrak{g}_n$ and $\mathfrak{g}_> = \bigoplus_{n>0} \mathfrak{g}_n$. A *relaxed highest weight vector* is then a simultaneous eigenvector of $\mathfrak{h} \subseteq \mathfrak{g}_0$ that is annihilated by $\mathfrak{g}_>$. A *relaxed highest weight module* is a module that is generated by a single relaxed highest weight vector. The *relaxed Borel subalgebra* is $\mathfrak{g}_\geq = \mathfrak{g}_0 \oplus \mathfrak{g}_>$ and a *relaxed Verma module* is a \mathfrak{g} -module isomorphic to $\mathcal{R}_\mathcal{M} = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_\geq} \mathcal{M}$, where \mathcal{M} is a simple weight module of \mathfrak{g}_0 upon which $\mathfrak{g}_>$ acts as 0.

These definitions have obvious analogues for Lie superalgebras and other more general structures, but we will not need this level of generality in what follows.

In this article, we will only consider triangular decompositions of a conformally graded Lie algebra \mathfrak{g} that satisfies $\mathfrak{g}_< \subseteq \mathfrak{g}_-$ and $\mathfrak{g}_> \subseteq \mathfrak{g}_+$. Thus, positive modes are always in \mathfrak{g}_+ and negative modes are always in \mathfrak{g}_- . When $\mathfrak{g}_0 = \mathfrak{h}$, the relaxed triangular decomposition then coincides with the (unique) triangular decomposition of this type. We will shortly see examples of relaxed highest weight modules which are not highest weight modules in the usual sense. First, however, we mention that relaxed highest weight modules may be identified as generalised highest weight modules with respect to the parabolic subalgebra $\mathfrak{g}_\geq = \mathfrak{g}_0 \oplus \mathfrak{g}_>$. We recall that a parabolic subalgebra is any subalgebra that contains a Borel subalgebra (see [47, 48] for a quick overview of parabolic subalgebras).

It will be occasionally convenient to take this a step further and introduce a category of modules, generalising the well known category \mathcal{O} , that contains the relaxed highest weight modules of \mathfrak{g} . This is essentially (a variant of) the parabolic category \mathcal{O} discussed, for example, in [48]. The explicit details of this category are not essential for understanding the results to follow, but we shall devote a few paragraphs to explaining their physical motivation.

Definition. Given a conformally graded Lie algebra $\mathfrak{g} = \mathfrak{g}_< \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_>$, define the *relaxed category* \mathcal{R} to consist of the \mathfrak{g} -modules \mathcal{M} satisfying the following axioms:

- \mathcal{M} is finitely generated.
- \mathcal{M} is a weight module (the action of the Cartan subalgebra \mathfrak{h} is semisimple).
- The action of $\mathfrak{g}_>$ is locally nilpotent: For each $v \in \mathcal{M}$, the space $\mathfrak{U}\mathfrak{g}_> \cdot v$ is finite-dimensional.

The morphisms are the \mathfrak{g} -module homomorphisms between these modules, as usual.

All highest weight and relaxed highest weight modules belong to category \mathcal{R} . Moreover, if \mathfrak{g} has finite-dimensional root spaces, then it follows that each module in \mathcal{R} will have finite-dimensional weight spaces. Another important consequence of these axioms is that every (non-zero) module in category \mathcal{R} possesses a relaxed highest weight vector, hence that every simple category \mathcal{R} module is a relaxed highest weight module.

One can, and should, ask whether the mathematical axioms that we impose on category \mathcal{R} will end up excluding modules relevant for applications. This is an important question and the answer is that they do exclude relevant modules, but in a well-controlled manner. Our motivation for introducing this category is that we want to classify the modules of the admissible level k vertex operator algebra $A_1(u, v)$ by identifying these modules as $\widehat{\mathfrak{sl}}(2)$ -modules. For the physical application of investigating the corresponding conformal field theories, we must insist that the category of $A_1(u, v)$ -modules be closed under the conjugation operation of $\widehat{\mathfrak{sl}}(2)$ (see Section 2.2) and fusion. Moreover, we want the characters to behave well under modular transformations so that one can identify modular invariant partition functions and compute (Grothendieck) fusion rules from a Verlinde-type formula.

Category \mathcal{O} is not sufficient for these purposes, in particular, the conjugate of an $A_1(u, v)$ -module from category \mathcal{O} need not lie in category \mathcal{O} . Relaxing to category \mathcal{R} alleviates this problem and has recently been shown [15] to lead to characters with excellent modular behaviour, provided that one extends the category again to take into account twists by the so-called spectral flow automorphisms. The upshot is that these spectrally-flowed modules are not in category \mathcal{R} , but the twisting is very well understood, justifying the above statement that the exclusion of these physically relevant modules is under control.

Axiomatically, accounting for the spectrally-flowed modules would require weakening the local nilpotency axiom above. However, this axiom has the advantage that category \mathcal{R} provides a very natural setting for the important, and very useful, technology of Zhu's algebra, discussed in Appendix B. We restrict to weight modules because the fusion coproduct formulae [49] for $\widehat{\mathfrak{sl}}(2)$ show that the fusion product of two weight modules will again be weight. Similarly, the conjugate of a weight module is weight and omitting non-weight modules does not restrict the characters in any way. Moreover, being weight does not preclude the Virasoro zero mode L_0 from acting non-semisimply as is required in logarithmic conformal field theories.

To summarise, the relaxed category \mathcal{R} is a rich source of modules for affine Kac-Moody (super)algebras that appears to be even more relevant to conformal field theory than the much more familiar category \mathcal{O} .

2.2. Examples. In this paper, we will use the Wakimoto free field realisation to study the relaxed highest weight modules of $\widehat{\mathfrak{sl}}(2)$ and determine which of these are modules for the admissible level vertex operator algebras $A_1(u, v)$, where $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Z}_{\geq 1}$ are coprime and the level k is determined by $\frac{u}{v} = t = k + 2$. We will therefore need to investigate the (relaxed) highest weight theory of the Heisenberg algebra \mathfrak{H} , the $c = 2$ bosonic $\beta\gamma$ ghost system \mathfrak{G} and $\widehat{\mathfrak{sl}}(2)$ itself. This investigation constitutes the rest of the section.

The Heisenberg algebra \mathfrak{H} . We will use the same notations and conventions for the Heisenberg algebra as in [33]. The free boson vertex operator algebra is generated by a single bosonic field $a(z)$, defined by the operator product expansion

$$a(z)a(w) \sim \frac{\mathbf{1}}{(z-w)^2}. \quad (2.3)$$

With the standard Fourier expansion, $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, the operator product expansion implies the following commutation relations:

$$[a_m, a_n] = m\delta_{m+n,0}\mathbf{1}, \quad m, n \in \mathbb{Z}. \quad (2.4)$$

The Heisenberg algebra \mathfrak{H} is then the infinite-dimensional Lie algebra spanned by the a_n and $\mathbf{1}$, the latter being identified with the unit of the universal enveloping algebra of \mathfrak{H} , as usual. We will assume that $\mathbf{1}$ acts as the identity

operator on any \mathfrak{h} -module. This is only a minor restriction since a simple rescaling of the generators lets this operator act as multiplication by any non-zero number.

As is well known, the free boson vertex operator algebra admits a one-parameter family of conformal structures. We will write the energy-momentum tensor and central charge in the form

$$T^{\text{bos.}}(z) = \frac{1}{2} : a(z)a(z) : - \frac{1}{\alpha} \partial a(z), \quad c^{\text{bos.}} = 1 - \frac{12}{\alpha^2}, \quad (2.5)$$

where α parametrises the conformal structure. We note that $\alpha \rightarrow \infty$ reproduces the standard free boson central charge $c^{\text{bos.}} = 1$. In Wakimoto's construction, this would correspond to $k \rightarrow \infty$, so it is permissible to ignore this case. It is worth recalling that $a(z)$ is not a Virasoro primary for α finite; instead we have

$$T^{\text{bos.}}(z)a(w) \sim \frac{2/\alpha \mathbf{1}}{(z-w)^3} + \frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{z-w}. \quad (2.6)$$

The Fourier expansion $T^{\text{bos.}}(z) = \sum_{n \in \mathbb{Z}} L_n^{\text{bos.}} z^{-n-2}$ defines the Virasoro modes and it is easy to check that the Lie algebra \mathfrak{g} spanned by the a_n , $L_n^{\text{bos.}}$ and $\mathbf{1}$ is conformally graded. It is likewise easy to check that (for finite α) the only adjoint on the Heisenberg algebra consistent with the standard Virasoro adjoint $(L_n^{\text{bos.}})^\dagger = L_{-n}^{\text{bos.}}$ is

$$a_n^\dagger = -a_{-n} - \frac{2}{\alpha} \delta_{n,0} \mathbf{1}. \quad (2.7)$$

Since $\mathfrak{g}_0 = \text{span}\{a_0, L_0, \mathbf{1}\}$ is abelian, it coincides with the Cartan subalgebra \mathfrak{h} , hence relaxed highest weight theory reduces to ordinary highest weight theory for the Heisenberg algebra. We remark that this is one example where we cannot insist that the Cartan subalgebra consist of self-adjoint elements.

The highest weight theory of the Heisenberg Lie algebra is well known. The Verma modules are known as Fock spaces and are parametrised by the a_0 -eigenvalue p of the highest weight vector. They are simple for all $p \in \mathbb{C}$. We will denote the Fock spaces by \mathcal{F}_p and their corresponding highest weight vectors by u_p . In accordance with (2.7), the module conjugate to the Fock space \mathcal{F}_p is $\mathcal{F}_{-p-2/\alpha}$.

The ghost algebra \mathfrak{G} . For the $\beta\gamma$ ghost system, we follow the notations and conventions of [13]. The ghost vertex operator algebra is generated by two bosonic fields, $\beta(z)$ and $\gamma(z)$, whose operator product expansions are

$$\beta(z)\beta(w) \sim 0, \quad \gamma(z)\beta(w) \sim \frac{\mathbf{1}}{z-w}, \quad \gamma(z)\gamma(w) \sim 0. \quad (2.8)$$

From these, one constructs a Heisenberg field $J(z)$ and an energy-momentum tensor $T^{\text{gh.}}(z)$ by

$$J(z) = : \beta(z)\gamma(z) :, \quad T^{\text{gh.}}(z) = - : \beta(z)\partial\gamma(z) :. \quad (2.9)$$

These give $\beta(z)$ and $\gamma(z)$ Heisenberg weights $+1$ and -1 , and conformal weights 1 and 0 , respectively. We remark that $J(z)$ is not normalised as in (2.3), nor is it primary with respect to $T^{\text{gh.}}(z)$:

$$J(z)J(w) \sim \frac{-\mathbf{1}}{(z-w)^2}, \quad T^{\text{gh.}}(z)J(w) \sim \frac{-\mathbf{1}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}. \quad (2.10)$$

As with the free boson, there is actually a one-parameter family of conformal structures; $T^{\text{gh.}}(z)$ has been chosen so that the ghost fields $\beta(z)$ and $\gamma(z)$ have the required conformal weights. This choice also fixes the central charge of the ghost system to be $c^{\text{gh.}} = 2$.

The Fourier expansions $\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1}$ and $\gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n}$ now yield the commutation relations

$$[\beta_m, \beta_n] = 0, \quad [\gamma_m, \beta_n] = \delta_{m+n,0} \mathbf{1}, \quad [\gamma_m, \gamma_n] = 0; \quad m, n \in \mathbb{Z}. \quad (2.11)$$

The infinite-dimensional Lie algebra \mathfrak{G} , spanned by the β_n , γ_n and $\mathbf{1}$, is called the ghost Lie algebra. Again, $\mathbf{1}$ is identified with the unit of $\mathfrak{U}\mathfrak{G}$ and we assume that it acts as the identity on all \mathfrak{G} -modules. As with the Heisenberg algebra, we will extend this algebra by the modes $J_n, L_n^{\text{gh.}} \in \mathfrak{U}\mathfrak{G}$, defined by $J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$ and $T^{\text{gh.}}(z) = \sum_{n \in \mathbb{Z}} L_n^{\text{gh.}} z^{-n-2}$. The Lie algebra \mathfrak{g} spanned by the $\beta_n, \gamma_n, J_n, L_n^{\text{gh.}}$ and $\mathbf{1}$ is then conformally graded with

relations including

$$[J_m, \beta_n] = \beta_{m+n}, \quad [J_m, \gamma_n] = -\gamma_{m+n}, \quad [L_m^{\text{gh}}, \beta_n] = -n\beta_{m+n}, \quad [L_m^{\text{gh}}, \gamma_n] = -(m+n)\gamma_{m+n}, \quad (2.12a)$$

$$[J_m, J_n] = -m\delta_{m+n,0}\mathbf{1}, \quad [L_m^{\text{gh}}, J_n] = -nJ_{m+n} - \frac{1}{2}m(m+1)\delta_{m+n,0}\mathbf{1}. \quad (2.12b)$$

The Cartan subalgebra $\mathfrak{h} = \text{span}\{J_0, L_0^{\text{gh}}, \mathbf{1}\}$ is a proper subalgebra of $\mathfrak{g}_0 = \text{span}\{\beta_0, \gamma_0, J_0, L_0^{\text{gh}}, \mathbf{1}\}$, so the relaxed and ordinary highest weight theories of the $\beta\gamma$ ghost system do not coincide. In this case, the ghost adjoint

$$\beta_n^\dagger = \gamma_{-n} \quad (2.13)$$

implies that all the elements of \mathfrak{h} are self-adjoint.

We start with the ordinary highest weight theory corresponding to the triangular decomposition in which $\beta_0 \in \mathfrak{g}_+$ annihilates the highest weight vector v and $\gamma_0 \in \mathfrak{g}_-$ need not. It turns out that this yields a unique Verma module because these conditions imply that $J_0 v = L_0^{\text{gh}} v = 0$ (and $\mathbf{1}$ acts as the identity, as always). This module is simple; in fact, we may take it to be the ghost vacuum module \mathcal{G} because these conditions also imply that $L_{-1}^{\text{gh}} v = 0$. Of course, one can also take the triangular decomposition in which γ_0 annihilates the highest weight vector and β_0 need not.³ The resulting Verma module is the simple module $c(\mathcal{G})$ conjugate to \mathcal{G} , obtained by twisting the action of \mathfrak{G} by c , the (order 4) conjugation automorphism that sends β_n to γ_n and γ_n to $-\beta_n$.

The relaxed highest weight theory of the ghost system is significantly more interesting as we no longer require that β_0 (or γ_0) annihilates the highest weight vector. In fact, we may induce from a fairly arbitrary simple weight module of \mathfrak{g}_0 . Given that J_0 and L_0^{gh} , as abstract elements of \mathfrak{g}_0 , are going to be identified with elements of $\mathfrak{L}\mathfrak{G}$ and that the \mathfrak{g}_0 -module weight vectors are going to be identified with the relaxed highest weight vectors of the induced module, we may restrict to \mathfrak{g}_0 -modules on which the action of J_0 is identified with that of $\gamma_0\beta_0$ and the action of L_0^{gh} is always 0.

Proposition 2.1 [13, Prop. 1]. *A simple weight module over \mathfrak{g}_0 upon which $J_0 = \gamma_0\beta_0$ and $L_0^{\text{gh}} = 0$ is isomorphic to one of the following:*

- The module $\overline{\mathcal{G}}$ generated by a vector v which is annihilated by β_0 and thus also J_0 . This module has a basis of weight vectors v_j , $j \in \mathbb{Z}_{\leq 0}$, where $J_0 v_j = jv_j$.
- The module $\overline{c(\mathcal{G})}$, conjugate to $\overline{\mathcal{G}}$, generated by a vector v which is annihilated by γ_0 and thus $J_0 v = v$. This module has a basis of weight vectors v_j , $j \in \mathbb{Z}_{\geq 1}$, where $J_0 v_j = jv_j$.
- The modules $\overline{\mathcal{G}}_q$, where $q \in \mathbb{C} \setminus \mathbb{Z}$, each of which is generated by a vector v satisfying $J_0 v = qv$; it follows that no non-zero vector is annihilated by β_0 or γ_0 . The eigenvalues of $J_0 = \gamma_0\beta_0$ all lie in $q + \mathbb{Z}$ and these modules have a basis of weight vectors v_j , $j \in q + \mathbb{Z}$, satisfying $J_0 v_j = jv_j$. The modules $\overline{\mathcal{G}}_q$ and $\overline{\mathcal{G}}_{q+1}$ are isomorphic.

Additionally, one can consider the indecomposable \mathfrak{g}_0 -modules $\overline{\mathcal{G}}_0^+$ and $\overline{\mathcal{G}}_0^-$ that likewise have a basis of weight vectors v_j , $j \in \mathbb{Z}$, satisfying $J_0 v_j = jv_j$, but they are not simple. They are determined (up to isomorphism) by the following non-split short exact sequences:

$$0 \longrightarrow \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}_0^+ \longrightarrow \overline{c(\mathcal{G})} \longrightarrow 0, \quad 0 \longrightarrow \overline{c(\mathcal{G})} \longrightarrow \overline{\mathcal{G}}_0^- \longrightarrow \overline{\mathcal{G}} \longrightarrow 0. \quad (2.14)$$

Inducing the \mathfrak{g}_0 -modules $\overline{\mathcal{G}}$, $\overline{c(\mathcal{G})}$ and $\overline{\mathcal{G}}_q$ ($q \notin \mathbb{Z}$) therefore results in relaxed Verma modules for \mathfrak{G} . The first two give the simple ghost vacuum module \mathcal{G} and its conjugate $c(\mathcal{G})$, respectively; the last gives new modules \mathcal{G}_q which are also simple and satisfy $\mathcal{G}_q \cong \mathcal{G}_{q+1}$. We may similarly induce the non-simple \mathfrak{g}_0 -modules $\overline{\mathcal{G}}_0^+$ and $\overline{\mathcal{G}}_0^-$ to obtain non-simple \mathfrak{G} -modules $\overline{\mathcal{G}}_0^+$ and $\overline{\mathcal{G}}_0^-$ which are likewise determined (up to isomorphism) by the following non-split short exact sequences:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_0^+ \longrightarrow c(\mathcal{G}) \longrightarrow 0, \quad 0 \longrightarrow c(\mathcal{G}) \longrightarrow \mathcal{G}_0^- \longrightarrow \mathcal{G} \longrightarrow 0. \quad (2.15)$$

³There are other triangular decompositions, but they will not concern us here. Indeed, they may be obtained from those already mentioned by twisting with a so-called spectral flow automorphism, see [13].

The affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$. We consider the universal vertex operator algebra of non-critical level $k \neq -2$ generated by three fields $e(z)$, $h(z)$ and $f(z)$ satisfying the operator product expansions

$$\begin{aligned} h(z)e(w) &\sim \frac{+2e(w)}{z-w}, & h(z)h(w) &\sim \frac{2k\mathbf{1}}{(z-w)^2}, & e(z)e(w) &\sim 0, \\ h(z)f(w) &\sim \frac{-2f(w)}{z-w}, & e(z)f(w) &\sim \frac{-k\mathbf{1}}{(z-w)^2} - \frac{h(w)}{z-w}, & f(z)f(w) &\sim 0, \end{aligned} \quad (2.16)$$

and no other (independent) relations. We denote this vertex operator algebra by V_k . The maximal proper ideal of V_k is non-trivial if and only if k is admissible. Moreover, this ideal is generated by a single primary field (singular vector) [41]; we do not set this primary field to zero. Note that we have chosen the $\mathfrak{sl}(2)$ basis $\{e, h, f\}$ to be consistent with previous work, [12] in particular, where it was necessary to tailor the basis to the $\mathfrak{sl}(2; \mathbb{R})$ adjoint rather than the (more traditional) $\mathfrak{su}(2)$ adjoint. This is reflected in the signs appearing in the formulae for $e(z)f(w)$ above.

With the usual Fourier expansions $g(z) = \sum_{n \in \mathbb{Z}} g_n z^{-n-1}$, where $g = e, h, f$, the commutation relations are

$$\begin{aligned} [h_m, e_n] &= +2e_{m+n}, & [h_m, h_n] &= 2m\delta_{m+n,0}k\mathbf{1}, & [e_m, e_n] &= 0, \\ [h_m, f_n] &= -2f_{m+n}, & [e_m, f_n] &= -h_{m+n} - m\delta_{m+n,0}k\mathbf{1}, & [f_m, f_n] &= 0, \end{aligned} \quad m, n \in \mathbb{Z} \quad (2.17)$$

and these make span $\{e_n, h_n, f_n, \mathbf{1}\}$ into a Lie algebra which we denote by $\widehat{\mathfrak{sl}}(2)$. Once again, we assume that the unit $\mathbf{1} \in \mathfrak{U}(\widehat{\mathfrak{sl}}(2))$ acts as the identity on each $\widehat{\mathfrak{sl}}(2)$ -module.

The standard conformal structure of V_k is uniquely determined by requiring that $e(z)$, $h(z)$ and $f(z)$ are Virasoro primaries of conformal weight 1 (this structure exists for all $k \neq -2$). The Sugawara construction then gives the explicit form of the energy-momentum tensor as

$$T(z) = \frac{1}{2t} \left(\frac{1}{2} : h(z)h(z) : - : e(z)f(z) : - : f(z)e(z) : \right), \quad (2.18)$$

where $t = k + 2$. With $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, one finds that the modes L_n generate a copy of the Virasoro algebra with central charge $c = 3 - 6/t$. The Lie algebra \mathfrak{g} spanned by the e_n, h_n, f_n, L_n and $\mathbf{1}$ is then conformally graded with Cartan subalgebra $\mathfrak{h} = \text{span}\{h_0, L_0, \mathbf{1}\}$. We have chosen the $\mathfrak{sl}(2)$ basis so that the $\mathfrak{sl}(2; \mathbb{R})$ adjoint becomes

$$e_n^\dagger = f_{-n}, \quad h_n^\dagger = h_{-n}. \quad (2.19)$$

Again, the Cartan subalgebra consists of self-adjoint elements.

The highest weight theory of $\widehat{\mathfrak{sl}}(2)$ is well known. The standard triangular decomposition splits the zero modes so that $e_0 \in \mathfrak{g}_+$ annihilates highest weight vectors but $f_0 \in \mathfrak{g}_-$ need not. Then, the Verma modules \mathcal{V}_λ are parametrised by the $\mathfrak{sl}(2)$ -weight (h_0 -eigenvalue) $\lambda \in \mathbb{C}$ of the highest weight vector because it follows from (2.18) that its conformal weight is then given by

$$\Delta_\lambda = \frac{\lambda(\lambda+2)}{4t}. \quad (2.20)$$

These Verma modules need not be simple. The quotient module $\mathcal{V}_0/\mathcal{V}_{-2}$, where the submodule \mathcal{V}_{-2} is generated by acting with f_0 on the (generating) highest weight vector of \mathcal{V}_0 , is the vacuum module; it carries the structure of the universal vertex operator algebra V_k defined by (2.16). As with the ghosts, one can also consider the triangular decomposition in which f_0 annihilates highest weight vectors but e_0 need not.⁴ The Verma modules with respect to this decomposition are then the conjugates $w(\mathcal{V}_\lambda)$ of the Verma modules \mathcal{V}_λ , where the (order 2) conjugation automorphism w sends e_n to $-f_n$ and h_n to $-h_n$.

Because $\mathfrak{g}_0 = \text{span}\{e_0, h_0, f_0, L_0, \mathbf{1}\}$ is non-abelian, it strictly contains \mathfrak{h} , so the relaxed highest weight theory of $\widehat{\mathfrak{sl}}(2)$ is strictly more general. We note that inducing from a \mathfrak{g}_0 -module reduces to choosing an $\mathfrak{sl}(2)$ -module

⁴And as with the ghosts, there are again other triangular decompositions that will not concern us here, being related to those discussed here by spectral flow automorphisms, see [12, 15].

because we require that $4tL_0$ acts as $h_0^2 - 2e_0f_0 - 2f_0e_0$ (this is how $4tL_0$ acts on relaxed highest weight vectors). The analogue of Proposition 2.1 is then the classification of weight modules for $\mathfrak{sl}(2)$ (see [50], for example):

Proposition 2.2. *The simple weight modules of $\mathfrak{sl}(2)$ are exhausted by the following:*

- The $(\lambda + 1)$ -dimensional modules $\overline{\mathcal{L}}_\lambda$, with $\lambda \in \mathbb{Z}_{\geq 0}$. The module $\overline{\mathcal{L}}_\lambda$ has a basis of weight vectors w_m , where $m = \lambda, \lambda - 2, \dots, -\lambda$ and $h_0w_m = mw_m$. It is both highest and lowest weight.
- The infinite-dimensional highest weight modules $\overline{\mathcal{D}}_\lambda$, with $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. The module $\overline{\mathcal{D}}_\lambda$ has a basis of weight vectors w_m , where $m = \lambda, \lambda - 2, \lambda - 4, \dots$ and $h_0w_m = mw_m$.
- The infinite-dimensional lowest weight modules $\overline{w}(\overline{\mathcal{D}}_{-\lambda})$, with $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Here, \overline{w} is the Weyl reflection of $\mathfrak{sl}(2)$ that sends e_0 to $-f_0$ and h_0 to $-h_0$. The module $\overline{w}(\overline{\mathcal{D}}_{-\lambda})$ has a basis of weight vectors w_m , where $m = \lambda, \lambda + 2, \lambda + 4, \dots$ and $h_0w_m = mw_m$.
- The infinite-dimensional weight modules $\overline{\mathcal{R}}_{\lambda;\Delta}$, with $\lambda, \Delta \in \mathbb{C}$ satisfying $4t\Delta \neq \mu(\mu + 2)$ for any $\mu \in \lambda + 2\mathbb{Z}$. The module $\overline{\mathcal{R}}_{\lambda;\Delta}$ has a basis of weight vectors w_μ , with $\mu \in \lambda + 2\mathbb{Z}$, satisfying $h_0w_\mu = \mu w_\mu$ and $L_0w_\mu = \Delta w_\mu$. It is neither highest nor lowest weight. Moreover, there are isomorphisms $\overline{\mathcal{R}}_{\lambda;\Delta} \cong \overline{\mathcal{R}}_{\lambda+2;\Delta}$.

As with Proposition 2.1, there are non-simple analogues of the $\overline{\mathcal{R}}_{\lambda;\Delta}$ when $4t\Delta = \mu(\mu + 2)$ for some $\mu \in \lambda + 2\mathbb{Z}$. The structures of these indecomposables depend upon precisely how many $\mu \in \lambda + 2\mathbb{Z}$ satisfy this constraint [50] and we shall defer their consideration until they are needed (Section 4.2).

Inducing each of the simple $\mathfrak{sl}(2)$ -modules now yields relaxed Verma modules for $\widehat{\mathfrak{sl}}(2)$. More precisely:

- Inducing $\overline{\mathcal{L}}_\lambda$, where $\lambda \in \mathbb{Z}_{\geq 0}$, results in the highest weight module $\mathcal{V}_\lambda/\mathcal{V}_{-\lambda-2}$, where the highest weight vectors of \mathcal{V}_λ and $\mathcal{V}_{-\lambda-2}$ are related by $w_{-\lambda-2} = f_0^{\lambda+1}w_\lambda$. This induced module need not be simple; its simple quotient will be denoted by \mathcal{L}_λ . These simple modules are self-conjugate: $w(\mathcal{L}_\lambda) = \mathcal{L}_\lambda$.
- Inducing $\overline{\mathcal{D}}_\lambda$, where $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, results in the Verma module \mathcal{V}_λ ; the simple quotient will be denoted by \mathcal{D}_λ .
- Inducing $\overline{w}(\overline{\mathcal{D}}_{-\lambda})$, where $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, results in $w(\mathcal{V}_{-\lambda})$; the simple quotient is $w(\mathcal{D}_{-\lambda})$.
- Inducing $\overline{\mathcal{R}}_{\lambda;\Delta}$, with $\lambda, \Delta \in \mathbb{C}$ satisfying $4t\Delta \neq \mu(\mu + 2)$ for any $\mu \in \lambda + 2\mathbb{Z}$, results in a new relaxed Verma module that we shall denote by $\mathcal{R}_{\lambda;\Delta}$; its simple quotient will be denoted by $\mathcal{E}_{\lambda;\Delta}$. As above, there are isomorphisms $\mathcal{R}_{\lambda;\Delta} \cong \mathcal{R}_{\lambda+2;\Delta}$ and $\mathcal{E}_{\lambda;\Delta} \cong \mathcal{E}_{\lambda+2;\Delta}$. Finally, the module conjugate to $\mathcal{E}_{\lambda;\Delta}$ is $\mathcal{E}_{-\lambda;\Delta}$.

We emphasise that whereas the simple highest weight modules (and their conjugates) are characterised by a single parameter, the highest weight, the $\mathcal{E}_{\lambda;\Delta}$ require two parameters in general. The three classes of simple $\widehat{\mathfrak{sl}}(2)$ -modules \mathcal{L}_λ , \mathcal{D}_λ and $w(\mathcal{D}_{-\lambda})$, and $\mathcal{E}_{\lambda;\Delta}$ are distinguished by their relaxed highest weight vectors: \mathcal{L}_λ has finitely many ($\lambda + 1$ in fact); \mathcal{D}_λ and $w(\mathcal{D}_{-\lambda})$ have infinitely many, but their $\mathfrak{sl}(2)$ -weights are bounded above and below, respectively; $\mathcal{E}_{\lambda;\Delta}$ has infinitely many with no bounds on the $\mathfrak{sl}(2)$ -weights.

3. THE WAKIMOTO FREE FIELD REALISATION

A free field realisation of $\widehat{\mathfrak{sl}}(2)$ for any level was constructed by Wakimoto in [20]. It shows that fields $e(z)$, $h(z)$ and $f(z)$, satisfying the operator product expansions (2.16), may be constructed in terms of a free boson and a pair of bosonic ghosts. We review this and the screening operator formalism of free field theories, concluding by deriving explicit formulae, in terms of Jack symmetric polynomials, for certain (relaxed) singular vectors in (relaxed) Wakimoto modules over $\widehat{\mathfrak{sl}}(2)$. A corollary of this analysis is that for admissible levels, Wakimoto's construction describes the universal vertex operator algebra \mathcal{V}_k of $\widehat{\mathfrak{sl}}(2)$, rather than its simple quotient $\mathcal{A}_1(u, v)$.

3.1. The Wakimoto construction. There is a one-parameter family of realisations of $\widehat{\mathfrak{sl}}(2)$ in terms of tensor products of the free boson and ghost fields. Given the operator product expansions (2.3) and (2.8), it is straightforward to verify that defining (we omit the tensor products for notational simplicity)

$$\begin{aligned} e(z) &= \beta(z), & h(z) &= 2 : \beta(z)\gamma(z) : + \alpha a(z), \\ f(z) &= : \beta(z)\gamma(z)\gamma(z) : + \alpha a(z)\gamma(z) + \left(\frac{\alpha^2}{2} - 2\right) \partial\gamma(z) \end{aligned} \tag{3.1}$$

reproduces the $\widehat{\mathfrak{sl}}(2)$ operator product expansions (2.16) with the level $k = t - 2$ being related to the parameter α by $\alpha^2 = 2t$. Moreover, the $\widehat{\mathfrak{sl}}(2)$ energy-momentum tensor (2.18) and central charge then decompose as

$$\begin{aligned} T(z) &= T^{\text{bos.}}(z) + T^{\text{gh.}}(z) = \frac{1}{2} : a(z)a(z) : - \frac{1}{\alpha} \partial a(z) - : \beta(z)\partial\gamma(z) :, \\ c &= c^{\text{bos.}} + c^{\text{gh.}} = 1 - \frac{12}{\alpha^2} + 2 = 3 - \frac{6}{t}, \end{aligned} \quad (3.2)$$

identifying α with the deformation parameter in the free boson conformal structure (2.5).

Definition. We define the *Wakimoto vertex operator algebra* to be the tensor product of the Heisenberg and ghost vertex operator algebras, equipped with the conformal structure given in (3.2).

Recall that when k is admissible, the universal $\widehat{\mathfrak{sl}}(2)$ vertex operator algebra is not simple, but has a unique maximal ideal that is generated by a single primary field (singular vector). One of the results of this section (Corollary 3.2) is that this field is non-zero in the Wakimoto vertex operator algebra.

The Wakimoto free field realisation endows the tensor product of a Heisenberg Fock space and a ghost module with the structure of an $\widehat{\mathfrak{sl}}(2)$ -module, by restriction. We refer to such modules as Wakimoto modules, distinguishing at least four types:

- The highest weight Wakimoto modules $\mathcal{W}_p = \mathcal{F}_p \otimes \mathcal{G}$, recalling that \mathcal{G} denotes the ghost vacuum module.
- The conjugate highest weight Wakimoto modules $c(\mathcal{W}_p) = \mathcal{F}_p \otimes c(\mathcal{G})$, obtained by conjugating the ghost module.
- The relaxed highest weight Wakimoto modules $\mathcal{W}_{p;q} = \mathcal{F}_p \otimes \mathcal{G}_q$, for $q \notin \mathbb{Z}$.
- The relaxed highest weight Wakimoto modules $\mathcal{W}_{p;0}^\pm = \mathcal{F}_p \otimes \mathcal{G}_0^\pm$.

The structure of the highest weight Wakimoto modules \mathcal{W}_p was determined in [4]. To the best of our knowledge, the relaxed modules $\mathcal{W}_{p;q}$ have not previously been considered in the literature.

Consider now the tensor product of a Heisenberg highest weight vector u_p and the ghost vacuum v , denoting it by $|p\rangle = u_p \otimes v$ for convenience. The free field realisations (3.1) of the $\widehat{\mathfrak{sl}}(2)$ fields imply that $|p\rangle$ is an $\widehat{\mathfrak{sl}}(2)$ highest weight vector of $\mathfrak{sl}(2)$ - and conformal weight

$$\lambda_p = \alpha p, \quad \Delta_p = \frac{1}{2}p \left(p + \frac{2}{\alpha} \right) = \frac{\lambda_p(\lambda_p + 2)}{4t}, \quad (3.3)$$

where we recall that $\alpha^2 = 2t$. Similarly, the tensor product $|p;q\rangle = u_p \otimes v_q$ of u_p with a relaxed highest weight vector v_q for the ghosts is an $\widehat{\mathfrak{sl}}(2)$ relaxed highest weight vector of $\mathfrak{sl}(2)$ - and conformal weight

$$\lambda_{p;q} = \alpha p + 2q, \quad \Delta_{p;q} = \frac{1}{2}p \left(p + \frac{2}{\alpha} \right) = \frac{(\lambda_{p;q} - 2q)(\lambda_{p;q} - 2q + 2)}{4t}. \quad (3.4)$$

This shows that all the simple $\widehat{\mathfrak{sl}}(2)$ -modules, highest weight and relaxed, may be realised as subquotients of Wakimoto modules.

3.2. Screening fields and operators. We begin by recalling the construction of vertex operators for the free boson. For this, one extends the Heisenberg algebra by introducing a generator \hat{a} satisfying

$$[a_m, \hat{a}] = \delta_{m,0} \mathbf{1}. \quad (3.5)$$

The vertex operators $\mathcal{Y}_p(z)$, parametrised by $p \in \mathbb{C}$, are then defined by

$$\mathcal{Y}_p(z) = e^{p\hat{a}} z^{pa_0} \prod_{m \geq 1} \exp \left(p \frac{\alpha_{-m}}{m} z^m \right) \exp \left(-p \frac{\alpha_m}{m} z^{-m} \right). \quad (3.6)$$

A standard computation shows that the vertex operators are free boson primaries of Heisenberg weight p and conformal weight $\frac{1}{2}p \left(p + \frac{2}{\alpha} \right)$, by virtue of the operator product expansions

$$a(z)\mathcal{Y}_p(w) \sim \frac{p\mathcal{Y}_p(w)}{z-w}, \quad T^{\text{bos.}}(z)\mathcal{Y}_p(w) \sim \frac{\frac{1}{2}p \left(p + \frac{2}{\alpha} \right) \mathcal{Y}_p(w)}{(z-w)^2} + \frac{\partial \mathcal{Y}_p(w)}{z-w}. \quad (3.7)$$

For later use, we record that the composition of k vertex operators is given by

$$\mathcal{Y}_{p_1}(z_1) \cdots \mathcal{Y}_{p_k}(z_k) = \prod_{i < j} (z_i - z_j)^{p_i p_j} \cdot e^{\sum_{i=1}^k p_i \hat{a}} \prod_{i=1}^k z_i^{p_i a_0} \cdot \prod_{m \geq 1} \exp \left(\frac{a-m}{m} \sum_{i=1}^k p_i z_i^m \right) \exp \left(-\frac{a_m}{m} \sum_{i=1}^k p_i z_i^{-m} \right). \quad (3.8)$$

We now define the notion of a screening field.

Definition. A *screening field* $\mathcal{Q}(w)$ for a vertex operator algebra \mathcal{V} is a field, generally not belonging to \mathcal{V} itself, which has the property that the singular terms of the operator product expansions of each of the generating fields of \mathcal{V} with $\mathcal{Q}(w)$ are total derivatives in w .

Our definition of a *screening operator*, in a moral sense at least, is then a vertex operator algebra module homomorphism that can be constructed from screening fields. This means that a screening operator commutes with the fields, and hence the mode algebra, of the vertex operator algebra. The standard way of constructing screening operators is as the residues (zero modes) of screening fields; these are guaranteed to commute with all the vertex operator algebra fields, provided that the residues are well defined. Their chief application stems from the fact that they map (relaxed) highest weight vectors to (relaxed) highest weight vectors and thereby explicitly construct (relaxed) singular vectors. We remark that if the screening field is a Virasoro primary (excluding the identity field), then its operator product expansion with the energy-momentum tensor forces its conformal weight to be 1.

The operator product expansions of the free field $\widehat{\mathfrak{sl}}(2)$ fields (3.1) with a free boson vertex operator are easily computed to be

$$e(z)\mathcal{Y}_p(w) \sim 0, \quad h(z)\mathcal{Y}_p(w) \sim \frac{\alpha p \mathcal{Y}_p(w)}{z-w}, \quad f(z)\mathcal{Y}_p(w) \sim \frac{\alpha p \mathcal{Y}_p(w) \gamma(w)}{z-w}. \quad (3.9)$$

For a vertex operator to be a non-trivial screening field, its Heisenberg weight would have to satisfy $\frac{1}{2}p(p + \frac{2}{\alpha}) = 1$. However, the above operator product expansions show that these vertex operators are not screening fields for $\widehat{\mathfrak{sl}}(2)$. However, the field $\mathcal{Q}(z) = \mathcal{Y}_{-2/\alpha}(z)\beta(z)$ is a screening field [4]:

$$e(z)\mathcal{Q}(w) \sim 0, \quad h(z)\mathcal{Q}(w) \sim 0, \quad f(z)\mathcal{Q}(w) \sim -t \partial_w \frac{V_{-2/\alpha}(w)}{z-w}. \quad (3.10)$$

It follows that the zero mode

$$\mathcal{Q}^{[1]} = \oint_0 \mathcal{Q}(z) \frac{dz}{2\pi i} = \oint_0 \mathcal{Y}_{-2/\alpha}(z)\beta(z) \frac{dz}{2\pi i} \quad (3.11)$$

is a screening operator, whenever the contour around 0 actually closes. This will be the case when $\mathcal{Q}^{[1]}$ acts on a state for which the relevant Fourier expansion of $\mathcal{Q}(z)$ has only integer powers of z . Equivalently, $\mathcal{Q}^{[1]}$ has a well defined action on a given state if and only if the operator product expansion of $\mathcal{Q}(z)$ with the corresponding field is a Laurent series. For example, $\mathcal{Q}^{[1]}$ only acts on $|p\rangle = u_p \otimes v$, the tensor product of a Heisenberg vacuum u_p with the ghost vacuum v , when $p = \frac{1}{2}m\alpha$, $m \in \mathbb{Z}$, because

$$\mathcal{Q}(z)\mathcal{Y}_p(w) = \mathcal{Y}_{-2/\alpha}(z)\mathcal{Y}_p(w)\beta(z) = \frac{\mathcal{Y}_{p-2/\alpha}(w)\beta(w)}{(z-w)^{2p/\alpha}} + \cdots. \quad (3.12)$$

This shows that the screening field $\mathcal{Q}(z)$ only defines module homomorphisms, hence constructs singular vectors, for certain vertex operator algebra modules.

To construct more module homomorphisms, it is natural to consider products of screening fields. To check that such products also yield screening operators, it is convenient to use the language of differential forms. Suppose then that $j(z)$ is a vertex operator algebra field and that $\mathcal{Q}_i(w_i)$ is a screening field, so that

$$[j_n, \mathcal{Q}_i(w_i) dw_i] = \partial_{w_i} \mathcal{P}_i(w_i) dw_i = d\mathcal{P}_i(w_i), \quad (3.13)$$

for some (n -dependent) $\mathcal{P}_i(w_i)$. Then,

$$\begin{aligned} [j_n, \mathcal{Q}_1(w_1) dw_1 \wedge \mathcal{Q}_2(w_2) dw_2] &= d\mathcal{P}_1(w_1) \wedge \mathcal{Q}_2(w_2) dw_2 + \mathcal{Q}_1(w_1) dw_1 \wedge d\mathcal{P}_2(w_2) \\ &= d(\mathcal{P}_1(w_1) \mathcal{Q}_2(w_2) dw_2 - \mathcal{Q}_1(w_1) dw_1 \mathcal{P}_2(w_2)) \end{aligned} \quad (3.14)$$

is exact, hence the commutator vanishes upon integrating over a closed cycle. The generalisation to more than two screening fields is immediate.

We therefore ask the question of when there exists a closed cycle over which some given product of the screening fields $\mathcal{Q}(z) = \mathcal{V}_{-2/\alpha}(z)\beta(z)$ can be integrated to define $\widehat{\mathfrak{sl}}(2)$ -homomorphisms. We introduce the shorthand

$$\mathcal{Q}^{[r]}(z) = \mathcal{Q}^{[r]}(z_1, \dots, z_r) = \mathcal{Q}(z_1) \cdots \mathcal{Q}(z_r) \quad (3.15)$$

and consider the action of this product of screening fields on the Wakimoto module \mathcal{W}_p whose Heisenberg highest weight is p . Using (3.8), this action can be written in the form

$$\begin{aligned} \mathcal{Q}^{[r]}(z) \Big|_{\mathcal{W}_p} &= \prod_{1 \leq i < j \leq r} (z_i - z_j)^{4/\alpha^2} \cdot \prod_{i=1}^r z_i^{-2p/\alpha} \\ &\quad \cdot \prod_{m \geq 1} \exp\left(-\frac{2}{\alpha} \mathfrak{p}_m(z) \frac{a_{-m}}{m}\right) \exp\left(\frac{2}{\alpha} \overline{\mathfrak{p}_m(z)} \frac{a_m}{m}\right) \cdot \prod_{i=1}^r \beta(z_i) \Big|_{\mathcal{W}_{p-2r/\alpha}}, \end{aligned} \quad (3.16)$$

where $\mathfrak{p}_m(z)$ denotes a power sum and the overline $\overline{\mathfrak{p}_m(z)} = \mathfrak{p}_m(z_1^{-1}, \dots, z_r^{-1})$ denotes variable inversion (see Appendix A for our conventions for power sums and other symmetric polynomials). The action of $\mathcal{Q}^{[r]}(z)$ on the other Wakimoto modules with Heisenberg highest weight p , for example the relaxed highest weight module $\mathcal{W}_{p,q}$, is identical — the ghost weight q is not changed.

Up to an unimportant phase factor, which we suppress, the first two factors on the right-hand side of (3.16) are

$$\prod_{1 \leq i < j \leq r} (z_i - z_j)^{4/\alpha^2} \cdot \prod_{i=1}^r z_i^{-2p/\alpha} = \prod_{1 \leq i \neq j \leq r} \left(1 - \frac{z_i}{z_j}\right)^{1/t} \cdot \prod_{i=1}^r z_i^{(r-1)2/\alpha^2 - 2p/\alpha}, \quad (3.17)$$

where we recall that $\alpha^2 = 2t$. The second factor on the right-hand side of this expression therefore isolates all the (potential) non-integer powers of the z_i in (3.16), hence a closed cycle over which $\mathcal{Q}^{[r]}(z)$ may be integrated will exist precisely when the common exponent of the z_i in this factor is an integer, $s \in \mathbb{Z}$ say. This requires the Heisenberg weight p of the Wakimoto module \mathcal{W} to have the form

$$p_{r,s} = \frac{r-1}{\alpha} - \frac{s\alpha}{2}, \quad r \in \mathbb{Z}_{\geq 1}, \quad s \in \mathbb{Z}. \quad (3.18)$$

We remark that the multivalued function

$$G_r(z; t) = \prod_{1 \leq i \neq j \leq r} \left(1 - \frac{z_i}{z_j}\right)^{1/t}, \quad (3.19)$$

appearing on the right-hand side of (3.17), is just the integration kernel of the inner product (A.18) with respect to which the Jack symmetric polynomials are orthogonal. Setting $p = p_{r,s}$, (3.16) now takes the form

$$\mathcal{Q}^{[r]}(z) \Big|_{\mathcal{W}_{p_{r,s}}} = G_r(z; t) \prod_{i=1}^r z_i^s \cdot \prod_{m \geq 1} \exp\left(-\frac{2}{\alpha} \mathfrak{p}_m(z) \frac{a_{-m}}{m}\right) \exp\left(\frac{2}{\alpha} \overline{\mathfrak{p}_m(z)} \frac{a_m}{m}\right) \cdot \prod_{i=1}^r \beta(z_i) \Big|_{\mathcal{W}_{p-r,s}}. \quad (3.20)$$

Again, the action on $\mathcal{W}_{p_{r,s};q}$ is identical except that the right-hand side now acts on $\mathcal{W}_{p-r,s;q}$.

3.3. Singular vectors. The existence of cycles over which the product $\mathcal{Q}^{[r]}(z)$ of screening fields may be integrated follows from the same arguments used in the analogous question for the free field realisation of the universal Virasoro vertex operator algebra, because the multivalued function $G_r(z; t)$ is the same in both cases. This question was answered for the Virasoro case (see Theorem A.4) by Tsuchiya and Kanie [26] to whom we refer for further details. We will use the cycles $[\Delta_r]$ that they construct in what follows, but normalised so that

$$\int_{[\Delta_r]} G_r(z; t) \frac{dz_1 \cdots dz_r}{z_1 \cdots z_r} = 1. \quad (3.21)$$

We mention that there are various explicit constructions of cycles, over which screening operators can be integrated, in the conformal field theory literature, but that the symmetric function literature uses different constructions again. However, Cohen and Varchenko [51] showed that, up to normalisation, there is only one non-trivial

homology class of cycles when the integrand is $G_r(z;t)$ times a symmetric function, bar some restrictions on t , and thus the various constructions in the literature are all essentially equivalent.⁵

The normalised cycles $[\Delta_r]$ let us construct $\widehat{\mathfrak{sl}}(2)$ -homomorphisms (screening operators) from the products $\mathcal{Q}^{[r]}(z)$ of screening fields. Our aim is to use these homomorphisms to explicitly construct singular vectors in (relaxed) highest weight modules over $\widehat{\mathfrak{sl}}(2)$ using symmetric function technology (we refer to Appendix A for a primer on what is needed). Of course, we need to verify that the screening operators are not just zero maps.

To this purpose, we introduce an algebra isomorphism ρ_δ , for each $\delta \in \mathbb{C}^*$, from the algebra Λ of symmetric polynomials in infinitely many variables to the universal enveloping algebra $\mathfrak{U}\mathfrak{h}_-$ of the negative subalgebra of the Heisenberg algebra. This isomorphism is given, on the power sum generators, by

$$\rho_\delta(p_m(y)) = \delta a_{-m}. \quad (3.22)$$

For convenience, we will always take ρ_δ to act only on symmetric polynomials of the variables y_i . For the ghosts, we analogously define injective linear maps σ_r and $\tilde{\sigma}_r$ from the algebra Λ_r of symmetric polynomials in r variables to the ghost universal enveloping algebra $\mathfrak{U}\mathfrak{G}$. These maps are defined on the basis $\{g_v^t : \ell(v) \leq r\}$ dual to the symmetric monomials m_v by

$$\sigma_r(g_v^t(x)) = \beta_{-v_1-1} \cdots \beta_{-v_r-1}, \quad \tilde{\sigma}_r(g_v^t(x)) = \beta_{-v_1} \cdots \beta_{-v_r}. \quad (3.23)$$

We will only take these maps to act on symmetric polynomials in the x_i . Here, the partition v may have length $\ell(v)$ strictly less than r ; in this case, we pad the partition with zeroes so that the images $\sigma_r(g_v^t)$ and $\tilde{\sigma}_r(g_v^t)$ are padded by β_{-1} or β_0 , respectively, so that the right hand sides of (3.23) each consist of r factors.

With these maps, we can prove that the $\mathcal{Q}^{[r]}(z)$ yield non-trivial screening operators and derive explicit formulae for certain (relaxed) singular vectors of the (relaxed) Verma modules over $\widehat{\mathfrak{sl}}(2)$.

Theorem 3.1. *Let $r \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{Z}$ and $t \in \mathbb{C}^*$ and suppose that $d(d+1)/t \notin \mathbb{Z}$ and $d(r-d)/t \notin \mathbb{Z}$, for all integers d satisfying $1 \leq d \leq r-1$. Then,*

$$\mathcal{Q}^{[r]} = \int_{[\Delta_r]} \mathcal{Q}^{[r]}(z_1, \dots, z_r) dz_1 \cdots dz_r \quad (3.24)$$

defines non-trivial $\widehat{\mathfrak{sl}}(2)$ -module homomorphisms between (relaxed) Wakimoto modules:

$$\mathcal{Q}^{[r]}: \mathcal{W}_{p_{r,s}} \rightarrow \mathcal{W}_{p_{-r,s}}, \quad \mathcal{Q}^{[r]}: \mathcal{W}_{p_{r,s};q} \rightarrow \mathcal{W}_{p_{-r,s};q}. \quad (3.25)$$

In particular, if $|p_{r,s}\rangle = u_{p_{r,s}} \otimes v$ and $|p_{r,s};q\rangle = u_{p_{r,s}} \otimes v_q$, where u_p denotes the Heisenberg highest weight vector of weight p , v denotes the ghost vacuum and v_q denotes a ghost relaxed highest weight vector of weight $q \notin \mathbb{Z}$, then

$$\mathcal{Q}^{[r]}|p_{r,s}\rangle = \begin{cases} (\rho_{-\alpha} \circ \sigma_r) \left(Q_{[-s-1]^r}^t(x,y) \right) |p_{-r,s}\rangle \neq 0 & \text{if } s \leq -1, \\ 0 & \text{if } s \geq 0, \end{cases} \quad (3.26a)$$

$$\mathcal{Q}^{[r]}|p_{r,s};q\rangle = \begin{cases} (\rho_{-\alpha} \circ \tilde{\sigma}_r) \left(Q_{[-s]^r}^t(x,y) \right) |p_{-r,s};q\rangle \neq 0 & \text{if } s \leq 0, \\ 0 & \text{if } s \geq 1, \end{cases} \quad (3.26b)$$

where Q_v^t denotes the symmetric polynomials dual to the Jack polynomials (see Appendix A). The $\mathcal{Q}^{[r]}$ -images of $|p_{r,s}\rangle$ ($|p_{r,s};q\rangle$) in $\mathcal{W}_{p_{-r,s}}$ ($\mathcal{W}_{p_{-r,s};q}$) are therefore (relaxed) singular vectors, for $s \leq -1$ ($s \leq 0$).

Proof. We first show the non-triviality of $\mathcal{Q}^{[r]}$. Let $\langle p_{-r,s}|$ and $\langle p_{-r,s};q|$ denote the functionals dual to $|p_{-r,s}\rangle$ and $|p_{-r,s};q\rangle$, respectively. Then, it follows from the ghost commutation relations (2.11) and adjoint (2.13) that

$$\langle v_q | \prod_{i=1}^r \beta(z_i) \gamma_{-s} v_q \rangle = (-1)^r r! \langle v_q | v_q \rangle \prod_{i=1}^r z_i^{-s-1}, \quad s \geq 1, q \in \mathbb{C}. \quad (3.27a)$$

⁵It is because of these minor restrictions on t that we state our choice of class of cycle explicitly. These restrictions are only relevant for the precise statement of Theorem 3.1. A more detailed understanding of these cycles is not required for reading the remainder of this article.

Replacing the relaxed highest weight vector v_q by the ghost vacuum v gives the same result, but for $s \geq 0$ (because β_0 annihilates v):

$$\langle v | \prod_{i=1}^r \beta(z_i) \gamma_{-s}^r v \rangle = (-1)^r r! \langle v | v \rangle \prod_{i=1}^r z_i^{-s-1}, \quad s \geq 0. \quad (3.27b)$$

The non-triviality of $\mathcal{Q}^{[r]}$ on $\mathcal{W}_{p_{r,s}}$, for $s \geq 0$, now follows from (3.20) and the normalisation of $[\Delta_r]$:

$$\begin{aligned} \langle p_{-r,s} | \mathcal{Q}^{[r]} \gamma_{-s}^r | p_{r,s} \rangle &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^s \cdot \langle p_{-r,s} | \prod_{i=1}^r \beta(z_i) \gamma_{-s}^r | p_{-r,s} \rangle dz_1 \cdots dz_r \\ &= (-1)^r r! \langle p_{-r,s} | p_{-r,s} \rangle \int_{[\Delta_r]} G_r(z;t) \frac{dz_1 \cdots dz_r}{z_1 \cdots z_r} \\ &= (-1)^r r! \langle p_{-r,s} | p_{-r,s} \rangle \neq 0. \end{aligned} \quad (3.28a)$$

Here, we note that the a_{-m} and a_m , with $m \geq 1$, appearing in (3.20) annihilate $\langle p_{-r,s} |$ and $| p_{r,s} \rangle$, respectively. A similar calculation gives the relaxed version for $s \geq 1$:

$$\langle p_{-r,s}; q | \mathcal{Q}^{[r]} \gamma_{-s}^r | p_{r,s}; q \rangle = (-1)^r r! \langle p_{-r,s}; q | p_{-r,s}; q \rangle \neq 0. \quad (3.28b)$$

We note that the conformal weight of $| p_{-r,s} \rangle$ ($\langle p_{-r,s}; q |$) is greater than that of $| p_{r,s} \rangle$ ($| p_{r,s}; q \rangle$) by rs . It follows that $\mathcal{Q}^{[r]} | p_{r,s} \rangle = 0$ ($\mathcal{Q}^{[r]} | p_{r,s}; q \rangle = 0$) for all $s \geq 1$ because $\mathcal{Q}^{[r]}$ is an $\widehat{\mathfrak{sl}}(2)$ -module homomorphism and hence it preserves conformal weights.

We settle the non-triviality of $\mathcal{Q}^{[r]}$ for the remaining values of s by explicitly computing the image of the (relaxed) highest weight vectors $| p_{r,s} \rangle$ and $| p_{r,s}; q \rangle$. In the former case, we obtain

$$\begin{aligned} \mathcal{Q}^{[r]} | p_{r,s} \rangle &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^s \cdot \prod_{m \geq 1} \exp\left(\frac{-2}{\alpha} \mathfrak{p}_m(z) \frac{a_{-m}}{m}\right) \cdot \prod_{i=1}^r \beta(z_i) | p_{-r,s} \rangle dz_1 \cdots dz_r \\ &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^s \cdot \prod_{m \geq 1} \exp\left(\frac{-2}{\alpha} \frac{\mathfrak{p}_m(z) a_{-m}}{m}\right) \sum_{\mathbf{v}: \ell(\mathbf{v}) \leq r} \beta_{-v_1-1} \cdots \beta_{-v_r-1} \mathfrak{m}_{\mathbf{v}}(z) | p_{-r,s} \rangle dz_1 \cdots dz_r \\ &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^s \cdot \rho_{-\alpha} \left(\prod_{m \geq 1} \exp\left(\frac{1}{t} \frac{\mathfrak{p}_m(y) \mathfrak{p}_m(z)}{m}\right) \right) \cdot \sigma_r \left(\sum_{\mathbf{v}: \ell(\mathbf{v}) \leq r} \mathfrak{g}_{\mathbf{v}}^t(x) \mathfrak{m}_{\mathbf{v}}(z) \right) | p_{-r,s} \rangle dz_1 \cdots dz_r \\ &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^{s+1} \cdot (\rho_{-\alpha} \circ \sigma_r) \left(\prod_{m \geq 1} \exp\left(\frac{1}{t} \frac{(\mathfrak{p}_m(x) + \mathfrak{p}_m(y)) \mathfrak{p}_m(z)}{m}\right) \right) | p_{-r,s} \rangle \frac{dz_1 \cdots dz_r}{z_1 \cdots z_r} \end{aligned}$$

(if $s = 0$, this vanishes, in agreement with (3.26a), as the powers of the z_i in the integrand are all positive)

$$\begin{aligned} &= \sum_{\mathbf{v}: \ell(\mathbf{v}) \leq r} \left\langle \mathbf{P}_{[(-1-s)r]}^t(z), \mathbf{P}_{\mathbf{v}}^t(z) \right\rangle_t^r (\rho_{-\alpha} \circ \sigma_r) (\mathbf{Q}_{\mathbf{v}}^t(x, y)) | p_{-r,s} \rangle \\ &= (\rho_{-\alpha} \circ \sigma_r) \left(\mathbf{Q}_{[(-1-s)r]}^t(x, y) \right) | p_{-r,s} \rangle. \end{aligned} \quad (3.29a)$$

Here, we have used the finite-variable version of (A.16) to rewrite the sum over the symmetric monomials $\mathfrak{m}_{\mathbf{v}}$ and their duals $\mathfrak{g}_{\mathbf{v}}^t$ as a product of exponentials of power sums. Then, we note that $\mathfrak{p}_m(x) + \mathfrak{p}_m(y)$ is the power sum $\mathfrak{p}_m(x, y)$ in both x_i and y_j variables and use (A.16) again to expand the product of exponentials in terms of Jack polynomials and their duals. Finally, we use Proposition A.5 to identify $\prod_i z_i^{-1-s}$ as a Jack polynomial in the z_i^{-1} , then apply the orthogonality of Jack polynomials, and lastly note that the Jack polynomial norm follows from the normalisation of $[\Delta_r]$. For $s \leq -1$, the result is non-vanishing since $\rho_{-\alpha}$ and σ_r are injective. This proves the non-triviality of $\mathcal{Q}^{[r]}$ on $\mathcal{W}_{p_{r,s}}$ for all $s \in \mathbb{Z}$, as well as (3.26a).

A similar computation in the relaxed sector results in

$$\begin{aligned} \mathcal{Q}^{[r]} | p_{r,s}; q \rangle &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^s \cdot \prod_{m \geq 1} \exp\left(\frac{-2}{\alpha} \frac{\mathfrak{p}_m(z) a_{-m}}{m}\right) \cdot \prod_{i=1}^r \beta(z_i) | p_{-r,s}; q \rangle dz_1 \cdots dz_r \\ &= \int_{[\Delta_r]} G_r(z;t) \prod_{i=1}^r z_i^s \cdot \prod_{m \geq 1} \exp\left(\frac{-2}{\alpha} \frac{\mathfrak{p}_m(z) a_{-m}}{m}\right) \cdot \sum_{\mathbf{v}: \ell(\mathbf{v}) \leq r} \beta_{-v_1} \cdots \beta_{-v_r} \mathfrak{m}_{\mathbf{v}}(z) | p_{-r,s}; q \rangle \frac{dz_1 \cdots dz_r}{z_1 \cdots z_r} \end{aligned}$$

$$= (\rho_{-\alpha} \circ \tilde{\sigma}_r) \left(Q_{[-s^r]}^t(x, y) \right) |p_{-r, s}; q\rangle, \quad (3.29b)$$

which is likewise non-vanishing for $s \leq 0$ as $\rho_{-\alpha}$ and $\tilde{\sigma}_r$ are injective. This proves (3.26b) and the non-triviality of $\mathcal{Q}^{[r]}$ on $\mathcal{W}_{p_{r, s}; q}$ for all $s \in \mathbb{Z}$. ■

Suppose now that the $\widehat{\mathfrak{sl}}(2)$ level k is admissible: $t = k + 2 = \frac{u}{v}$, where $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Z}_{\geq 1}$ are coprime. The $\widehat{\mathfrak{sl}}(2)$ vacuum may be identified with the Wakimoto highest weight vector $|0\rangle = |p_{1, 0}\rangle = |p_{-u+1, -v}\rangle \in \mathcal{W}_{p_{-u+1, -v}}$ (we note the symmetry $p_{r, s} = p_{r+u, s+v}$). Theorem 3.1 then guarantees that the $\widehat{\mathfrak{sl}}(2)$ -module homomorphism $\mathcal{Q}^{[u-1]}$ acts non-trivially on $|p_{u-1, -v}\rangle$ to give a non-trivial singular vector in the vacuum module. The corresponding field then generates the non-trivial proper ideal of the universal vertex operator algebra, proving the following result:

Corollary 3.2. *The universal vertex operator algebra \mathcal{V}_k of $\widehat{\mathfrak{sl}}(2)$ at non-critical level $k \neq -2$ may be realised as a subalgebra of the Wakimoto vertex operator algebra.*

This is, of course, obvious if k is not admissible. What it means in the admissible level case is that calculations requiring the singular vector of the vacuum module of $\widehat{\mathfrak{sl}}(2)$ may be equivalently carried out in the free field realisation using Jack symmetric polynomials.

4. CLASSIFYING ADMISSIBLE MODULES

In this section, we specialise to admissible levels $k = t - 2$, where $t = \frac{u}{v}$ and $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Z}_{\geq 1}$ are coprime. Then, the universal vertex operator algebra \mathcal{V}_k of $\widehat{\mathfrak{sl}}(2)$ is not simple and the unique maximal proper ideal is generated by the field that corresponds to the singular vector of the vacuum module (see Section 2.2). As we saw in Theorem 3.1 and Corollary 3.2, the vacuum module may be constructed as a submodule of the Wakimoto vacuum module \mathcal{W}_0 and its singular vector is then explicitly given by $\mathcal{Q}^{[u-1]}|p_{u-1, -v}\rangle$.

Quotienting by this maximal proper ideal, that is, setting the singular vector $\mathcal{Q}^{[u-1]}|p_{u-1, -v}\rangle$ to zero, amounts to replacing the universal vertex operator algebra \mathcal{V}_k by its simple counterpart $\mathcal{A}_1(u, v)$. Our aim in this section is to use the explicit expression for the vacuum singular vector to classify the possible (relaxed) highest weights and thereby determine the spectrum of $\mathcal{A}_1(u, v)$ -modules. By Corollary 3.2, these calculations may be performed in Wakimoto's free field realisation. More specifically, we will use symmetric polynomial technology to compute a generator of the annihilating ideal in Zhu's algebra. For readers unfamiliar with Zhu's algebra, we refer to Appendix B for motivation, basic definitions and a very short primer.

An old result of Frenkel and Zhu [52] states that Zhu's algebra $\text{Zhu}[\mathcal{V}_k]$ for \mathcal{V}_k is nothing but the universal enveloping algebra $\mathcal{U}\mathfrak{sl}(2)$ of (non-affine) $\mathfrak{sl}(2)$ (see Proposition B.3). By Proposition B.2, Zhu's algebra $\text{Zhu}[\mathcal{A}_1(u, v)]$ for the quotient $\mathcal{A}_1(u, v)$ is then the quotient of $\mathcal{U}\mathfrak{sl}(2)$ by the annihilating ideal generated by the representative of the singular vector $\mathcal{Q}^{[u-1]}|p_{u-1, -v}\rangle$. Since Zhu's algebra is filtered by conformal weight, whereas the conformal weight of $e(z)$, $h(z)$ and $f(z)$ is 1 and the conformal weight of the singular vector is $(u-1)v$, it follows that the image of the singular vector in $\text{Zhu}[\mathcal{V}_k]$ is a polynomial in the $\mathfrak{sl}(2)$ generators of total degree at most $(u-1)v$. Furthermore, as Zhu's algebra is just the algebra of zero modes acting on relaxed highest weight vectors, the polynomial corresponding to the singular vector can be determined by evaluating the zero mode of the singular vector on general relaxed highest weight vectors, as in [53], since this is equivalent to evaluating the polynomial at infinitely many points.

However, the $\mathfrak{sl}(2)$ -weight of the vacuum singular vector $\mathcal{Q}^{[u-1]}|p_{u-1, -v}\rangle$ is $\lambda_{u-1, -v} = \lambda_{p_{u-1, -v}} = 2(u-1)$, which means that the corresponding field and its zero mode shifts the $\mathfrak{sl}(2)$ -weight of any relaxed highest weight vector upon which it acts by this amount. It is far more convenient to work with a field that does not shift $\mathfrak{sl}(2)$ -weights and so we instead consider the field corresponding to the $\mathfrak{sl}(2)$ -weight 0 vector

$$\begin{aligned} f_0^{u-1} \mathcal{Q}^{[u-1]}|p_{u-1, -v}\rangle &= \mathcal{Q}^{[u-1]} f_0^{u-1}|p_{u-1, -v}\rangle \\ &= \lambda_{u-1, -v} (\lambda_{u-1, -v} - 1) \cdots (\lambda_{u-1, -v} - u + 2) \mathcal{Q}^{[u-1]} \gamma_0^{u-1}|p_{u-1, -v}\rangle. \end{aligned} \quad (4.1)$$

Since $u \geq 2$, we may renormalise this vector by dividing by the non-zero $\lambda_{u-1,-v}$ -dependent factors on the right-hand side. The field corresponding to this renormalised vector is then

$$\begin{aligned} \chi(w) &= \int_{[\Delta_r]} \mathcal{Q}^{[u-1]}(z_1 + w, \dots, z_{u-1} + w) \mathcal{Y}_{p_{u-1,-v}}(w) \gamma(w)^{u-1} dz_1 \cdots dz_{u-1} \\ &= \int_{[\Delta_r]} \mathcal{Y}_{-2/\alpha}(z_1 + w) \cdots \mathcal{Y}_{-2/\alpha}(z_{u-1} + w) \mathcal{Y}_{p_{u-1,-v}}(w) \beta(z_1 + w) \cdots \beta(z_{u-1} + w) \gamma(w)^{u-1} dz_1 \cdots dz_{u-1}. \end{aligned} \quad (4.2)$$

We note that e_0^{u-1} acting on the vector $f_0^{u-1} \mathcal{Q}^{[u-1]}|p_{u-1,-v}\rangle$ gives a non-zero multiple of the singular vector $\mathcal{Q}^{[u-1]}|p_{u-1,-v}\rangle$. It follows that the annihilating ideal that we obtain from $\chi(w)$ will be the same as that which we would have obtained if we had instead worked with the singular vector directly.

4.1. Admissible highest weight modules. In this section, we will evaluate the action of the zero mode of $\chi(w)$ on a general highest weight vector before moving on to the evaluation of $\chi(w)$ on relaxed highest weight vectors. This highest weight computation will not yet yield sufficient information to determine the image of $\chi(w)$ in Zhu's algebra $\text{Zhu}[\mathbb{V}_k]$ as a polynomial in the $\mathfrak{sl}(2)$ generators e, h and f , but it will constrain the weights of admissible highest weight vectors to a finite set.

Every highest weight vector of \mathbb{V}_k may be realised as a highest weight vector of some Wakimoto module \mathcal{W}_p . It follows that the Heisenberg weights p of the admissible highest weight vectors of $\widehat{\mathfrak{sl}}(2)$, that is, the highest weight vectors of the $A_1(u, v)$ -modules, are zeroes of

$$\begin{aligned} \langle p | \chi(w) | p \rangle &= \int_{[\Delta_r]} \langle u_p, \mathcal{Y}_{-2/\alpha}(z_1 + w) \cdots \mathcal{Y}_{-2/\alpha}(z_{u-1} + w) \mathcal{Y}_{p_{u-1,-v}}(w) u_p \rangle \\ &\quad \cdot \langle v, \beta(z_1 + w) \cdots \beta(z_{u-1} + w) \gamma(w)^{u-1} v \rangle dz_1 \cdots dz_{u-1}. \end{aligned} \quad (4.3)$$

The ghost contribution is easily evaluated by computing the operator product expansion of the ghost fields. Since v is the ghost vacuum, only the fully contracted part of the operator product expansion contributes:

$$\begin{aligned} \langle v, \beta(z_1 + w) \cdots \beta(z_{u-1} + w) \gamma(w)^{u-1} v \rangle &= \langle v, : \beta(z_1 + w) \cdots \beta(z_{u-1} + w) : : \gamma(w) \cdots \gamma(w) : v \rangle \\ &= \frac{(-1)^{u-1} (u-1)!}{z_1 \cdots z_{u-1}} \langle v, v \rangle. \end{aligned} \quad (4.4)$$

The contribution from the free boson part of the Wakimoto realisation is likewise easily determined. Up to non-zero constant factors, which obviously do not affect the zeroes of $\langle p | \chi(w) | p \rangle$, this contribution is

$$\begin{aligned} &\langle u_p, \mathcal{Y}_{-2/\alpha}(z_1 + w) \cdots \mathcal{Y}_{-2/\alpha}(z_{u-1} + w) \mathcal{Y}_{2(u-1)/\alpha}(w) u_p \rangle \\ &= \prod_{1 \leq i \neq j \leq u-1} (z_i - z_j)^{1/t} \cdot \prod_{i=1}^{u-1} (z_i + w)^{-2p/\alpha} z_i^{-2(u-1)/t} \cdot w^{2(u-1)p/\alpha} \langle u_p, u_p \rangle \\ &= G_{u-1}(z; t) \prod_{i=1}^{u-1} z_i^{-v} \cdot \prod_{i=1}^{u-1} \left(1 + \frac{z_i}{w}\right)^{-2p/\alpha} \langle u_p, u_p \rangle, \end{aligned} \quad (4.5)$$

where we have used (3.8), noting that all the terms involving the a_n with $n \neq 0$ either annihilate the Heisenberg vacuum u_p or its dual, and recognised the kernel of the symmetric polynomial inner product from (3.19).

Putting these contributions together, we find that the admissible Heisenberg weights of an $\widehat{\mathfrak{sl}}(2)$ highest weight vector (in the Wakimoto free field realisation) must satisfy

$$0 = \langle p | \chi(w) | p \rangle = \int_{[\Delta_r]} G_{u-1}(z; t) \overline{P_{[v^{u-1}]}^t(z)} \prod_{i=1}^{u-1} \left(1 + \frac{z_i}{w}\right)^{-2p/\alpha} \frac{dz_1 \cdots dz_{u-1}}{z_1 \cdots z_{u-1}} \langle p | p \rangle, \quad (4.6)$$

where we have recognised the product of the z_i as a Jack polynomial using Proposition A.5 (recall that the overline indicates a symmetric polynomial in the inverse variables z_i^{-1}). We have also, again, neglected an overall non-zero constant factor. This expression has the form of an inner product for symmetric polynomials; to evaluate it, we only need to decompose the product over i into Jack polynomials. For this, we use specialisation in the form given

in (A.28) with $x_i = -z_i/w$ for $i \leq u-1$, $x_i = 0$ for $i > u-1$, and $X = \alpha p$:

$$\prod_{i=1}^{u-1} \left(1 + \frac{z_i}{w}\right)^{-2p/\alpha} = \sum_{\tau} P_{\tau}^t(-z_1/w, \dots, -z_{u-1}/w, 0, 0, \dots) \Xi_{\alpha p}(Q_{\tau}^t(y)) = \sum_{\tau} \frac{(-1)^{|\tau|}}{w^{|\tau|}} P_{\tau}^t(z) \Xi_{\alpha p}(Q_{\tau}^t(y)). \quad (4.7)$$

Here, we have also used the homogeneity of the Jack polynomials ($|\tau|$ is the sum of the parts of the partition τ).

Using the orthogonality of Jack polynomials, (4.6) now becomes

$$\begin{aligned} 0 &= \sum_{\tau} w^{-|\tau|} \left\langle P_{[v^{u-1}]}^t, P_{\tau}^t \right\rangle_t^{u-1} \Xi_{\alpha p}(Q_{\tau}^t(y)) = w^{-|[v^{u-1}]|} \left\langle P_{[v^{u-1}]}^t, P_{[v^{u-1}]}^t \right\rangle_t^{u-1} \Xi_{\alpha p}(Q_{[v^{u-1}]}^t(y)) \\ &= w^{-(u-1)v} \Xi_{\alpha p}(Q_{[v^{u-1}]}^t(y)) = w^{-(u-1)v} \prod_{b \in [v^{u-1}]} \frac{\alpha p + t a'(b) - l'(b)}{t(a(b) + 1) + l(b)} \\ &= w^{-(u-1)v} \prod_{i=1}^{u-1} \prod_{j=1}^v \frac{\alpha p + t(j-1) - (i-1)}{t(v-j+1) + u-1-i}, \end{aligned} \quad (4.8)$$

where the result of specialising the dual Jack polynomials Q_{τ}^t is given in Proposition A.5 (or (A.28)) and the arm and leg (co)lengths of (A.2) are easy to determine for the rectangular partition $[v^{u-1}]$. As the denominators of the factors appearing in this expression are all strictly positive, we arrive at the constraint

$$\prod_{i=1}^{u-1} \prod_{j=0}^{v-1} (\alpha p + t j - (i-1)) = \prod_{i=1}^{u-1} \prod_{j=0}^{v-1} (\alpha p - \alpha p_{i,j}) = 0. \quad (4.9)$$

This proves the following result:

Proposition 4.1. *Let $\lambda_{r,s} = \lambda_{p_{r,s}} = \alpha p_{r,s} = r-1-ts$. Then, every highest weight vector of an $A_1(u, v)$ -module has $\widehat{\mathfrak{sl}}(2)$ -weight of the form $\lambda_{r,s}$, where $r = 1, 2, \dots, u-1$ and $s = 0, 1, \dots, v-1$.*

Note that when $v = 1$, so that $k = t-2 = u-2 \in \mathbb{Z}_{\geq 0}$, the allowed $\widehat{\mathfrak{sl}}(2)$ -weights $\lambda_{r,s} = \lambda_{r,0} = r-1$ belong to the set $\{0, 1, \dots, k\}$. These are, of course, the highest weights of the integrable $\widehat{\mathfrak{sl}}(2)$ -modules that are well known to be the $A_1(k+2, 1)$ -modules that arise in the Wess-Zumino-Witten models defined on the Lie group $SU(2)$.

At this point, we cannot say whether all these highest weight vectors do actually appear in an $A_1(u, v)$ -module. For this, we need to work out the constraints on an arbitrary relaxed highest weight vector because it is these constraints which allow us to write down the generator of the annihilating ideal of Zhu's algebra. It is, however, well known that the Verma module $\mathcal{V}_{\lambda_{r,s}}$, with $r = 1, 2, \dots, u-1$ and $s = 0, 1, \dots, v-1$, has infinitely many linearly independent singular vectors; however, none have $\widehat{\mathfrak{sl}}(2)$ -weights belonging to the allowed set except the generator of $\widehat{\mathfrak{sl}}(2)$ -weight $\lambda_{r,s}$. It follows that there are only finitely many highest weight $A_1(u, v)$ -modules:

Corollary 4.2. *Every highest weight $A_1(u, v)$ -module is isomorphic to one of the simple $\widehat{\mathfrak{sl}}(2)$ -modules $\mathcal{L}_{\lambda_{r,0}}$ or $\mathcal{D}_{\lambda_{r,s}}$, where $r = 1, 2, \dots, u-1$ and $s = 1, 2, \dots, v-1$.*

Again, we have not yet proven that all these \mathcal{V}_k -modules are actually $A_1(u, v)$ -modules. However, it is germane to point out, at this point, that these highest weight modules are precisely the admissible modules first discussed by Kac and Wakimoto [2].

4.2. Admissible relaxed highest weight modules. We now turn to the more intricate, but ultimately more rewarding, analysis of the relaxed highest weight modules of $A_1(u, v)$. The goal is to determine the image of the field $\chi(w)$ in Zhu's algebra $\text{Zhu}[\mathcal{V}_k] \cong \mathfrak{U} \widehat{\mathfrak{sl}}(2)$ as a polynomial $I_{u,v}(e, h, f)$ in the $\widehat{\mathfrak{sl}}(2)$ generators. Here, we identify the $\widehat{\mathfrak{sl}}(2)$ generators e, h and f with the images of $e(z), h(z)$ and $f(z)$, respectively, in $\text{Zhu}[\mathcal{V}_k]$. Since Zhu's algebra is nothing but the algebra of zero modes acting on relaxed highest weight vectors (Appendix B), $I_{u,v}(e, h, f)$ is identical to the polynomial $I_{u,v}(e_0, h_0, f_0)$ that describes the action of the zero mode χ_0 on relaxed highest weight vectors. We remark that as the $\widehat{\mathfrak{sl}}(2)$ -weight of $\chi(w)$ is 0, the polynomial $I_{u,v}$ may be expressed as a polynomial in h_0 and the Virasoro zero mode L_0 .

So as in the previous section, we evaluate matrix elements containing the field $\chi(w)$, but this time the ‘‘bra’’ and the ‘‘ket’’ will be relaxed highest weight vectors from a general relaxed Wakimoto module:

$$\begin{aligned} \langle p; q | \chi(w) | p; q \rangle &= \int_{[\Delta_r]} \langle u_p, \mathcal{V}_{-2/\alpha}(z_1 + w) \cdots \mathcal{V}_{-2/\alpha}(z_{u-1} + w) \mathcal{V}_{p_{u-1}, -v}(w) u_p \rangle \\ &\quad \cdot \langle v_q, : \beta(z_1 + w) \cdots \beta(z_{u-1} + w) : : \gamma(w)^{u-1} : v_q \rangle dz_1 \cdots dz_{u-1}. \end{aligned} \quad (4.10)$$

We recall that v_q satisfies $J_0 v_q = \gamma_0 \beta_0 v_q = q v_q$.

The contribution from the free boson is exactly the same as in the non-relaxed case and was given in (4.5). The ghost contribution, however, requires more work. Wick's theorem lets us write this contribution in terms of contractions and normally-ordered products:

$$\begin{aligned} \langle v_q, : \beta(z_1 + w) \cdots \beta(z_{u-1} + w) : : \gamma(w)^{u-1} : v_q \rangle \\ = \sum_{I \subseteq \{1, \dots, u-1\}} \frac{(-1)^{|I|} (u-1)!}{(u-1-|I|)!} \prod_{i \in I} z_i^{-1} \cdot \langle v_q, : \prod_{i \notin I} \beta(z_i + w) \cdot \gamma(w)^{u-1-|I|} : v_q \rangle \end{aligned} \quad (4.11)$$

Here, each factor of $-z_i^{-1}$ is the contraction of $\beta(z_i + w)$ and $\gamma(w)$ and the factorials count how many such contractions are needed. As relaxed highest weight vectors are not necessarily annihilated by β_0 , the normally-ordered factor is quite non-trivial:

$$\begin{aligned} \langle v_q, : \prod_{i \notin I} \beta(z_i + w) \cdot \gamma(w)^{u-1-|I|} : v_q \rangle &= \langle v_q, \gamma_0^{u-1-|I|} \beta_0^{u-1-|I|} v_q \rangle \prod_{i \notin I} (z_i + w)^{-1} \\ &= (u-1-|I|)! \binom{u-2-|I|+q}{u-1-|I|} \prod_{i \notin I} (z_i + w)^{-1} \langle v_q, v_q \rangle. \end{aligned} \quad (4.12)$$

Up to an overall non-zero constant factor, the total ghost contribution is therefore

$$\begin{aligned} \sum_{I \subseteq \{1, \dots, u-1\}} (-1)^{|I|} \binom{u-2-|I|+q}{u-1-|I|} \prod_{i \in I} z_i^{-1} \cdot \prod_{i \notin I} (z_i + w)^{-1} \\ = \prod_{i=1}^{u-1} (z_i + w)^{-1} \cdot \sum_{n=0}^{u-1} (-1)^n \binom{u-2-n+q}{u-1-n} e_n \left(1 + \frac{w}{z_1}, \dots, 1 + \frac{w}{z_{u-1}} \right) \\ = \prod_{i=1}^{u-1} \left(1 + \frac{z_i}{w} \right)^{-1} \cdot \sum_{n=0}^{u-1} (-1)^n \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n \binom{u-1-m}{n-m} \overline{e_m(z)} w^{m-(u-1)}, \end{aligned} \quad (4.13)$$

where e_m denotes the m -th elementary symmetric polynomial and we have used the identity

$$e_n(1 + x_1, \dots, 1 + x_{u-1}) = \sum_{m=0}^n \binom{u-1-m}{n-m} e_m(x) \quad (4.14)$$

to get from the second to the third line.

Combining this with the free boson contribution (4.5), the matrix element (4.10) is thus proportional to

$$\begin{aligned} \int_{[\Delta_r]} G_{u-1}(z; t) \prod_{i=1}^{u-1} z_i^{-(v-1)} \cdot \prod_{i=1}^{u-1} \left(1 + \frac{z_i}{w} \right)^{-2p/\alpha-1} \\ \cdot \sum_{n=0}^{u-1} (-1)^n \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n \binom{u-1-m}{n-m} \overline{e_m(z)} w^m \frac{dz_1 \cdots dz_{u-1}}{z_1 \cdots z_{u-1}} \\ = \sum_{n=0}^{u-1} (-1)^n \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n \binom{u-1-m}{n-m} w^m \left\langle P_{[(v-1)u-1]}^t(z) P_{[1^m]}^t(z), \prod_{i=1}^{u-1} \left(1 + \frac{z_i}{w} \right)^{-2p/\alpha-1} \right\rangle_t^{u-1}, \end{aligned} \quad (4.15)$$

where we recall that elementary symmetric polynomials are examples of Jack polynomials (Proposition A.5).

Using (A.25) and specialisation as in (4.7), but with $X = \alpha p + t$, this reduces to

$$\begin{aligned} \sum_{n=0}^{u-1} (-1)^n \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n \binom{u-1-m}{n-m} w^m \sum_{\tau} \frac{(-1)^{|\tau|}}{w^{|\tau|}} \left\langle P_{[(v-1)u-1]+[1^m]}^t(z), P_{\tau}^t(z) \right\rangle_t^{u-1} \Xi_{\alpha p+t}(\mathcal{Q}_{\tau}^t(y)) \\ = \sum_{n=0}^{u-1} (-1)^n \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n \binom{u-1-m}{n-m} \frac{(-1)^{|\mu|}}{w^{|\mu|-m}} \left\langle P_{\mu}^t(z), P_{\mu}^t(z) \right\rangle_t^{u-1} \Xi_{\alpha p+t}(\mathcal{Q}_{\mu}^t(y)) \end{aligned}$$

$$= w^{-(u-1)(v-1)} \sum_{n=0}^{u-1} \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n (-1)^{m+n} \binom{u-1-m}{n-m} \left\langle P_{[1^m]}^t(z), P_{[1^m]}^t(z) \right\rangle_t^{u-1} \Xi_{\alpha p+t}(\mathcal{Q}_\mu^t(y)), \quad (4.16)$$

where $\mu = [(v-1)^{u-1}] + [1^m] = [v^m, (v-1)^{u-1-m}]$ and $|\mu| = (u-1)(v-1) + m$. In the last step, we have used (A.25) again and the definition (A.18) of the symmetric polynomial inner product.

Proposition A.5 gives the norm squared of $P_{[1^m]}^t = e_m$ and the specialisation of \mathcal{Q}_μ^t as

$$\left\langle P_{[1^m]}^t(z), P_{[1^m]}^t(z) \right\rangle_t^{u-1} = \prod_{i=1}^m \frac{(u-i)(t+m-i)}{(m-i+1)(t+u-1-i)} = \binom{u-1}{m} \prod_{i=1}^m \frac{t+m-i}{t+u-1-i}, \quad (4.17a)$$

$$\begin{aligned} \Xi_{\alpha p+t}(\mathcal{Q}_\mu^t(y)) &= \prod_{j=1}^{v-1} \left[\prod_{i=1}^m \frac{\alpha p + t j - i + 1}{2u-1-i-t(j-1)} \cdot \prod_{i=m+1}^{u-1} \frac{\alpha p + t j - i + 1}{2u-1-i-t j} \right] \cdot \prod_{i=1}^m \frac{\alpha p + u - i + 1}{t+m-i} \\ &= \frac{\prod_{r=1}^{u-1} \prod_{s=1}^{v-1} (\lambda_p - \lambda_{r,s}) \cdot \prod_{i=0}^{m-1} (\alpha p + u - i)}{\prod_{i=1}^{u-1} \prod_{j=2}^{v-1} (2u-1-i-t(j-1)) \cdot \prod_{i=0}^{m-1} (2(u-1)-i) \cdot \prod_{i=m+1}^{u-1} (t+u-1-i) \cdot \prod_{i=1}^m (t+m-i)}. \end{aligned} \quad (4.17b)$$

Here, we have assumed that $v > 1$; if $v = 1$, then the denominator of (4.17b) is just $\prod_{i=1}^m (t+m-i)$. Noting that the double product in the denominator of (4.17b) is a constant, independent of m, n, p and q , the matrix element (4.10) is thus proportional to

$$\begin{aligned} \prod_{r=1}^{u-1} \prod_{s=1}^{v-1} (\lambda_p - \lambda_{r,s}) \cdot \sum_{n=0}^{u-1} \binom{u-2-n+q}{u-1-n} \sum_{m=0}^n (-1)^{m+n} \binom{u-1-m}{n-m} \binom{u-1}{m} \prod_{i=0}^{m-1} \frac{\alpha p + u - i}{2(u-1)-i} \\ = \prod_{r=1}^{u-1} \prod_{s=1}^{v-1} (\lambda_p - \lambda_{r,s}) \cdot \sum_{\ell=0}^{u-1} \binom{q-1}{\ell} \binom{u-1+\ell}{u-1} \binom{\alpha p + u}{u-1-\ell}, \end{aligned} \quad (4.18)$$

where in addition to suppressing the denominator we have also suppressed an overall non-zero constant factor that arises when simplifying the binomial expressions.

The double product in this expression can be interpreted by noting that

$$(\lambda_p - \lambda_{r,s})(\lambda_p - \lambda_{u-r,v-s}) = 4t(\Delta_p - \Delta_{r,s}), \quad \Delta_{r,s} = \Delta_{p_{r,s}} = \frac{(r-ts)^2 - 1}{4t} = \frac{(vr-us)^2 - v^2}{4uv}, \quad (4.19)$$

by (3.4). If we define $K(u, v)$ to be the set of pairs $(r, s) \in \{1, \dots, u-1\} \times \{1, \dots, v-1\}$ with (r, s) and $(u-r, v-s)$ identified, then the matrix element (4.10) is given by

$$\langle p; q | \chi(w) | p; q \rangle = \text{const} \cdot \prod_{(r,s) \in K(u,v)} (\Delta_p - \Delta_{r,s}) \cdot \sum_{\ell=0}^{u-1} \binom{q-1}{\ell} \binom{u-1+\ell}{u-1} \binom{\alpha p + u}{u-1-\ell}. \quad (4.20)$$

In order to use this to find a generator of the annihilating ideal in Zhu's algebra, the sum factor in (4.20) needs to be expressible in terms of $\mathfrak{sl}(2)$ data. To demonstrate this, we define a function f of u, p and q by

$$f_{p;q}(u) = \sum_{\ell=0}^{u-1} \binom{q-1}{\ell} \binom{u-1+\ell}{u-1} \binom{\alpha p + u}{u-1-\ell}. \quad (4.21)$$

For small values of u , it gives polynomials in $\lambda_{p;q}$ and Δ_p :

$$f_{p;q}(2) = \lambda_{p;q}, \quad f_{p;q}(3) = \frac{3}{4} \lambda_{p;q}^2 - t \Delta_p, \quad f_{p;q}(4) = \frac{5}{12} \lambda_{p;q}^3 + \left(\frac{1}{3} - t \Delta_p \right) \lambda_{p;q}. \quad (4.22)$$

Of course, $\alpha = \sqrt{2t} = \sqrt{2u/v}$ also depends upon u , but may be regarded as an independent variable for the following analysis because of its v -dependence.

Proposition 4.3. *For each $u \in \mathbb{Z}_{\geq 0}$, $f_{p;q}(u)$ is a polynomial in $\lambda_{p;q}$ and Δ_p that satisfies the recursion relation*

$$f_{p;q}(u+2) = \frac{(2u+1)\lambda_{p;q}}{(u+1)^2} f_{p;q}(u+1) - \frac{4t\Delta_p - (u-1)(u+1)}{(u+1)^2} f_{p;q}(u). \quad (4.23)$$

Proof. We apply Zeilberger's creative telescoping algorithm, see [54] for background. If we set

$$F_{p;q}(u, \ell) = \binom{q-1}{\ell} \binom{u-1+\ell}{u-1} \binom{\alpha p + u}{u-1-\ell}, \quad (4.24)$$

then the algorithm constructs the recursion relation

$$((\alpha p + 1)^2 - u^2)F_{p;q}(u, \ell) - (2u + 1)\lambda_{p;q}F_{p;q}(u + 1, \ell) + (u + 1)^2F_{p;q}(u + 2, \ell) = G(u, \ell + 1) - G(u, \ell), \quad (4.25)$$

where $G(u, \ell) = R(u, \ell)F_{p;q}(u, \ell)$ and

$$R(u, \ell) = -\frac{2\ell^2(\alpha p + 1 + \ell)(2u^2 + (2\alpha p + 3)u + \alpha p + 1)}{u(u - \ell)(u - \ell + 1)}. \quad (4.26)$$

Summing this recursion relation over ℓ then yields (4.23), upon noting that $(\alpha p + 1)^2 = 4t\Delta_p + 1$. Since $f_{p;q}(0) = 0$ and $f_{p;q}(1) = 1$, it follows from (4.23) that $f_{p;q}(u)$ is a polynomial in $\lambda_{p;q}$ and Δ_p , as claimed. ■

Let $g_{u,v}(\lambda, \Delta)$ denote the polynomial for which $f_{p;q}(u) = g_{u,v}(\lambda_{p;q}, \Delta_p)$ and let

$$I_{u,v}(\lambda, \Delta) = \prod_{(r,s) \in K(u,v)} (\Delta - \Delta_{r,s}) \cdot g_{u,v}(\lambda, \Delta). \quad (4.27)$$

It is a simple corollary of (4.22) and (4.23) that $g_{u,v}$ has degree $u - 1$ as a polynomial in λ . If we regard Δ as having degree 2, then the total degree of $g_{u,v}$ is also $u - 1$ and that of $I_{u,v}$ is therefore $(u - 1)v$.

Theorem 4.4. *Zhu's algebra of $A_1(u, v)$ is given by the quotient*

$$\text{Zhu}[A_1(u, v)] = \frac{\mathfrak{Usl}(2)}{\langle I_{u,v}(h, T) \rangle}, \quad (4.28)$$

where $T = \frac{1}{2t}(\frac{1}{2}h^2 - ef - fe)$ denotes the image of $T(z)$ in $\text{Zhu}[V_k] \cong \mathfrak{Usl}(2)$.

Proof. As noted above, the ideal of $\mathfrak{Usl}(2)$ by which one quotients to get the Zhu algebra $\text{Zhu}[A_1(u, v)]$ is generated by the image of the null field $\chi(z)$. This image is a polynomial in e, f and h of total degree at most $(u - 1)v$. We have evaluated the action of the zero mode of $\chi(z)$ on a continuum of relaxed highest weight vectors and thus the image of $\chi(z)$ in Zhu's algebra is, up to non-zero constant factors, equal to the polynomial $I_{u,v}(h, T)$ of total degree $(u - 1)v$. ■

Before we can use this presentation of Zhu's algebra of $A_1(u, v)$, we need to know a little more about the zeroes of the polynomial $g_{u,v}$.

Proposition 4.5. *For each $u \in \mathbb{Z}_{\geq 1}$, the polynomials $g_{u,v}(\lambda, \Delta_{r,0})$ evaluate to zero when $r = 1, 2, \dots, u - 1$ and $\lambda = r - 1, r - 3, \dots, -r + 3, -r + 1$.*

Proof. This is trivial for $u = 1$ as there are then no r or λ to check. For $u = 2$, (4.22) gives $g_{2,v}(\lambda, \Delta) = \lambda$ and we need only check $r = 1$ and $\lambda = 0$. We may therefore assume, inductively, that the statement of the proposition is true for $g_{1,v}, g_{2,v}, \dots, g_{u+1,v}$. Then, the recursion relation (4.23) shows that

$$g_{u+2,v}(\lambda, \Delta_{r,0}) = \frac{(2u + 1)\lambda}{(u + 1)^2} g_{u+1,v}(\lambda, \Delta_{r,0}) - \frac{4t\Delta_{r,0} - (u - 1)(u + 1)}{(u + 1)^2} g_{u,v}(\lambda, \Delta_{r,0}) \quad (4.29)$$

will vanish for all $r = 1, 2, \dots, u - 1$ and $\lambda = r - 1, r - 3, \dots, -r + 3, -r + 1$, because $g_{u+1,v}$ and $g_{u,v}$ do. Moreover, because $4t\Delta_{u,0} = (u - 1)(u + 1)$, $g_{u+2,v}$ also vanishes for $r = u$ and $\lambda = r - 1, r - 3, \dots, -r + 3, -r + 1$.

The only remaining case is $r = u + 1$. Then, $4t\Delta_{u+1,0} = u(u + 2)$, hence we may identify αp with u or $-u - 2$, hence $q = \frac{1}{2}(\lambda - \alpha p)$ with $\frac{1}{2}(\lambda - u)$ or $\frac{1}{2}(\lambda + u + 2)$, respectively. From (4.21) and $\alpha p = u$, we now obtain

$$\begin{aligned} g_{u+2,v}(\lambda, \Delta_{u+1,0}) &= f_{u/\alpha;(\lambda-u)/2}(u + 2) = \sum_{\ell=0}^{u+1} \binom{\frac{1}{2}(\lambda - u) - 1}{\ell} \binom{u + 1 + \ell}{\ell} \binom{2(u + 1)}{u + 1 - \ell} \\ &= \binom{2(u + 1)}{u + 1} \sum_{\ell=0}^{u+1} \binom{\frac{1}{2}(\lambda - u) - 1}{\ell} \binom{u + 1}{u + 1 - \ell} = \binom{2(u + 1)}{u + 1} \binom{\frac{1}{2}(\lambda + u)}{u + 1} \end{aligned}$$

$$= \binom{2(u+1)}{u+1} \frac{(\lambda+u)(\lambda+u-2)\cdots(\lambda-u+2)(\lambda-u)}{2^{u+1}(u+1)!}, \quad (4.30)$$

which clearly vanishes for $\lambda = u, u-2, \dots, -u+2, -u$. The result is the same for $\alpha p = -u-2$. \blacksquare

We are now in a position to classify the simple weight modules over $\text{Zhu}[A_1(u, v)]$. Recall that $\text{Zhu}[A_1(u, v)]$ -modules are automatically $\mathfrak{sl}(2)$ -modules; by a weight module over $\text{Zhu}[A_1(u, v)]$, we mean that it is a weight module over $\mathfrak{sl}(2)$. The classification of simple $\mathfrak{sl}(2)$ weight modules was summarised in Proposition 2.2. We also recall that the quadratic Casimir operator

$$Q = \frac{1}{2}h^2 - ef - fe = 2tT \quad (4.31)$$

acts as a scalar multiple of the identity on any simple $\mathfrak{sl}(2)$ weight module.

Theorem 4.6. *The following $\mathfrak{sl}(2)$ -modules provide a complete list of the inequivalent isomorphism classes of simple weight modules of $\text{Zhu}[A_1(u, v)]$:*

- *The finite-dimensional highest and lowest weight modules $\overline{\mathcal{L}}_{\lambda_{r,0}}$, where $1 \leq r \leq u-1$.*
- *The infinite-dimensional highest weight modules $\overline{\mathcal{D}}_{\lambda_{r,s}}$, where $1 \leq r \leq u-1$ and $1 \leq s \leq v-1$.*
- *The infinite-dimensional lowest weight modules $\overline{\mathcal{w}}(\overline{\mathcal{D}}_{\lambda_{r,s}})$, where $1 \leq r \leq u-1$ and $1 \leq s \leq v-1$.*
- *The infinite-dimensional relaxed highest weight modules $\overline{\mathcal{R}}_{\lambda, \Delta_{r,s}}$, where $(r, s) \in K(u, v)$ and $4t\Delta_{r,s} \neq \mu(\mu+2)$ for all $\mu \in \lambda + 2\mathbb{Z}$.*

Proof. We first consider a simple finite-dimensional $\text{Zhu}[A_1(u, v)]$ -module M , which must therefore also be a finite-dimensional $\mathfrak{sl}(2)$ -module. As the quadratic Casimir takes the value $\frac{1}{2}(r^2-1) = 2t\Delta_{r,0}$ on the r -dimensional simple $\mathfrak{sl}(2)$ module, it follows that $g_{u,v}(h, T)$ must act trivially on M because the remaining λ -independent factors of $I_{u,v}(\lambda, \Delta)$ do not have $\Delta_{r,0}$ as a root for any positive integer r . By Proposition 4.5, $g_{u,v}(m, \Delta_{r,0}) = 0$ if $1 \leq r \leq u-1$ and $m = r-1, r-3, \dots, -r+1$. Conversely if $r \geq u$, then $g_{u,v}(m, \Delta_{r,0}) \neq 0$ for some $m = r-1, r-3, \dots, -r+1$, because the λ -degree of $g_{u,v}(\lambda, \Delta_{r,0})$ is $u-1$, so there cannot be more than $u-1$ zeroes. Thus, M must be isomorphic to one of the $\overline{\mathcal{L}}_{\lambda_{r,0}}$ for some $1 \leq r \leq u-1$.

Next, we consider a simple infinite-dimensional $\text{Zhu}[A_1(u, v)]$ -module M , which must therefore also be an infinite-dimensional $\mathfrak{sl}(2)$ weight module. Because there must be an infinite number of weight vectors in M with distinct $\mathfrak{sl}(2)$ -weights, $g_{u,v}(h, T)$ cannot vanish identically on M . In order for $I_{u,v}(h, T)$ to then vanish, T must act as multiplication by $\Delta_{r,s}$ for some $(r, s) \in K(u, v)$. Referring to Proposition 2.2, it follows that the last three cases of Theorem 4.6 exhaust all the possible isomorphism classes for M . \blacksquare

The correspondence (Theorem B.1) between simple modules of a vertex operator algebra and its Zhu algebra then proves the following classification result:

Theorem 4.7. *The following $\widehat{\mathfrak{sl}}(2)$ -modules provide a complete list of the inequivalent isomorphism classes of simple relaxed highest weight modules of $A_1(u, v)$:*

- *The highest weight modules $\mathcal{L}_{\lambda_{r,0}}$, where $1 \leq r \leq u-1$.*
- *The highest weight modules $\mathcal{D}_{\lambda_{r,s}}$, where $1 \leq r \leq u-1$ and $1 \leq s \leq v-1$.*
- *The conjugates $\mathfrak{w}(\mathcal{D}_{\lambda_{r,s}})$, where $1 \leq r \leq u-1$ and $1 \leq s \leq v-1$.*
- *The relaxed highest weight modules $\mathcal{E}_{\lambda, \Delta_{r,s}}$, where $(r, s) \in K(u, v)$ and $4t\Delta_{r,s} \neq \mu(\mu+2)$ for all $\mu \in \lambda + 2\mathbb{Z}$.*

These are therefore the simple modules of the vertex operator algebra $A_1(u, v)$ which belong to the category \mathcal{R} that was introduced in Section 2.1.

Zhu's correspondence also extends to non-simple $\text{Zhu}[A_1(u, v)]$ -modules. In particular, $\mathfrak{sl}(2)$ admits reducible, but indecomposable, modules similar to the $\overline{\mathcal{R}}_{\lambda, \Delta}$ of Proposition 2.2 whenever

$$4t\Delta = \mu(\mu+2) \quad \text{for some } \mu \in \lambda + 2\mathbb{Z}. \quad (4.32)$$

For $\Delta = \Delta_{r,s}$, where $(r,s) \in K(u,v)$, the only solutions are $\mu = r - 1 - ts = \lambda_{r,s}$ and $\mu = -r - 1 + ts = \lambda_{u-r,v-s}$. As $0 < s < v$, we find that $\lambda_{r,s} - \lambda_{u-r,v-s} \notin 2\mathbb{Z}$, concluding that an indecomposable with $\Delta = \Delta_{r,s}$ may have at most one weight μ satisfying (4.32). Thus, there are [50] precisely two reducible, but indecomposable, $\mathfrak{sl}(2)$ -modules $\overline{\mathcal{R}}_{\lambda_{r,s};\Delta_{r,s}}^+$ and $\overline{\mathcal{R}}_{\lambda_{r,s};\Delta_{r,s}}^-$ for each $1 \leq r \leq u-1$ and $1 \leq s \leq v-1$. They are determined (up to isomorphism) by the following non-split short exact sequences:

$$0 \longrightarrow \overline{\mathcal{D}}_{\lambda_{r,s}} \longrightarrow \overline{\mathcal{R}}_{\lambda_{r,s};\Delta_{r,s}}^+ \longrightarrow \overline{\mathfrak{w}}(\overline{\mathcal{D}}_{\lambda_{u-r,v-s}}) \longrightarrow 0, \quad 0 \longrightarrow \overline{\mathfrak{w}}(\overline{\mathcal{D}}_{\lambda_{u-r,v-s}}) \longrightarrow \overline{\mathcal{R}}_{\lambda_{r,s};\Delta_{r,s}}^- \longrightarrow \overline{\mathcal{D}}_{\lambda_{r,s}} \longrightarrow 0. \quad (4.33)$$

Applying Zhu's construction now leads to the following result:

Theorem 4.8. *For each $1 \leq r \leq u-1$ and $1 \leq s \leq v-1$, there exist two reducible, but indecomposable, $A_1(u,v)$ -modules $\mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}^+$ and $\mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}^-$, obtained by inducing $\overline{\mathcal{R}}_{\lambda_{r,s};\Delta_{r,s}}^+$ and $\overline{\mathcal{R}}_{\lambda_{r,s};\Delta_{r,s}}^-$ to level $k \widehat{\mathfrak{sl}}(2)$ -modules and quotienting by the sum of all the submodules that trivially intersect the space of conformal weight $\Delta_{r,s}$. They are determined (up to isomorphism) by the following non-split short exact sequences:*

$$0 \longrightarrow \mathcal{D}_{\lambda_{r,s}} \longrightarrow \mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}^+ \longrightarrow \mathfrak{w}(\mathcal{D}_{\lambda_{u-r,v-s}}) \longrightarrow 0, \quad 0 \longrightarrow \mathfrak{w}(\mathcal{D}_{\lambda_{u-r,v-s}}) \longrightarrow \mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}^- \longrightarrow \mathcal{D}_{\lambda_{r,s}} \longrightarrow 0. \quad (4.34)$$

A theorem of Kac and Wakimoto [2, Prop. 1] asserts that the $A_1(u,v)$ -modules in category \mathcal{O} are all semisimple. The category \mathcal{O} $A_1(u,v)$ -modules therefore consist of finite direct sums of the highest weight modules of Theorem 4.7. By way of contrast, a corollary of Theorem 4.8 is that the $A_1(u,v)$ -modules of category \mathcal{R} need not be semisimple (when $v \neq 1$). We remark that we have not excluded the possibility that there exist relaxed highest weight modules over $A_1(u,v)$ that extend the $\mathcal{E}_{\lambda;\Delta_{r,s}}$, or the $\mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}^\pm$, non-trivially; this would seem to require more information about the submodule structure of the relaxed Verma modules than is currently available, see [39, 55]. However, the highest weight result given in Corollary 4.2 and the analogous results for the Virasoro minimal models suggest the following conjecture:

Conjecture. The $A_1(u,v)$ -modules of Theorem 4.7 and Theorem 4.8 exhaust the indecomposable $A_1(u,v)$ -modules of category \mathcal{R} .

We close by demonstrating that the non-semisimplicity of relaxed highest weight modules over $A_1(u,v)$ does not imply that the Virasoro mode L_0 acts non-semisimply, a fact that is of interest to logarithmic conformal field theory studies. This result requires the finite-dimensionality of the weight spaces of the category \mathcal{R} modules, discussed in Section 2.1.

Theorem 4.9.

- 1) *The image of the energy momentum tensor T in $\text{Zhu}[A_1(u,v)] \cong \mathfrak{sl}(2) / \langle I_{u,v}(h,T) \rangle$ acts semisimply on every weight module of $\text{Zhu}[A_1(u,v)]$ with finite-dimensional weight spaces.*
- 2) *The Virasoro zero mode L_0 acts semisimply on every relaxed highest weight module of $A_1(u,v)$.*

Proof.

- 1) Let \mathcal{M} be an indecomposable weight module of $\text{Zhu}[A_1(u,v)]$ on which (the image of) T acts non-semisimply. As T is proportional to the image of the quadratic Casimir Q , it follows that T has a single (generalised) eigenvalue on \mathcal{M} . Then, by Weyl's theorem for $\mathfrak{sl}(2)$, \mathcal{M} must be infinite-dimensional with an infinite number of distinct $\mathfrak{sl}(2)$ weights. Let \mathcal{W} be the submodule of \mathcal{M} spanned by the eigenvectors of T . As \mathcal{W} is non-zero, \mathcal{W} must also be infinite-dimensional as it possesses an infinite number of distinct $\mathfrak{sl}(2)$ -weights. Thus, \mathcal{W} is an eigenspace of T with eigenvalue $\Delta_{r,s}$, for some $(r,s) \in K(u,v)$.

Next, assume that there exists a generalised eigenvector v of T , so that $(T - \Delta_{r,s})v \neq 0$, with $\mathfrak{sl}(2)$ -weight λ_v . The existence of v would imply that $I_{u,v}(\lambda_v, \Delta)$ has a zero of order at least 2 at $\Delta = \Delta_{r,s}$. Since there are only finitely many $\mathfrak{sl}(2)$ -weights λ for which $\Delta = \Delta_{r,s}$ is a zero of $I_{u,v}(\lambda, \Delta)$ of order at least 2, the quotient \mathcal{M}/\mathcal{W} must be finite-dimensional. But, the eigenvalue of T on $v + \mathcal{W}$ is then $\Delta_{r,0}$, for some $1 \leq r \leq u-1$, which is a contradiction. It follows that no such generalised eigenvectors exist, hence that T acts semisimply on \mathcal{M} .

- 2) On any relaxed highest weight module over $A_1(u, v)$, the action of L_0 on the relaxed highest weight vectors coincides with that of T on the corresponding Zhu $[A_1(u, v)]$ -module. As the latter action is semisimple, so is that of L_0 on the relaxed highest weight vectors. As these generate the whole module, L_0 acts semisimply. ■

We stress that this result does not imply that the conformal field theories corresponding to the vertex operator algebras $A_1(u, v)$ are non-logarithmic. Indeed, it has been known for some time that there are models with $k = -\frac{4}{3}$ [7] and $k = -\frac{1}{2}$ [9, 11] that are logarithmic. The loophole is that we may also twist by the so-called spectral flow automorphisms which, when $v \neq 1$, lead to infinitely many new simple (and indecomposable) $A_1(u, v)$ -modules that do not belong to category \mathcal{R} . For $k = -\frac{4}{3}$ and $k = -\frac{1}{2}$, there exist indecomposable $A_1(u, v)$ -modules that are formed from relaxed highest weight modules from different spectral flow sectors and the action of L_0 on these is non-semisimple. In fact, these modules are *staggered* in the sense of [16, 56]; a detailed discussion may be found in [14]. We expect that there exist staggered $A_1(u, v)$ -modules whenever $v \neq 1$, hence that the associated conformal field theories are logarithmic, and hope to report on this in the future.

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APPENDIX A. SYMMETRIC POLYNOMIALS

The standard reference for the parts of symmetric function theory most applicable to the work reported here is Macdonald’s book [28]. In this appendix, we summarise the results from Chapters 1 and 6 of this book that we use freely throughout.

A.1. Partitions of integers. A number of bases of the ring of symmetric polynomials are indexed by partitions. We therefore fix some notation on partitions before going on to discuss symmetric polynomials. A partition $\lambda = [\lambda_1, \dots, \lambda_m]$ is a weakly descending sequence of positive integers called parts. The length $\ell(\lambda) = m$ is the length of the sequence and the weight $|\lambda| = \sum_i \lambda_i$ is the sum over all elements of the sequence. Sometimes, it is convenient to define the λ_i for $i > \ell(\lambda)$ to be 0. A partition λ is often also referred to as a partition of the integer $|\lambda|$; $[3, 3, 2, 1, 1, 1]$ is thus a partition of 11. It is customary to regard the empty partition $[\]$ as a partition of 0.

A convenient shorthand for partitions is to indicate repeated parts using a superscript. Thus, $[3, 3, 2, 1, 1, 1]$ and $[3^2, 2, 1^3]$ denote the same partition. The multiplicity in λ of a given part i , that is, the superscript in the convenient shorthand notation, will be denoted by $m_\lambda(i)$. For every partition λ , one may then introduce the following number:

$$z_\lambda = \prod_{i \geq 1} m_\lambda(i)! \cdot i^{m_\lambda(i)}. \tag{A.1}$$

These numbers play a small role in what follows, see (A.15) below for example.

One also associates, to each partition λ , a diagram called a Young diagram. This consists of $\ell(\lambda)$ rows of left-aligned boxes for which the length of the i -th row is λ_i . We draw the first row at the top and the $\ell(\lambda)$ -th at the bottom. With this convention, the conjugate partition λ' is defined to be the partition whose Young diagram is the reflection of that of λ along the diagonal line from top left to bottom right. This reflection exchanges the lengths of the columns and the rows.

Each box b of the Young diagram of λ , generally written as $b \in \lambda$, may be parametrised by a pair $b = (i, j)$, where i is the row number counted from top to bottom and j is the column number counted from left to right. Given a partition λ and a box $b \in \lambda$, the arm length $a(b)$, the leg length $l(b)$, the arm colength $a'(b)$ and the leg

colength $l'(b)$ are the distances from b to the right, bottom, left and top edges of the Young diagram, respectively. In formulae,

$$a(b) = \lambda_i - j, \quad a'(b) = j - 1, \quad l(b) = \lambda'_j - i, \quad l'(b) = i - 1. \quad (\text{A.2})$$

Finally, we remark that partitions admit a number of partial orderings, among which the dominance ordering, which we denote by \geq , will prove useful. Two partitions λ and μ satisfy $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \mu_i, \quad \text{for all } m \geq 1. \quad (\text{A.3})$$

A.2. Symmetric polynomials. Let Λ_n be the ring of n -variable polynomials with complex coefficients that are invariant under arbitrary permutations of the variables. Λ_n is called the ring of symmetric polynomials in n variables. This ring is graded,

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k, \quad (\text{A.4})$$

where Λ_n^k is the space of homogeneous symmetric polynomials of degree k .

Examples of symmetric polynomials include:

1) *Power sums*: The symmetric polynomial

$$p_k(x_1, \dots, x_n) = \sum_{i=1}^n x_i^k, \quad k \geq 1, \quad (\text{A.5})$$

is called the k -th power sum. For each partition $\lambda = (\lambda_1, \dots, \lambda_m)$, we define

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_m}. \quad (\text{A.6})$$

2) *Elementary symmetric polynomials*: The symmetric polynomials

$$e_0(x_1, \dots, x_n) = 1, \quad e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad 1 \leq k \leq n, \quad (\text{A.7})$$

are called the elementary symmetric polynomials. The elementary symmetric polynomials appear in the expansion of the generating function

$$\prod_{i=1}^n (y + x_i) = \sum_{i=0}^n e_i(x_1, \dots, x_n) y^{n-i}. \quad (\text{A.8})$$

3) *Monomial symmetric polynomials*: Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ be an n -tuple of non-negative integers. The symmetric polynomials

$$m_\alpha(x_1, \dots, x_n) = \sum_{\sigma} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad (\text{A.9})$$

where the sum runs over all distinct permutations σ of α , are called the monomial symmetric polynomials or symmetric monomials for short. Given α , there is always precisely one distinct permutation σ whose entries are in descending order, hence by omitting any trailing zeroes, we may assume that α is a partition of length at most n .

Proposition A.1.

1) *The power sums p_k are algebraically independent for $k \leq n$ and generate Λ_n :*

$$\Lambda_n = \mathbb{C}[p_1, \dots, p_n]. \quad (\text{A.10})$$

2) *The elementary symmetric polynomials are algebraically independent and generate Λ_n :*

$$\Lambda_n = \mathbb{C}[e_1, \dots, e_n]. \quad (\text{A.11})$$

3) *The symmetric monomials m_λ with $\ell(\lambda) \leq n$ form a basis of Λ_n .*

When working with symmetric polynomials, it is a remarkable fact that the number of variables is often irrelevant, assuming only that this number is sufficiently large. For this reason, it is convenient to work with symmetric polynomials in infinitely many variables. The ring of symmetric polynomials in infinitely many variables, x_i say,

is given by an inverse limit:

$$\Lambda = \varprojlim_n \Lambda_n. \quad (\text{A.12})$$

Often the elements of Λ are distinguished by referring to them as symmetric functions. The projection

$$\pi_n: \Lambda \rightarrow \Lambda_n, \quad (\text{A.13})$$

defined by setting $x_i = 0$ for all $i > n$, recovers the case of finitely many variables.

Proposition A.2.

1) *The power sums are algebraically independent for all $k \geq 1$ and generate Λ :*

$$\Lambda = \mathbb{C}[p_1, p_2, \dots]. \quad (\text{A.14})$$

In addition, the p_λ form a basis of Λ as λ runs over all partitions of all non-negative integers.

2) *The symmetric monomials m_λ form a basis of Λ as λ runs over all partitions of all non-negative integers. The projection $\pi_n: \Lambda \rightarrow \Lambda_n$ satisfies $\pi_n(m_\lambda) = 0$ if and only if $\ell(\lambda) > n$.*

The basis of Λ which is most interesting for the purposes of this article is that consisting of the Jack symmetric polynomials. These polynomials are orthogonal with respect to the following inner product on Λ :

$$\langle p_\lambda, p_\mu \rangle_t = z_\lambda t^{\ell(\lambda)} \delta_{\lambda, \mu}. \quad (\text{A.15})$$

In fact, there are infinitely many families of Jack polynomials, each forming a basis of Λ , parametrised by the complex number $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ appearing in this inner product.

Proposition A.3.

1) *Let $\{u_m\}$ be a basis of Λ and let $\{v_m\}$ be the dual basis with respect to the inner product (A.15). Then, the following identities hold:*

$$\prod_{k \geq 1} \exp\left(\frac{1}{t} \frac{p_k(x_1, x_2, \dots) p_k(y_1, y_2, \dots)}{k}\right) = \prod_{i, j \geq 1} (1 - x_i y_j)^{-1/t} = \sum_m u_m(x_1, x_2, \dots) v_m(y_1, y_2, \dots). \quad (\text{A.16})$$

Analogous identities hold for finite numbers of variables x_i and/or y_j by projection.

2) *For every partition λ and every $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, there exists a unique basis of symmetric polynomials P_λ^t such that $\langle P_\lambda^t, P_\mu^t \rangle_t = 0$, whenever $\lambda \neq \mu$, and such that they satisfy upper-triangular decompositions into symmetric monomials,*

$$P_\lambda^t = \sum_{\mu \leq \lambda} u_{\lambda, \mu}(t) m_\mu, \quad (\text{A.17})$$

where $u_{\lambda, \mu}(t) \in \mathbb{C}$ and $u_{\lambda, \lambda}(t) = 1$. Here, the sum runs over all μ that are dominated by λ . It follows that the P_λ^t are homogeneous symmetric polynomials of degree $|\lambda|$.

Definition. The P_λ^t determined by the previous proposition are called the *Jack symmetric polynomials* or, when working in the infinite-variable ring Λ , the *Jack symmetric functions*.

We remark that if $t \in \mathbb{Q}_{\leq 0}$, then some of the coefficients $u_{\lambda, \mu}(t)$ in (A.17) will diverge for certain λ , hence some of the Jack polynomials will not be defined.

The Jack polynomials P_λ^t satisfy a number of remarkable properties. Before we can list those that we shall require, we need to introduce one more inner product, this time that of two symmetric polynomials $f, g \in \Lambda_n$ of n variables:

$$\langle f, g \rangle_t^n = \int_{[\Delta_n]} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{1/t} \overline{f(x_1, \dots, x_n)} g(x_1, \dots, x_n) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}. \quad (\text{A.18})$$

Here, the overline indicates that the arguments of the function have been inverted: $\overline{f(x_1, \dots, x_n)} = f(x_1^{-1}, \dots, x_n^{-1})$.

We remark that the integral in (A.18) may be thought of as an n -variable generalisation of taking the residue at 0 of a meromorphic function. In particular, the integral vanishes if f and g are homogeneous of different degrees.

In [28], an inner product is defined for a deformation of the Jack polynomials (now) called the Macdonald polynomials. The definition is almost identical to (A.18), utilising a cycle $[\Delta_n]$ that is just a (normalised) product of n unit circles. However, there are some subtleties when passing to the Jack polynomial limit and, in particular, this cycle is no longer suitable. Instead, we will use the (normalised) cycles constructed by Tsuchiya and Kanie which also have the advantage of being supported in the domains required for radially-ordered expansions of screening operators.

Theorem A.4 [26]. *Let $r \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{C}^*$ and suppose that $d(d+1)/t \notin \mathbb{Z}$ and $d(r-d)/t \notin \mathbb{Z}$, for all integers d satisfying $1 \leq d \leq r-1$. Then, there exists a cycle Δ_r such that for each symmetric Laurent polynomial $f(z_1, \dots, z_r)$, the integral*

$$\int_{\Delta_r} G_r(z; t) f(z_1, \dots, z_r) \frac{dz_1 \cdots dz_r}{z_1 \cdots z_r} \quad (\text{A.19})$$

is equal to

$$\int_{|z|=1} \int_{\sigma_{r-1}} \prod_{i=1}^{r-1} (1-y_i)^{2/t} \cdot G_{r-1}(y; t) f(z, zy_1, zy_2, \dots, zy_{r-1}) \frac{dy_1 \cdots dy_{r-1} dz}{y_1 \cdots y_{r-1} z}, \quad (\text{A.20})$$

where σ_{r-1} is (a regularisation of) the $(r-1)$ -simplex $\{1 > y_1 > \cdots > y_{r-1} > 0\}$. If $r=1$, then $G_1(z; t) = 1$ and Δ_1 is just the unit circle. In particular, if $f(z_1, \dots, z_r) = 1$, then (A.19) may be evaluated as a Selberg integral:

$$S_n(t) = \int_{\Delta_r} G_r(z; t) \frac{dz_1 \cdots dz_r}{z_1 \cdots z_r} = \frac{2\pi i}{(r-1)!} \prod_{j=1}^{r-1} \frac{\Gamma(1+(j+1)/t) \Gamma(-j/t)}{\Gamma(1+1/t)}. \quad (\text{A.21})$$

This integral is non-zero, hence the cycle Δ_r is non-trivial.

The normalised cycle $[\Delta_n]$ is then given by $[\Delta_n] = \Delta_n / S_n(t)$, so that $\langle 1, 1 \rangle_t^n = 1$.

We finish by summarising the properties of the Jack symmetric polynomials that we will use in this article.

Proposition A.5.

- 1) The elementary symmetric polynomials are Jack polynomials (for all values of t): $e_k = P_{[1^k]}^t$.
- 2) The norm squared of P_λ^t with respect to the infinite variable inner product $\langle \cdot, \cdot \rangle_t$ is

$$\langle P_\lambda^t, P_\lambda^t \rangle_t = \prod_{b \in \lambda} \frac{t(a(b)+1) + l(b)}{ta(b) + l(b) + 1}. \quad (\text{A.22})$$

The corresponding dual basis of the Jack polynomials will be denoted by

$$Q_\lambda^t = \frac{P_\lambda^t}{\langle P_\lambda^t, P_\lambda^t \rangle_t}. \quad (\text{A.23})$$

- 3) For any $X \in \mathbb{C}$, one defines a ring homomorphism $\Xi_X: \Lambda \rightarrow \mathbb{C}$, called the specialisation map, by $\Xi_X(p_k) = X$, for all $k \geq 1$. The Jack polynomials and their duals specialise to

$$\Xi_X(P_\lambda^t) = \prod_{b \in \lambda} \frac{X + ta'(b) - l'(b)}{ta(b) + l(b) + 1}, \quad \Xi_X(Q_\lambda^t) = \prod_{b \in \lambda} \frac{X + ta'(b) - l'(b)}{t(a(b)+1) + l(b)}. \quad (\text{A.24})$$

- 4) The projection onto n variables satisfies $\pi_n(P_\lambda^t) = 0$ if and only if $\ell(\lambda) > n$.
- 5) In Λ_n , one has $P_{[m^n]}^t(x_1, \dots, x_n) = m_{[m^n]}(x_1, \dots, x_n) = \prod_{i=1}^n x_i^m$.
- 6) Suppose that λ satisfies $\ell(\lambda) \leq n$ and let $\lambda + [m^n]$ denote the partition $[\lambda_1 + m, \dots, \lambda_n + m]$. Then, in Λ_n ,

$$\prod_{i=1}^n x_i^m \cdot P_\lambda^t = P_{\lambda + [m^n]}^t. \quad (\text{A.25})$$

- 7) Suppose that λ satisfies $\ell(\lambda) \leq n$. Then, the norm squared of the P_λ^t with respect to the finite variable inner product $\langle \cdot, \cdot \rangle_t^n$ is

$$\langle P_\lambda^t, P_\lambda^t \rangle_t^n = \prod_{b \in \lambda} \frac{(t(a(b)+1) + l(b))(n + ta'(b) - l'(b))}{(ta(b) + l(b) + 1)(n + t(a'(b)+1) - l'(b) - 1)}. \quad (\text{A.26})$$

The specialisation map of Proposition A.5, item 3), seems somewhat mysterious at first glance and deserves some additional explanation. Our chief use for it is to expand products of the form $\prod_{i \geq 1} (1 - x_i)^{-X/t}$ in terms of

Jack polynomials. To do this, we consider (A.16) with $y_1 = 1$ and $y_i = 0$ for all $i > 1$:

$$\prod_{i \geq 1} (1 - x_i)^{-1/t} = \prod_{k \geq 1} \exp\left(\frac{1}{t} \frac{p_k(x_1, \dots, x_n)}{k}\right). \quad (\text{A.27})$$

It follows that we may now write

$$\begin{aligned} \prod_{i \geq 1} (1 - x_i)^{-X/t} &= \prod_{k \geq 1} \exp\left(\frac{X}{t} \frac{p_k(x_1, x_2, \dots)}{k}\right) = \mathbb{E}_X \left(\prod_{k \geq 1} \exp\left(\frac{1}{t} \frac{p_k(x_1, x_2, \dots) p_k(y_1, y_2, \dots)}{k}\right) \right) \\ &= \mathbb{E}_X \left(\sum_{\lambda} P'_{\lambda}(x_1, x_2, \dots) Q'_{\lambda}(y_1, y_2, \dots) \right) = \sum_{\lambda} P'_{\lambda}(x_1, x_2, \dots) \mathbb{E}_X(Q'_{\lambda}(y_1, y_2, \dots)) \\ &= \sum_{\lambda} \left[\prod_{b \in \lambda} \frac{X + t a'(b) - l'(b)}{t(a(b) + 1) + l(b)} \right] P'_{\lambda}(x_1, x_2, \dots), \end{aligned} \quad (\text{A.28})$$

where we take the specialisation map \mathbb{E}_X to act only on the symmetric polynomials in the y_i .

APPENDIX B. ZHU'S ALGEBRA

From any vertex operator algebra \mathcal{V} , one can construct a unital associative algebra $\text{Zhu}[\mathcal{V}]$ called Zhu's algebra. The representation theory of this associative algebra is closely related that of \mathcal{V} and is a crucial tool for classifying \mathcal{V} -modules, especially simple \mathcal{V} -modules. In this appendix, we motivate Zhu's algebra by using generalised commutation rules to relate it to the algebra of zero modes of the fields of \mathcal{V} . From this point of view, the technology of Zhu may be regarded as an abstract formalisation of the "annihilating ideals" discussed in the physics literature [53]. A part of this motivational discussion may be found in [57]; another means of motivating Zhu's algebra via deforming the standard conventions for normal ordering is the subject of [58].

A weight module \mathcal{M} of \mathcal{V} is said to be \mathbb{N} -graded if its decomposition into generalised L_0 -eigenspaces is bounded below, that is, if there exists an $h \in \mathbb{C}$ so that

$$\mathcal{M} = \bigoplus_{n \geq 0} \mathcal{M}_n, \quad \mathcal{M}_n = \{u \in \mathcal{M} : (L_0 - h - n)^m u = 0 \text{ for some } m \in \mathbb{Z}_{\geq 1}\}. \quad (\text{B.1})$$

The space $\overline{\mathcal{M}} = \mathcal{M}_0$ is often called the space of ground states; these states are clearly annihilated by every positive mode of every field. We remark that in the language of Section 2.1, the space $\overline{\mathcal{M}}$ is spanned by relaxed highest weight vectors (though \mathcal{M} may contain other such vectors that are not in $\overline{\mathcal{M}}$).

Our preferred means to motivate the construction of Zhu's algebra is the observation that a simple weight module \mathcal{M} of \mathcal{V} is, rather generally, completely determined by its space of ground states $\overline{\mathcal{M}}$. This space admits an action of the zero modes of the fields and Zhu's algebra is an abstract realisation of this action. To be more precise, consider [57] the generalised commutation relation obtained from the contour integral

$$\oint_0 \oint_w \frac{A(z)B(w)z^{h_A}w^{h_B-1}}{z-w} \frac{dz}{2\pi i} \frac{dw}{2\pi i}, \quad (\text{B.2})$$

where $A(z)$ and $B(z)$ are two fields of \mathcal{V} of conformal weights h_A and h_B , respectively. (We denote their modes by A_j and B_j , respectively, and the corresponding states by A and B .) Without the denominator in the integrand, the usual procedure of breaking the inner contour in two or employing the operator product expansion would lead to the commutation rules of the modes A_1 and B_0 . With the denominator, the generalised commutation relation is

$$\sum_{j \geq 0} [A_{-j}B_j + B_{-j-1}A_{j+1}] = \sum_{j \geq -h_A} \binom{h_A}{j+h_A} (A_j B)_0, \quad (\text{B.3})$$

where $(A_j B)_0$ denotes the zero mode of the field corresponding to the state $A_j B$. Letting this generalised commutation relation act on a ground state $v \in \overline{\mathcal{M}}$ (or a relaxed highest weight vector), we arrive at

$$A_0 B_0 v = \sum_{j \geq -h_A} \binom{h_A}{j+h_A} (A_j B)_0 v. \quad (\text{B.4})$$

This motivates the definition [46] of Zhu's product $*$ on the vertex operator algebra \mathcal{V} (we identify its elements with the states of the vacuum module for convenience):

$$A * B = \sum_{j \geq -h_A} \binom{h_A}{j+h_A} A_j B = \oint_0 A(z) B \frac{(1+z)^{h_A}}{z} \frac{dz}{2\pi i}. \quad (\text{B.5})$$

The vacuum Ω is easily verified to be a (two-sided) unit with respect to this operation. To illustrate, we tabulate the products of the generators e , h and f of the level k universal vertex operator algebra \mathcal{V}_k of $\widehat{\mathfrak{sl}}(2)$:

$*$	e	h	f	
e	$:ee:$	$:eh: - 2e$	$:ef: - h$	
h	$:he: + 2e$	$:hh:$	$:hf: - 2f$	
f	$:fe: + h$	$:fh: + 2f$	$:ff:$	

(B.6)

For example, $h * e = h_{-1}e + h_0e = :he: + 2e$. We see immediately that, unlike the algebra of zero modes on the ground states, this $*$ operation does not respect the $\mathfrak{sl}(2)$ commutation rules, for example, $h * e - e * h = 4e + 2\partial e$. Moreover, it is not even associative, for example, $(h * e) * e - h * (e * e) = 2 : \partial ee : + 2 : ee :$.

Consider now, for each \mathbb{N} -graded \mathcal{V} -module \mathcal{M} , the map taking a vertex operator algebra element $A \in \mathcal{V}$ to the linear endomorphism A_0 , restricted to the space of ground states $\overline{\mathcal{M}}$. From (B.4) and (B.5), we see that

$$\pi_{\mathcal{M}}(A * B) = \pi_{\mathcal{M}}(A)\pi_{\mathcal{M}}(B). \quad (\text{B.7})$$

Because the algebra of zero modes is associative, it follows that the failure of $*$ to be associative on \mathcal{V} must be explained by the discrepancies being mapped to zero:

$$\pi_{\mathcal{M}}((A * B) * C - A * (B * C)) = 0, \quad (\text{B.8})$$

for all $A, B, C \in \mathcal{V}$ and all \mathbb{N} -graded \mathcal{V} -modules \mathcal{M} . Similarly, discrepancies between the commutation rules of the zero modes and the $*$ -commutation rules must also map to zero. In accord with these observations, it is easy to check that $\pi_{\mathcal{M}}$ does map $h * e - e * h - 2e = 2(e + \partial e)$ and $(h * e) * e - h * (e * e) = 2(: \partial ee : + : ee :)$ to zero.

To accurately reflect the algebra of zero modes acting on the ground states, we should therefore quotient the vertex operator algebra by the intersection (over all \mathcal{M}) of the kernels of the $\pi_{\mathcal{M}}$. One can obtain many elements of this kernel by again appealing to generalised commutation relations, in this case that obtained from

$$\oint_0 \oint_w \frac{A(z)B(w)z^{h_A+1}w^{h_B-1}}{(z-w)^2} \frac{dz}{2\pi i} \frac{dw}{2\pi i}. \quad (\text{B.9})$$

The resulting generalised commutation relation, applied to $v \in \overline{\mathcal{M}}$, takes the form

$$A_0 B_0 v = \sum_{j \geq -h_A - 1} \binom{h_A + 1}{j + h_A + 1} (A_j B)_0 v \quad \Rightarrow \quad \sum_{j \geq -h_A - 1} \binom{h_A}{j + h_A + 1} (A_j B)_0 v = 0, \quad (\text{B.10})$$

where we have combined this relation with that of (B.4) in order to obtain the vanishing condition.

This motivates the definition [46] of Zhu's other product \circ on \mathcal{V} :

$$A \circ B = \sum_{j \geq -h_A - 1} \binom{h_A}{j + h_A + 1} A_j B = \oint_0 A(z) B \frac{(1+z)^{h_A}}{z^2} \frac{dz}{2\pi i}. \quad (\text{B.11})$$

It is clear from (B.10) and (B.11) that all elements of the form $A \circ B$ belong to the kernel of every $\pi_{\mathcal{M}}$. Indeed, Zhu showed that the space $O(\mathcal{V})$ spanned by the elements of this form is a two-sided ideal of \mathcal{V} . There appears to be some confusion in the literature as to whether Zhu proved explicitly that $O(\mathcal{V})$ is in fact the intersection of the kernels of the $\pi_{\mathcal{M}}$ (for example, a remark amounting to this is stated without proof in [52]), but this result may be found in [59, App. A.2]. In any case, the quotient $\text{Zhu}[\mathcal{V}] = \mathcal{V}/O(\mathcal{V})$ is what is now referred to as Zhu's algebra. It is a unital associative algebra with respect to Zhu's $*$ product.

Consideration of the elements of $O(\mathcal{V})$, for example, $2T + \partial T = T \circ \Omega \in O(\mathcal{V})$, shows that Zhu's algebra is not graded by conformal weight. It is, however, filtered by conformal weight in the sense that it has an increasing

sequence of subspaces $\text{Zhu}_0[V] \subseteq \text{Zhu}_1[V] \subseteq \text{Zhu}_2[V] \subseteq \dots$, where $\text{Zhu}_m[V]$ is the image in $\text{Zhu}[V]$ of $\bigoplus_{n=0}^m V_n$ (and V is here regarded as an \mathbb{N} -graded module over itself).

It should now be clear that the space $\overline{\mathcal{M}}$ of ground states of a V -module \mathcal{M} is a $\text{Zhu}[V]$ -module. The converse to this statement is of great importance to the representation theory of vertex operator algebras.

Theorem B.1 [46, Thms. 2.2.1–2]. *There is a bijective correspondence between isomorphism classes of simple $\text{Zhu}[V]$ -modules $\overline{\mathcal{M}}$ and simple \mathbb{N} -graded V -modules. More precisely, for every simple \mathbb{N} -graded V -module \mathcal{M} , the ground states form a simple $\text{Zhu}[V]$ -module $\overline{\mathcal{M}}$ and for every simple $\text{Zhu}[V]$ -module $\overline{\mathcal{M}}$, there exists a simple \mathbb{N} -graded V module \mathcal{M} with $\overline{\mathcal{M}}$ as its space of ground states.*

Zhu actually gives a construction of a universal \mathbb{N} -graded V -module \mathcal{M} from a simple $\text{Zhu}[V]$ -module $\overline{\mathcal{M}}$ so that any \mathbb{N} -graded V -module \mathcal{M}' with $\overline{\mathcal{M}}$ as its space of ground states is a quotient of \mathcal{M} . If the vertex operator algebra V is rational, then its modules are semisimple and thus \mathcal{M} is simple. However, for more general vertex operator algebras the universal \mathcal{M} that Zhu constructs may be reducible. Moreover, one may also start with a non-simple $\text{Zhu}[V]$ -module; the result is then always reducible.

We close this appendix with two results of Frenkel and Zhu that will be used in this paper.

Proposition B.2 [52, Prop. 1.4.2]. *Let V be a vertex operator algebra with an ideal I that does not contain the vacuum Ω or the conformal vector T . Then, the image $\text{Zhu}[I]$ of I in $\text{Zhu}[V]$ is a two-sided ideal satisfying*

$$\text{Zhu}[V/I] = \frac{\text{Zhu}[V]}{\text{Zhu}[I]}. \quad (\text{B.12})$$

This result allows us to describe Zhu's algebra for the simple admissible level vertex operator algebras $A_1(u, v)$ of $\widehat{\mathfrak{sl}}(2)$ in terms of that of the universal vertex operator algebras V_k and the generators (singular vectors) of their maximal ideals. The next result determines the latter Zhu algebras.

Proposition B.3 [52, Thm. 3.1.1]. *Zhu's algebra for the universal $\widehat{\mathfrak{sl}}(2)$ vertex operator algebra at level $k \neq -2$ is isomorphic to the universal enveloping algebra of non-affine $\mathfrak{sl}(2)$:*

$$\text{Zhu}[V_k] \cong \mathfrak{U}\mathfrak{sl}(2). \quad (\text{B.13})$$

Moreover, the image of T in $\text{Zhu}[V_k]$ may be identified with the quadratic Casimir of $\mathfrak{sl}(2)$:

$$T = \frac{1}{2(k+2)}Q, \quad Q = \frac{1}{2}h^2 - ef - fe. \quad (\text{B.14})$$

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