Indecomposable Modules for the Virasoro Algebra

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May 2, 2011

K Kytola & DR, arXiv:0905.0108 [math-ph]

Crap version: JMP 50:123503, 2009.

Virasoro and CFT

The Virasoro algebra $\mathfrak{vir}$ is the most fundamental example of a chiral algebra in 2D conformal field theory:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C,$$
$$[L_m, C] = 0.$$

The quantum state space is built from $\mathfrak{vir}$-modules and the partition function $Z(q)$ is the trace of $q^{L_0 - C/24}$ (often multiplied by its antiholomorphic analogue) over this state space.

Applications to statistical models, string theory and pure mathematics are legion.

Virasoro Module Definitions

A module is irreducible if its only submodules are 0 and itself.

A module is indecomposable if it cannot be written as the direct sum of two (non-zero) submodules.

A highest weight state is an eigenstate of L_0 (and C) which is annihilated by L_n for $n > 0$.

A highest weight module is one that is generated, under the $\mathfrak{vir}$-action, by a highest weight state.

The modules relevant to the state space of a standard CFT are irreducible and highest weight. They are characterised by the conformal dimension (L_0-eigenvalue) h of their highest weight states and the central charge (C-eigenvalue) c.

Verma Modules

Even in standard CFT, it is useful to consider modules which need not be irreducible.

The paradigm

Null States in Representations	$\longleftrightarrow$	Differential Equations for Correlators

motivates the introduction of Verma modules.

Verma modules are indecomposable highest weight modules. Every highest weight module is a quotient of a Verma module.

The “null states” are the highest weight states which are set to zero upon forming the irreducible quotient.

As they generate a (maximal) submodule, the submodule structure of Verma modules is relevant to the computation of correlation functions in CFT.

Highest Weight Module Structure

Highest weight modules have a given central charge c and the states are graded by the conformal dimension. A basis is

$$L_{-n_1} L_{-n_2} \cdots L_{-n_k} v_{h,c} \quad (k \geqslant 0, n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1),$$

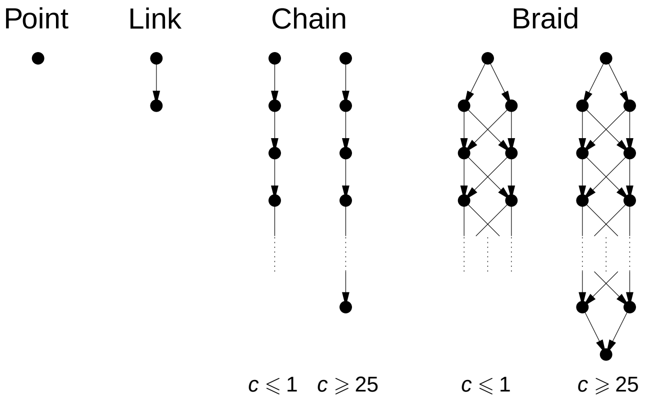
where $v_{h,c}$ is the (generating) highest weight state.

Every submodule of a Verma module is generated by at most two highest weight vectors called singular vectors.

Kac's determinant formula gives the conformal dimension of each singular vector in any given Verma module. There is at most one singular vector for each conformal dimension.

Feigin and Fuchs determined the submodule structure of Verma modules. There are several cases.

Feigin–Fuchs Classification of Verma Modules



Possibilities for the singular vector structure (black circles) of a Verma module. Arrows indicate that the latter vector is a descendant of the former and not vice-versa. Point and link-type modules occur for all central charges. Chain and braid-type modules occur only when a certain rationality condition is met.

Category $\mathscr{O}$

Mathematically, it is convenient to replace $\mathfrak{vir}$ by a specialisation of its universal enveloping algebra:

$$\mathscr{U} = \mathscr{U}_c = \frac{\mathscr{U}(\mathfrak{vir})}{\langle C - c\,\mathbf{1} \rangle}.$$

Denote the subalgebra generated by the L_n with $n > 0$ ($n < 0$) by $\mathscr{U}^+$ ($\mathscr{U}^-$).

Abstracting the properties of highest weight modules leads to the category $\mathscr{O}$, consisting of $\mathscr{U}$-modules V which are

- finitely generated,
- L_0-diagonalisable,
- $\mathscr{U}^+$-finite: *ie*. $\dim(\mathscr{U}^+ v) < \infty$ for all $v \in V$.

The morphisms are the usual module homomorphisms.

What does this buy us?

From the perspective of standard CFT: Not much!

Non-highest weight modules in $\mathscr{O}$ are relevant to advanced topics such as projective covers, BGG-resolutions, *etc*...

[In highest weight categories, some modules may fail to have a projective cover!]

This extension beyond the highest weight category streamlines (and beautifies) the analysis of the relationship between a Verma module and its irreducible quotient.

It is mathematically natural, but physically inessential.

However, advanced CFT requires a broader module category and category $\mathscr{O}$ is not quite big enough for the job.

Beyond Category $\mathscr{O}$

One of the important generalisations of standard CFT is logarithmic CFT.

This is characterised by a state space on which L_0 is not diagonalisable.

We are then led to consider a mild generalisation $\bar{\mathscr{O}}$ of category $\mathscr{O}$ in which L_0-diagonalisability is relaxed: L_0 is allowed to have Jordan cells of finite rank.

To motivate this relaxation mathematically, remember that CFT endows its module category with a product, the fusion product. Neither the highest weight category nor category $\mathscr{O}$ is closed under fusion. $\bar{\mathscr{O}}$ appears to be.

[Of course, one should do this with vertex algebras...]

Questions in Category $\bar{\mathscr{O}}$

From a logarithmic CFT perspective, we would like to understand this enhanced module category.

Every $V \in \mathscr{O}$ has a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$$

in which each V_i / V_{i-1} is a highest weight module.

- What do we need, along with these quotients, to characterise an indecomposable element of $\bar{\mathscr{O}}$?
- Is there an upper bound to the rank of a Jordan cell for L_0?
- Can we construct suitable projective covers (and injective hulls) for the irreducible modules of this category?

I think these questions are hard...

An easier question in $\bar{\mathscr{O}}$

We consider a first step towards addressing these questions: Classifying the indecomposables of $\bar{\mathscr{O}}$ which are built from two highest weight modules.

So, let $\mathscr{H}^{\mathrm{L}}$ and $\mathscr{H}^{\mathrm{R}}$ be highest weight modules, the left and right modules, respectively, and consider an exact sequence

$$0 \longrightarrow \mathscr{H}^{\mathrm{L}} \xrightarrow{\iota} \mathscr{S} \xrightarrow{\pi} \mathscr{H}^{\mathrm{R}} \longrightarrow 0.$$

We call $\mathscr{S}$ a staggered module (following Rohsiepe) if L_0 acts non-semisimply on $\mathscr{S}$.

This last condition is inessential but derives from the application to logarithmic CFT. The maximal rank for a Jordan cell of L_0 on $\mathscr{S}$ is 2.

This is not the same as asking for $\mathrm{Ext}^1_{\mathscr{U}}\left(\mathscr{H}^{\mathrm{R}}, \mathscr{H}^{\mathrm{L}}\right)$ in $\bar{\mathscr{O}}$.

Staggered Module Notation

Let $\mathscr{S} \in \bar{\mathscr{O}}$ be staggered with left and right modules $\mathscr{H}^{\mathrm{L}}$, $\mathscr{H}^{\mathrm{R}}$ with highest weight states x^{L}, x^{R} of conformal dimension and central charge h^{L}, h^{R} and $c^{\mathrm{L}} = c^{\mathrm{R}} \equiv c$.

The generalised eigenspaces of L_0 are finite-dimensional, so we can grade $\mathscr{S}$ by its generalised eigenvalues (conformal dimensions).

Let $x = \iota\left(x^{\mathrm{L}}\right)$, recalling that $\iota \colon \mathscr{H}^{\mathrm{L}} \hookrightarrow \mathscr{S}$.

Let y be an element of $\pi^{-1}\left(x^{\mathrm{R}}\right)$ of conformal dimension h^{R}, recalling that $\pi \colon \mathscr{S} \twoheadrightarrow \mathscr{H}^{\mathrm{R}}$.

Set $\omega_0 = \left(L_0 - h^{\mathrm{R}}\right) y$, $\omega_1 = L_1 y$ and $\omega_2 = L_2 y$.

[Since L_1 and L_2 generate $\mathscr{U}^+$, we need not consider $\omega_n = L_n y$ for $n > 0$.]

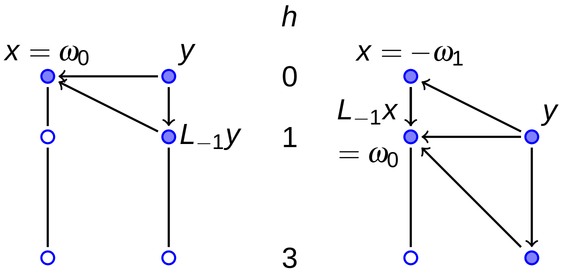
Basic Theory

- ω_0, ω_1 and ω_2 are elements of $\imath\left(\mathscr{H}^{\mathrm{L}}\right)$, and ω_0 is non-zero and singular.
- Staggered modules can only exist when $\ell \equiv h^{\mathrm{R}} - h^{\mathrm{L}} \in \mathbb{N}$. Moreover, if $\ell \in \mathbb{Z}_+$, then h^{L} must have the form

$$h^{\mathrm{L}} = \frac{r^2-1}{4} t - \frac{rs-1}{2} + \frac{s^2-1}{4} t^{-1}, \text{ where } c = 13 - 6\left(t + t^{-1}\right),$$

  for some $r, s \in \mathbb{Z}_+$.
- Indecomposable modules $\mathscr{S}$ with $\ell < 0$ can exist (reducible Verma modules are conspicuous examples), but they cannot be staggered (L_0 will be diagonalisable).

Examples I



$\mathscr{H}^{\mathrm{L}} = \frac{\mathscr{V}_0}{\mathscr{V}_1}$ $\mathscr{H}^{\mathrm{R}} = \frac{\mathscr{V}_0}{\mathscr{V}_3}$ $\mathscr{H}^{\mathrm{L}} = \frac{\mathscr{V}_0}{\mathscr{V}_3}$ $\mathscr{H}^{\mathrm{R}} = \frac{\mathscr{V}_1}{\mathscr{V}_6}$

These staggered modules arise in the description of the conformal limit of the loop model of critical dense polymers, a $c = -2$ logarithmic CFT.

More Basic Theory

If $\mathscr{H}^{\mathrm{R}}$ is not a Verma module, we have $\overline{X}x^{\mathrm{R}} = 0$ for some combination(s) $\overline{X} \in \mathscr{U}^{-}$. This defines vector(s) ϖ by $\varpi = \overline{X}y$.

- $\varpi = \overline{X}y$ is an element of $\iota\left(\mathscr{H}^{\mathrm{L}}\right)$ and $\overline{X}\omega_0 = 0$.
- ϖ is completely determined by $\mathscr{H}^{\mathrm{L}}$, $\mathscr{H}^{\mathrm{R}}$, ω_1 and ω_2.

The first result gives structural constraints upon a staggered module: If $\mathscr{H}^{\mathrm{R}}$ has a singular vector set to zero at dimension h, then the dimension h singular vector of $\mathscr{H}^{\mathrm{L}}$ must also be zero.

The second result implies that a staggered module is determined by its exact sequence and the pair (ω_1, ω_2). We call this pair the data of the staggered module.

When $h^{\mathrm{L}} = h^{\mathrm{R}}$, the data can only be $(0,0)$, hence a given exact sequence can result in *at most* one staggered module.

Isomorphic Staggered Modules

Staggered modules with different data can be isomorphic!

We normalise the singular vector ω_0 so that it has the form

$$\omega_0 = Xx, \qquad X = L_{-1}^{\ell} + \dots$$

This normalises y too, but does not fix it entirely. Replacing y by $y + u$ for $u \in \iota\left(\mathscr{H}^{\mathrm{L}}\right)$ of conformal dimension h^{R} does not change the isomorphism class of the module.

Such a replacement changes the data:

$$(\omega_1, \omega_2) \longrightarrow (\omega_1 + L_1 u, \omega_2 + L_2 u),$$

where $u \in \mathscr{H}^{\mathrm{L}}$ has conformal dimension h^{R}. This defines an equivalence relation on data pairs.

Staggered Module Invariants

- Two staggered modules with the same left and right modules are isomorphic iff their data are equivalent.

Which invariants characterise the isomorphism classes of staggered modules?

When $\ell > 0$, define the beta-invariant β by

$$X^\dagger y = \beta x,$$

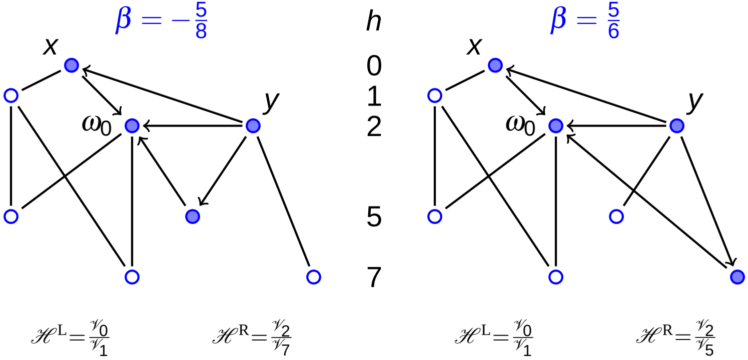
where $\omega_0 = Xx$ and $L_n^\dagger = L_{-n}$.

- The number β does not depend on the choice of y (it is invariant under $y \to y + u$). *ie*. isomorphic staggered modules have identical beta-invariants.

There can be up to two independent beta-invariants.

Examples II

Our previous $c = -2$ example had $\beta = -1$ (the first example has no β as ℓ was 0). A similar module arises in the abelian sandpile model, but there, β seems to be $\frac{1}{2}$.



These modules arise in the study of percolation and the self-avoiding random walk as $c = 0$ logarithmic CFTs.

Constructing Staggered Modules

To attack existence, note that $\mathscr{V}_h = \mathscr{U} / \langle L_0 - h\mathbf{1}, L_1, L_2 \rangle$.

We try to construct staggered modules with given $\mathscr{H}^{\mathrm{L}}$ and $\mathscr{H}^{\mathrm{R}}$ as $\left(\mathscr{H}^{\mathrm{L}} \oplus \mathscr{U}\right) / \mathscr{N}$, where

$$\mathscr{N} = \left\langle \left(-\omega_0, L_0 - h^{\mathrm{R}}\mathbf{1}\right), \left(-\omega_1, L_1\right), \left(-\omega_2, L_2\right), \left(-\varpi, \overline{X}\right) \right\rangle.$$

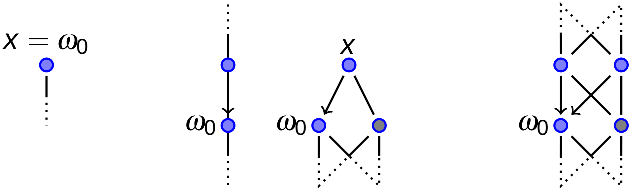
- The quotient $\left(\mathscr{H}^{\mathrm{L}} \oplus \mathscr{U}\right) / \mathscr{N}$ is staggered with left module $\mathscr{H}^{\mathrm{L}}$ and right module $\mathscr{H}^{\mathrm{R}}$ iff $\mathscr{N} \cap \mathscr{H}^{\mathrm{L}} = \{0\}$.
- Every staggered module is a quotient of a staggered module with the same left module, the same data, but the right module Verma.

If $\mathscr{N} \cap \mathscr{H}^{\mathrm{L}}$ is non-trivial, the left module of the quotient will be a proper quotient module of $\mathscr{H}^{\mathrm{L}}$ (maybe trivial).

Results ($\mathscr{H}^{\mathrm{R}}$ Verma)

Theorem

The dimension of the vector space $\mathbb{S}$ of isomorphism classes of staggered modules with right module Verma is 0, 1 or 2 depending on where ω_0 appears in the singular vectors of $\mathscr{H}^{\mathrm{L}}$:



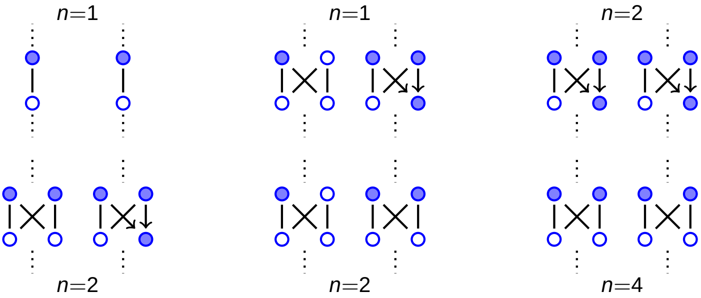
$\dim \mathbb{S} = 0$ $\dim \mathbb{S} = 1$ $\dim \mathbb{S} = 2$

Moreover, the beta-invariant β (or rather its generalisations) parametrise the vector space $\mathbb{S}$ (when $\dim \mathbb{S} > 0$).

Results (General $\mathscr{H}^{\mathrm{R}}$)

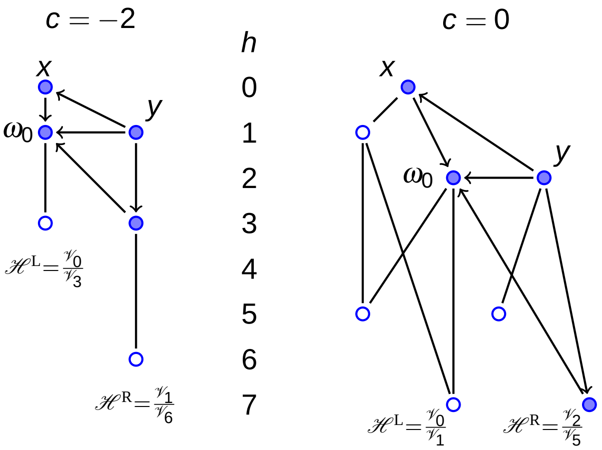
Theorem

The dimension of the vector space $\mathbb{S}$ of isomorphism classes of staggered modules matches the dimension when $\mathscr{H}^{\mathrm{R}}$ is Verma except when the singular vector structure of $\mathscr{H}^{\mathrm{L}}$ and $\mathscr{H}^{\mathrm{R}}$ is:



In these six cases, the dimension is generically reduced by n (and negative dimensions indicate non-existence).

Examples III



$\dim \mathbb{S} = 1$, $\beta \in \mathbb{C}$.

$\dim \mathbb{S} = 0$, $\beta = \frac{5}{6}$.

An annoying case!

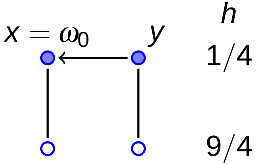
Proposition

Let $\mathscr{H}^{\mathrm{L}} = \mathscr{H}^{\mathrm{R}}$ be irreducible but not Verma. Then, the corresponding staggered module exists (and is unique) iff

$$h^L = h^R = \frac{1}{4}\left(r^2 - 1\right)t - \frac{1}{2}(rs - 1) + \frac{1}{4}\left(s^2 - 1\right)t^{-1},$$

where $c = 13 - 6\left(t + t^{-1}\right)$ and $t = q/p \in \mathbb{Q}$ with $\gcd\{p, q\} = 1$, $p \mid r$, $q \mid s$ and $|p|s \neq |q|r$.

eg. a staggered module exists with $c = 1$ and $\mathscr{H}^{\mathrm{L}} = \mathscr{H}^{\mathrm{R}} = \frac{1}{4}$, despite the fact that our generic expectations forbid it.



These are essentially the only known counterexamples to our generic expectations.

Summary

- The space $\mathbb{S}$ of isomorphism classes of staggered modules with given left and right modules is either empty or is an affine space of dimension 0, 1 or 2.
- This space is determined by the singular vector structure of the left and right modules, and is parametrised by the beta-invariants.
- These results answer the question of what is needed to characterise the staggered modules in category $\bar{\mathscr{O}}$.
- They therefore give researchers in logarithmic CFT (and SLE) the means to identify the staggered modules they encounter and check if the staggered modules they propose actually exist.
- These results also allow one to determine which highest weight modules are projective in $\bar{\mathscr{O}}$.

Outlook

- We need to generalise this study to consider indecomposables formed from more than two highest weight modules, especially those with rank 3 Jordan cells.
- Similar results need to be developed for many other algebras of interest in CFT, *eg*. affine Kac-Moody algebras and superalgebras, so-called W-algebras, and quantised enveloping algebras.
- Projective covers of irreducibles appear to be fundamental building blocks of bulk logarithmic CFTs. Our work gives a first step towards understanding these rigorously.
- A mathematical characterisation of the modules relevant to a description of a boundary logarithmic CFT is currently lacking. Results generalising the above will be very useful for explorations of this topic.
- Clearly, there is much to do...