Indecomposable Modules for the Virasoro Algebra

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Background

- Conformal field theory (CFT) is one of the success stories of modern physics, finding application in both statistical mechanics and string theory.
- Crucial to standard CFT is the theory of irreducible highest weight modules of certain infinite-dimensional Lie algebras, *eg*. the Virasoro algebra $\mathfrak{vir}$:

$$\left[L_m, L_n\right] = (m-n) L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C,$$
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- But then, people started constructing CFTs (in various degrees) from more general indecomposable modules...

[Rozansky–Saleur NPB 376 (1992), Gaberdiel–Kausch 9604026, Flohr 9605151]

Logarithmic CFT

- Gurarie noticed that certain fundamental quantities in CFT, the correlation functions, could exhibit logarithmic singularities.
- This was traced to the mode L_0 acting non-diagonalisably on the corresponding Virasoro module (not possible for highest weight modules).

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- Gaberdiel and Kausch managed to explicitly construct the first examples of such non-diagonalisable modules.
- Rohsiepe then initiated the study of such modules, referring to them as staggered modules.

[Gurarie 9303160, Gaberdiel–Kausch 9604026, Rohsiepe 9611160]

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  - Free field realisations and quantum groups.
  - Explicit lattice realisations and statistical models.
  - Fusion and algebraic structure.
  - Logarithmic vertex operator algebras.
- Moreover, these methods are all giving roughly similar answers!

[Fjelstad *et al* 0201091, Eberle–Flohr 0604097, Feigin *et al* 0606196, Pearce *et al* 0607232, Read–Saleur 0701117, Adamovic–Milas 0707.1857, Ruelle 0707.3766, Mathieu–Ridout 0708.0802, Lepowsky *et al* 0710.2687, Huang 0712.4109, Gaberdiel *et al* 0905.0916]

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- Since then, both physicists and mathematicians have been building bridges between the CFT and SLE descriptions.
- Indeed, the local martingales of the stochastic processes associated to certain SLE-variants carry a representation of the Virasoro algebra.
- Subsequent investigations have proven that these representations sometimes admit a non-diagonalisable action of L_0. These are then staggered modules.

[Schramm Isr. J. Math. 11 (2000), Bauer–Bernard 0301064, Kytölä math-ph/0604047, Kytölä 0804.2612]

Our Question

We come from different sides of this business (DR — CFT and KK — SLE). But we are both interested in the underlying representation theory of the Virasoro algebra, in particular in developing a more complete theory of the staggered modules of Rohsiepe (definition later!).

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Unfortunately, Rohsiepe's pioneering efforts in this direction fall far short of what is needed, and we know of no other works on this topic...

Highest Weight States and Modules

The Virasoro algebra admits a triangular decomposition,

$$\mathfrak{vir} = \mathfrak{vir}^- \oplus \mathfrak{vir}^0 \oplus \mathfrak{vir}^+,$$

where $\mathfrak{vir}^\pm$ is spanned by the $L_{\pm n}$ with $n > 0$ and $\mathfrak{vir}^0$ is spanned by L_0 and C.

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A highest weight state is a vector $v_{h,c}$ in a $\mathfrak{vir}$-module which is an eigenvector of $\mathfrak{vir}^0$ and is annihilated by $\mathfrak{vir}^+$:

$$L_0 v = h v_{h,c}, \qquad C v = c v_{h,c}, \qquad L_n v_{h,c} = 0 \quad (n > 0).$$

If a $\mathfrak{vir}$-module is generated by a highest weight state under the action of $\mathfrak{vir}^-$ then it is called a highest weight module.

Here, h is the conformal dimension of $v_{h,c}$ and c is its central charge.

Verma Modules

The "biggest" highest weight modules are constructed from highest weight states on which $\mathfrak{vir}^-$ acts freely. They are known as Verma modules, denoted by $\mathscr{V}_{h,c}$.
Verma modules are generically irreducible. But for certain h and c, $\mathscr{V}_{h,c}$ contains other highest weight states besides $v_{h,c}$. These are known as singular vectors and they generate (the only) proper submodules of the Verma module.

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Any highest weight module may be realised as a quotient of a Verma module, so understanding highest weight modules reduces to understanding Verma modules, and therefore their singular vectors. In particular, every Verma module has a unique irreducible quotient $\mathscr{L}_{h,c}$.
Physical applications usually prefer irreducibles which are not Verma modules.

Verma Modules (cont.)

The Verma module $\mathscr{V}_{h,c}$ is infinite-dimensional and has a basis of states of the form

$$L_{-n_1} L_{-n_2} \cdots L_{-n_k} v_{h,c} \quad (k \geqslant 0,\, n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1).$$

The above state has conformal dimension (L_0-eigenvalue) $h + n_1 + n_2 + \ldots + n_k$. Thus:

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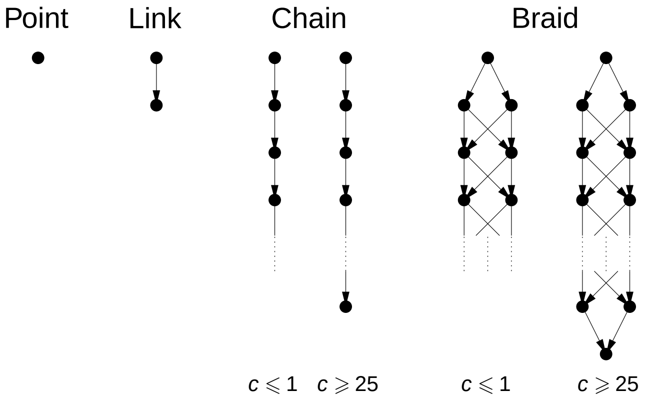
The above state has conformal dimension (L_0-eigenvalue) $h + n_1 + n_2 + \ldots + n_k$. Thus:

- L_0 is diagonalisable on $\mathscr{V}_{h,c}$, hence on any highest weight module.
- $\mathscr{V}_{h,c}$ is graded by the conformal dimension (relative to h) and the homogeneous subspaces are finite-dimensional.

This finite-dimensionality allowed Kac and Feigin–Fuchs to understand the singular vector structure of any Verma module. [Kac Lect. Notes Phys. (1979), Feigin–Fuchs Func. Anal. Appl. 16 (1982)]

Feigin–Fuchs Classification of Verma Modules



Possibilities for the singular vector structure (black circles) of $\mathscr{V}_{h,c}$. Arrows indicate that the latter vector is a descendant of the former and not vice-versa. Point and link-type modules occur for all central charges. Chain and braid-type modules occur only when a certain rationality condition is met.

Staggered Modules

We define a staggered module $\mathscr{S}$ to be an indecomposable Virasoro module for which we have a short exact sequence

$$0 \longrightarrow \mathscr{H}^{\mathrm{L}} \stackrel{\iota}{\longrightarrow} \mathscr{S} \stackrel{\pi}{\longrightarrow} \mathscr{H}^{\mathrm{R}} \longrightarrow 0,$$

in which:

- $\mathscr{H}^{\mathrm{L}}$ and $\mathscr{H}^{\mathrm{R}}$ are highest weight modules, the left and right module (respectively), of the same central charge c and (respective) conformal dimensions h^{L} and h^{R},
- ι and π are module homomorphisms, and
- L_0 is not diagonalisable on $\mathscr{S}$, possessing instead Jordan cells of rank at most 2.

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- L_0 is not diagonalisable on $\mathscr{S}$, possessing instead Jordan cells of rank at most 2.

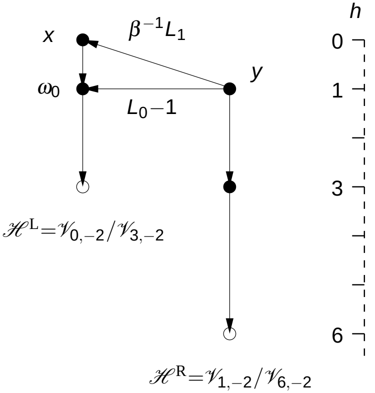
$\mathscr{S}$ still admits a grading by decomposing L_0 into its semisimple and nilpotent parts — the eigenvalue of the former on the states of $\mathscr{S}$ is now the conformal dimension.

Notation and Depiction

Let:

- x be the highest weight state of the submodule $\iota\left(\mathscr{H}^{\mathrm{L}}\right)$,
- y be an element of $\pi^{-1}\left(\mathscr{H}^{\mathrm{R}}\right)$ of conformal dimension h^{R},
- $\omega_0 = \left(L_0 - h^{\mathrm{R}}\right) y$,
- $\omega_1 = L_1 y$, and
- $\omega_2 = L_2 y$.

[Since L_1 and L_2 generate $\mathfrak{vir}^+$, we do not need to consider $L_n y$ for $n > 2$.]



An example of a staggered module. Black circles indicate singular vectors and white circles indicate those which have been set to zero.

Some Simple Consequences

Proposition

ω_0, ω_1 and ω_2 are elements of $\mathscr{H}^{\mathrm{L}}$, and ω_0 is non-zero and singular.

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Staggered modules can only exist when $\ell \equiv h^{\mathrm{R}} - h^{\mathrm{L}} \in \mathbb{N}$. Moreover, if $\ell > 0$, then h^{L} must have the form

$$h^{\mathrm{L}} = \frac{r^2-1}{4}t - \frac{rs-1}{2} + \frac{s^2-1}{4}t^{-1}, \quad \text{where } c = 13 - 6\left(t + t^{-1}\right),$$

for some $r, s \in \mathbb{Z}_+$.

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Indecomposable modules $\mathscr{S}$ with $\ell < 0$ can exist (reducible Verma modules are conspicuous examples), but they cannot be staggered (L_0 will be diagonalisable).

More Simple Consequences

If $\mathscr{H}^{\mathrm{R}}$ is not a Verma module, we have $\overline{X} x^{\mathrm{R}} = 0$ for some combination(s) $\overline{X}$ of negative Virasoro modes (x^{R} is the highest weight state of $\mathscr{H}^{\mathrm{R}}$). This defines vector(s) ϖ by

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Proposition

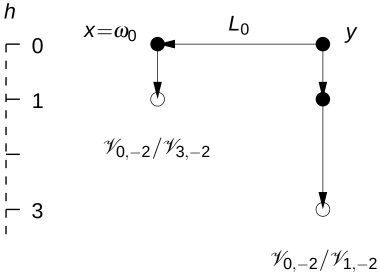
ϖ is completely determined by $\mathscr{H}^{\mathrm{L}}$, $\mathscr{H}^{\mathrm{R}}$, ω_1 and ω_2.

We call the pair (ω_1, ω_2) the data of the staggered module.

More Simple Consequences (cont.)

Corollary

If $\ell = 0$, then there is at most one staggered module $\mathscr{S}$ for any choice of left and right module (up to isomorphism).



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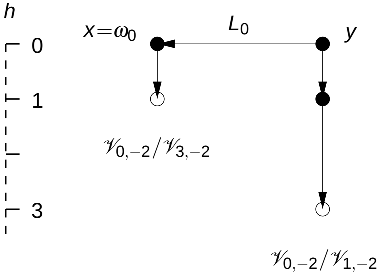

A staggered module with $\ell = 0$.

We can construct a wide variety of $\ell = 0$ staggered modules using:

- CFT methods (fusion),
- SLE techniques.

This proves existence in at least some cases.

Beta-Invariants

We normalise [Astashkevich, 9511032] the singular vector ω_0 so that it has the form

$$\omega_0 = Xx, \qquad \text{where} \qquad X = L_{-1}^{\ell} + \dots$$

Because $\omega_0 = (L_0 - h^{\mathrm{R}})y$, this normalises y too. But there is still freedom in choosing y. Replacing y by $y + u$ for $u \in \mathscr{H}^{\mathrm{L}}$ of conformal dimension h^{R} does not change the module structure. We call this a gauge transformation.

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Gauge invariant quantities should characterise the module, so when $\ell > 0$, define [Mathieu-Ridout 0708.0802] the beta-invariant β by

$$X^{\dagger} y = \beta x,$$

where † is the antiautomorphism of (the universal enveloping algebra of) $\mathfrak{vir}$ defined by

$$L_n^{\dagger} = L_{-n} \qquad \text{and} \qquad C^{\dagger} = C.$$

Even More Simple Consequences

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Unfortunately, the data of a staggered module is not gauge-invariant. Rather,

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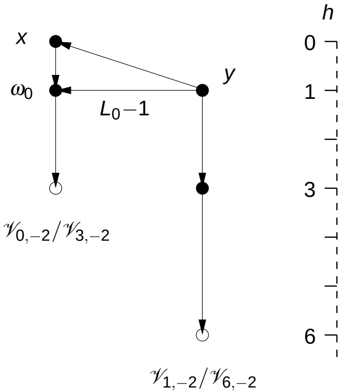
Proposition

Two staggered modules with the same left and right modules are isomorphic if and only if their data are gauge-equivalent.

Two Examples — #1



$$\omega_1 = \beta x, \qquad \omega_2 = 0.$$

Such a module arises in the "triplet" model [Gaberdiel–Kausch 9604026] where we measure $\beta = -1$. A similar module arises in the "abelian sandpile" model [Ruelle *et al* cond-mat/0609284] but there it seems that $\beta = \frac{1}{2}$.

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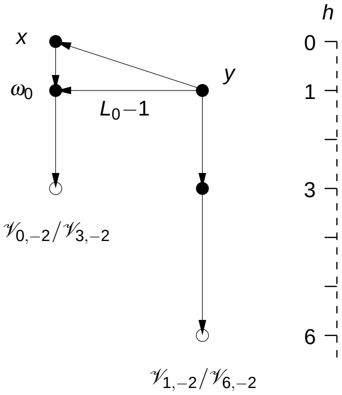

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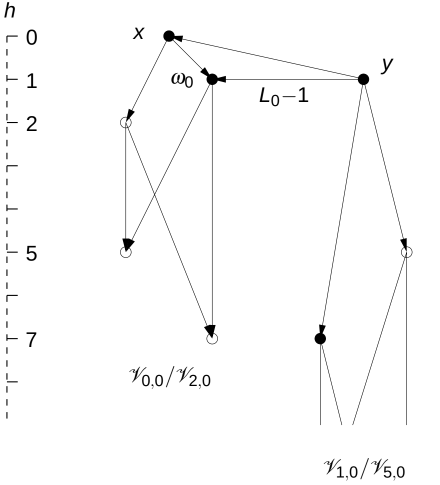
Gaberdiel and Kausch suggested that such a module exists for all β. Can we prove this?

Two Examples — #2



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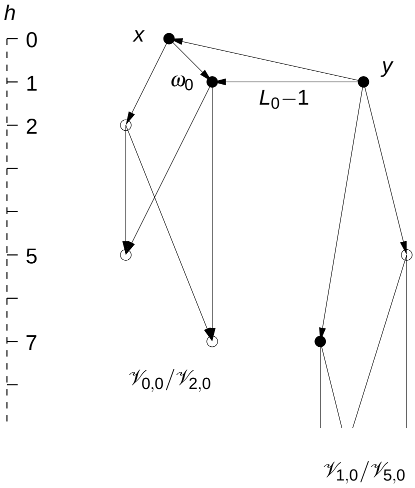

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However, this is the only β for which such a module exists [Mathieu–Ridout 0711.3541]. Why does this differ to the previous example?

Constructing Staggered Modules

To attack the existence question, we can try to construct a given staggered module explicitly as follows. Take $\mathscr{H}^{\mathrm{L}} \oplus \mathscr{U}$, where $\mathscr{U}$ is the universal enveloping algebra of $\mathfrak{vir}$, and quotient by the submodule $\mathscr{N}$ generated by

$$\left(\omega_0, h^{\mathrm{R}} - L_0\right), \quad \left(\omega_1, -L_1\right), \quad \left(\omega_2, -L_2\right), \quad \text{and} \quad \left(\varpi, -\overline{X}\right).$$

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Theorem

The quotient $\left(\mathscr{H}^{\mathrm{L}} \oplus \mathscr{U}\right) / \mathscr{N}$ *is a staggered module with left module* $\mathscr{H}^{\mathrm{L}}$ *and right module* $\mathscr{H}^{\mathrm{R}}$ *if and only if* $\mathscr{N} \cap \mathscr{H}^{\mathrm{L}} = \{0\}$.

[If $\mathscr{N} \cap \mathscr{H}^{\mathrm{L}}$ is non-trivial, the left module of the quotient will be a proper quotient module of $\mathscr{H}^{\mathrm{L}}$ (and may be itself trivial).]

Comparing Staggered Modules

Proposition

If $\mathscr{S}$ is a staggered module with left module $\mathscr{H}^{\mathrm{L}}$, right module $\mathscr{H}^{\mathrm{R}}$ and data (ω_1, ω_2), and $\mathscr{M}$ is a submodule of $\mathscr{H}^{\mathrm{L}}$ not containing ω_0, then there exists a staggered module $\hat{\mathscr{S}}$ with left module $\mathscr{H}^{\mathrm{L}} / \mathscr{M}$, right module $\mathscr{H}^{\mathrm{R}}$ and data $([\omega_1], [\omega_2])$.

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This suggests that we first study the case in which the right module is a Verma module.

What could go wrong?

Given left and right modules, which data (ω_1, ω_2) correspond to staggered modules? One way to rule out some data would be [Rohsiepe 9611160] if there is a $U \in \mathscr{U}$ such that

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Proposition

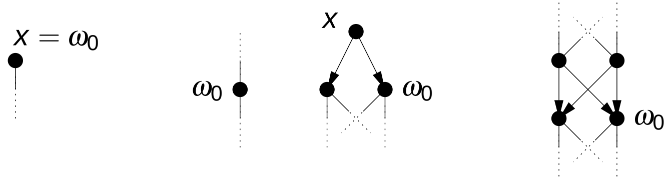
If the right module is Verma, this is the only way that data can fail to correspond to a staggered module.

The space $\mathbb{S}$ of staggered modules with right module Verma is then the vector space of data for which there are no such U (above) modulo the action of the gauge transformations.

Results

Theorem

The dimension of the vector space $\mathbb{S}$ of isomorphism classes of staggered modules with right module Verma is 0, 1 or 2 depending on where ω_0 appears in the singular vectors of $\mathscr{H}^{\mathrm{L}}$:



$\dim \mathbb{S} = 0$ $\dim \mathbb{S} = 1$ $\dim \mathbb{S} = 2$

Moreover, the beta-invariant β (or rather its generalisations) parametrise the vector space $\mathbb{S}$ (when $\dim \mathbb{S} > 0$).

Singular Vectors

As we understand staggered modules $\check{\mathscr{S}}$ with right module Verma, we now want to quotient this Verma module by a proper submodule to get the desired $\mathscr{H}^{\mathrm{R}}$. This submodule is generated by one or two singular vectors $w = \overline{X} x^{\mathrm{R}}$. The existence of the corresponding staggered module $\mathscr{S}$ is determined by the following:

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$\mathscr{S}$ exists (with the same left module and data as $\check{\mathscr{S}}$) if and only if each singular vector w "lifts" to a singular vector of $\check{\mathscr{S}}$.

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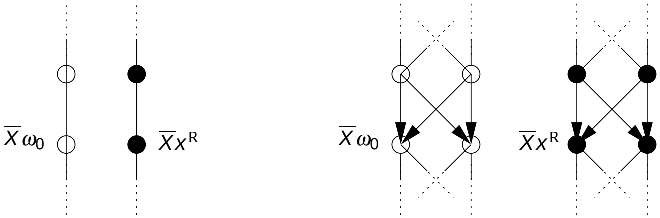
We can use this in our two examples to prove that in the former case, a staggered module exists for all β, whereas in the latter case, β is uniquely determined.

Singular Vectors (cont.)

Proposition

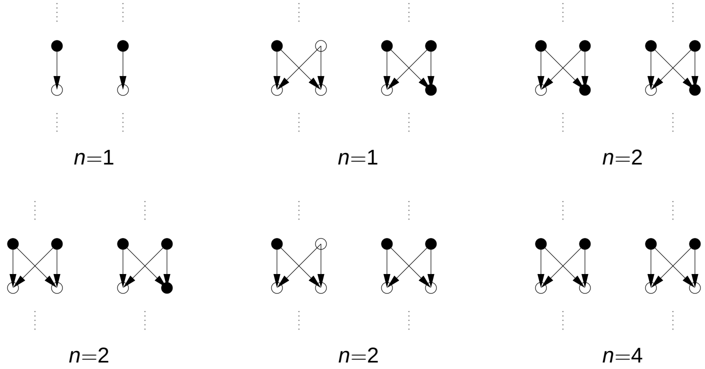
A singular vector $w = \overline{X} x^{\mathrm{R}}$ *of the Verma module* $\mathscr{V}_{h^{\mathrm{R}},c}$ *always lifts to a singular vector of* $\check{\mathscr{S}}$ *when* $\mathscr{H}^{\mathrm{L}}$ *and* $\mathscr{V}_{h^{\mathrm{R}},c}$ *have the following singular vector structures:*



Thus, in these cases $\dim \mathbb{S}$ *does not change when we replace* $\mathscr{V}_{h^{\mathrm{R}},c}$ *by its quotient* $\mathscr{H}^{\mathrm{R}}$*.*

What's left?

There are six remaining troublesome configurations of singular vectors for the left and right modules:



Generically, $\dim \mathbb{S} = \dim \check{\mathbb{S}} - n$ where n is as above and $\check{\mathbb{S}}$ refers to the space of staggered modules with right module Verma.

What's left (cont.)?

This expected correction to the dimension comes about because these configurations impose n inhomogeneous linear constraints upon the beta-invariants. But, we cannot prove that what we expect always comes to pass!

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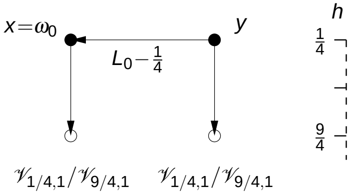

This configuration corresponds to $n = 1$ constraints, but there are no beta-invariants to constrain.

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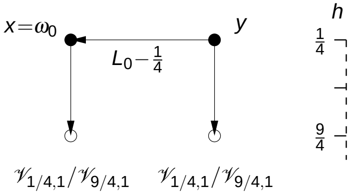

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Nevertheless, the module exists (and is unique): The vector

$$\left(L_{-1}^2 - L_{-2}\right) y - \frac{4}{3} L_{-2} x$$

is singular in the corresponding $\check{\mathcal{S}}$.

An annoying case!

Proposition

Let $\mathscr{H}^{\mathrm{L}}$ and $\mathscr{H}^{\mathrm{R}}$ be the same irreducible module. Then, the corresponding staggered module exists if and only if

- $h^{\mathrm{L}} = h^{\mathrm{R}} = \frac{1}{4}\left(r^2-1\right)t - \frac{1}{2}\left(rs-1\right) + \frac{1}{4}\left(s^2-1\right)t^{-1}$,
- $t = q/p \in \mathbb{Q}$ *with* $\gcd\{p,q\} = 1$,
- *p divides r, q divides s, and*
- $|p|s \neq |q|r$,

where the central charge is $c = 13 - 6\left(t + t^{-1}\right)$.

We can actually prove somewhat stronger results, but this covers pretty much all the known counterexamples to our "generic" expectations.

Summary

- The space $\mathbb{S}$ of isomorphism classes of staggered modules with given left and right modules is either empty or is an affine space of dimension 0, 1 or 2.
- This space is completely (up to a conjecture) determined by the singular vector structure of the left and right modules, and is parametrised by the (0, 1 or 2) beta-invariants (when this makes sense).
- $\mathbb{S}$ is empty unless ω_0 is a non-zero singular vector of the left module and $\overline{X} y = 0$ implies $\overline{X} \omega_0 = 0$ for all $\overline{X} \in \mathscr{U}$.
- $\dim \mathbb{S}$ is determined by imposing $n \in \{0, 1, 2, 4\}$ constraints upon the beta-invariants. We conjecture that these constraints are linearly independent except in the case where the left and right module are the same irreducible module (and simple generalisations of this case).

Outlook

- This gives researchers in LCFT and SLE a way to identify the staggered modules they encountered and check if staggered modules they propose actually exist.
- We would like to generalise this study to consider representations formed from more than two highest weight modules (composition series) and higher-rank Jordan cells.
- Similarly, staggered module theory needs to be developed for many other algebras of interest in CFT, *eg*. affine Kac-Moody algebras, their super-analogues, so-called W-algebras, and quantised enveloping algebras.
- We believe that the results reported here form a rigorous first step towards understanding the representation theory beyond the highest weight category. Such understanding is vital for future progress in modern mathematical physics.