# $\widehat{\mathfrak{sl}}(2)_{-1/2}$, $\beta\gamma$ Ghosts and Logarithmic CFT

David Ridout

DESY Theory Group;
Centre de Recherches Mathématiques, Montréal;
Department of Theoretical Physics
& Mathematical Sciences Institute,
Australian National University

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# History

WZW models describe non-critical strings propagating on a suitable (compact) Lie group. For the Feynman amplitudes to be single-valued, the level $k$ should be integral.

These are unitary, which can be used to prove the unitarity of their coset theories (*eg.* $\mathcal{M}(k+2, k+3)$).

Fractional level theories were postulated in order to provide a coset construction of the non-unitary minimal models:

$$\mathcal{M}(p,q) \sim \frac{\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{sl}}(2)_1}{\widehat{\mathfrak{sl}}(2)_{k+1}} \quad \text{if } k = \frac{2q-3p}{p-q}.$$

This makes sense at the level of chiral algebra representations. Do fractional level models exist as conformal field theories? If so, what is their nature?

Kac and Wakimoto discovered that at the required fractional levels, there are a finite number of admissible irreducibles whose characters carry a representation of $\mathrm{SL}(2;\mathbb{Z})$.

This led to many attempts to "construct" fractional level models from these irreducibles, *eg.* Koh-Sorba, Bernard-Felder, Mathieu-Walton, Awata-Yamada, Ramgoolam, Feigin-Malikov, Andreev, ...

There were a few problems:

1. The Verlinde formula gave negative fusion coefficients.
2. The admissible irreducibles did not close under conjugation.
3. Other methods of computing fusion rules gave different fusion coefficients (with their own problems).

Many "solutions" proclaimed — but none were universally agreed upon. CFT textbooks regarded the fractional level theories as "intrinsically sick".

# Logarithmic CFT to the Rescue!

Gaberdiel used the NGK algorithm to compute the fusion rules at $k = -\frac{4}{3}$. The results were that:

- Fusion does not close on the admissible irreducibles.
- An infinite number of distinct irreducibles are generated.
- Almost all of these irreducibles do not have a lower bound to the conformal dimensions of their states.
- Fusion also generates indecomposables, leading to a logarithmic CFT structure.

Lesage, Mathieu, Rasmussen and Saleur then proposed a similar story for $k = -\frac{1}{2}$, except that the admissibles do not then generate indecomposables (*ie.* no logarithmic structure).

However, they also proposed a “logarithmic lift” in which indecomposables are (naturally) put in by hand.

# Why should we care?

Fractional level theories give another source of logarithmic CFTs beyond superalgebras and W-algebras. Here, the algebras are familiar, even if the required representation theory may not be.

These models were supposed to be fundamental building blocks for rational non-unitary CFTs. Perhaps they may also represent fundamental building blocks for quasi-rational logarithmic CFTs.

Non-compact WZW studies, *eg*. on $SL(2;\mathbb{R})$ or $AdS_3$, may benefit from fractional level results: Can indecomposables be avoided in the absence of unitarity?

We shall see that fractional level theories can enjoy some of the properties of non-compact theories (*eg*. continuous spectrum) while maintaining quasi-rationality.

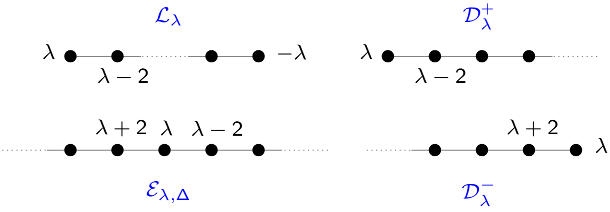
Modularity!

# $\mathfrak{sl}(2)$ and its Representations

This is the Lie algebra of traceless $2 \times 2$ matrices. A convenient basis is $e = \frac{1}{2}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$, $h = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$, $f = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}$, giving a triangular decomposition of $\mathfrak{sl}(2;\mathbb{R})$. Note:

$$\left[h, e\right] = 2e, \qquad \left[e, f\right] = -h, \qquad \left[h, f\right] = -2f.$$

The (weight) representations fall into four classes: Those with a highest weight state ($e\left|v\right\rangle = 0$), those with a lowest weight state ($f\left|w\right\rangle = 0$), those with both and those with neither.

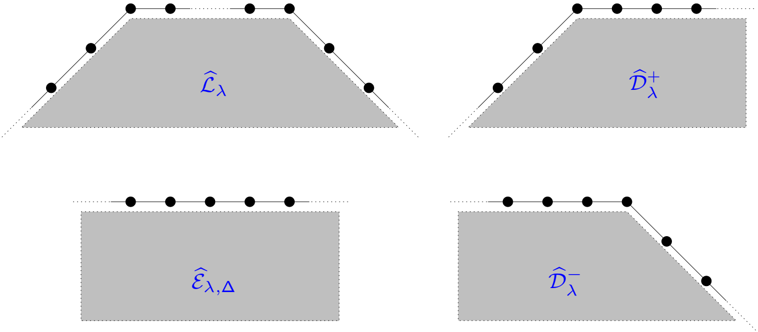


# The Affine Kac-Moody Algebra $\widehat{\mathfrak{sl}}(2)$

The affinisation comes with the standard Sugawara construction:

$$T(z) = \frac{1}{2(k+2)} : h(z)\,h(z) - e(z)\,f(z) - f(z)\,e(z) : .$$

One gets irreducible $\widehat{\mathfrak{sl}}(2)$-modules by inducing those of $\mathfrak{sl}(2)$ and quotienting by the maximal submodule:



# Automorphisms of $\widehat{\mathfrak{sl}}(2)$

The Weyl reflection of $\mathfrak{sl}(2)$ lifts to conjugation:

$$\mathsf{w}(e_n) = f_n, \quad \mathsf{w}(h_n) = -h_n, \quad \mathsf{w}(f_n) = e_n,$$
$$\mathsf{w}(K) = K, \quad \mathsf{w}(L_0) = L_0.$$

Meanwhile, the dual root translations generate the spectral flow automorphisms $\sigma^\ell$ ($\ell \in \mathbb{Z}$):

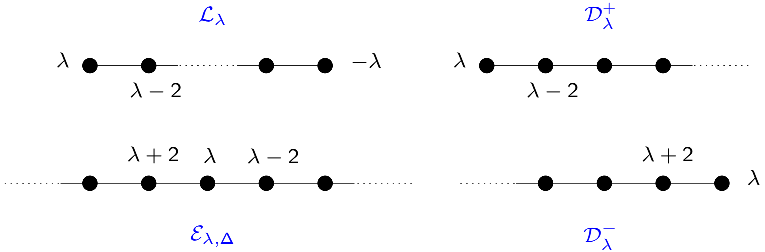
$$\sigma^\ell(e_n) = e_{n-\ell}, \quad \sigma^\ell(h_n) = h_n - \frac{1}{2}\ell\delta_{n,0}, \quad \sigma^\ell(f_n) = f_{n+\ell},$$
$$\sigma^\ell(K) = K, \quad \sigma^\ell(L_0) = L_0 + \tfrac{1}{2}\ell h_0 + \tfrac{1}{4}\ell^2 K.$$

$\sigma^2$ generates the coroot translations of the affine Weyl group.

# Twisted Representations

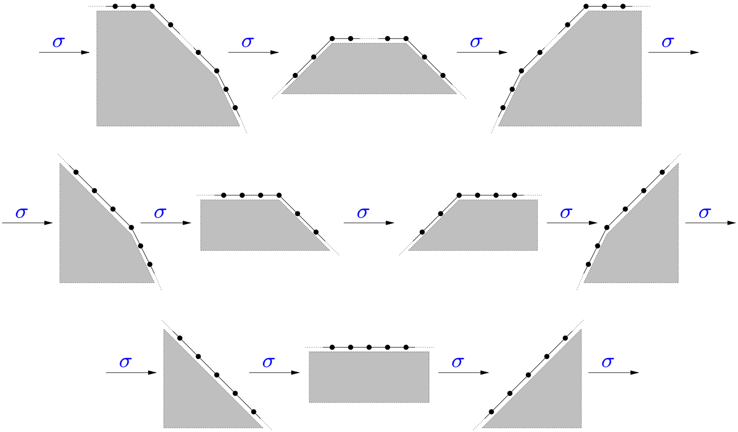
Twisting a representation by w amounts to taking the conjugate representation. For $\mathfrak{sl}(2)$, this gives

$$\mathcal{L}_\lambda \longleftrightarrow \mathcal{L}_\lambda, \qquad \mathcal{E}_{\lambda,\Delta} \longleftrightarrow \mathcal{E}_{-\lambda,\Delta}, \qquad \mathcal{D}^+_\lambda \longleftrightarrow \mathcal{D}^-_{-\lambda}.$$




The induced $\widehat{\mathfrak{sl}}(2)$-modules behave identically.

Twisting our induced $\widehat{\mathfrak{sl}}(2)$-modules by $\sigma$ is far less trivial!




We get infinitely many distinct representations, most of which have conformal dimensions which are unbounded below.

# Constructions at $k = -\frac{1}{2}$ $(c = -1)$

This level is interesting because:

- The $\beta\gamma$ ghost system has $\widehat{\mathfrak{sl}}(2)_{-1/2}$ symmetry.
- LMRS suggest that this theory is non-logarithmic (unlike $k = -\frac{4}{3}$).
- LMRS suggest that the logarithmic structure of symplectic fermions should "lift" to this theory.
- The issues with the Verlinde formula may be examined explicitly.

More to the point, the theory is sufficiently "small" that one can expect to be able to analyse it algebraically without getting overwhelmed by details.

We begin with the (irreducible) vacuum module $\widehat{\mathcal{L}}_0$. Since

$$\left(156e_{-3}e_{-1}-71e_{-2}^2+44e_{-2}h_{-1}e_{-1}-52h_{-2}e_{-1}^2+16f_{-1}e_{-1}^3-4h_{-1}^2e_{-1}^2\right)\left|0\right\rangle=0,$$

the state-field correspondence (or Zhu's algebra) restricts the "allowed modules" (among the "relaxed" highest weight modules) to the irreducibles

$$\widehat{\mathcal{L}}_0, \quad \widehat{\mathcal{L}}_1, \quad \widehat{\mathcal{D}}_{-1/2}^+, \quad \widehat{\mathcal{D}}_{-3/2}^+, \quad \widehat{\mathcal{D}}_{1/2}^-, \quad \widehat{\mathcal{D}}_{3/2}^-, \quad \widehat{\mathcal{E}}_{\lambda,-1/8}.$$

The highest weight modules appearing, $\widehat{\mathcal{L}}_0$, $\widehat{\mathcal{L}}_1$, $\widehat{\mathcal{D}}_{-1/2}^+$ and $\widehat{\mathcal{D}}_{-3/2}^+$, are the admissibles of Kac-Wakimoto.

For the $\widehat{\mathcal{E}}_{\lambda,-1/8}$, any $\lambda$ is allowed. However, $\lambda = \frac{1}{2}, \frac{3}{2}$ do not give irreducibles. Rather, one gets four allowed indecomposables corresponding to the four ways of coupling $\widehat{\mathcal{D}}_{\mp 1/2}^{\pm}$ with $\widehat{\mathcal{D}}_{\pm 3/2}^{\mp}$.

The conformal dimensions of the zero-grade states of $\widehat{\mathcal{L}}_0$ and $\widehat{\mathcal{L}}_1$ are 0 and $\frac{1}{2}$. For the other modules, such states have conformal dimension $-\frac{1}{8}$.

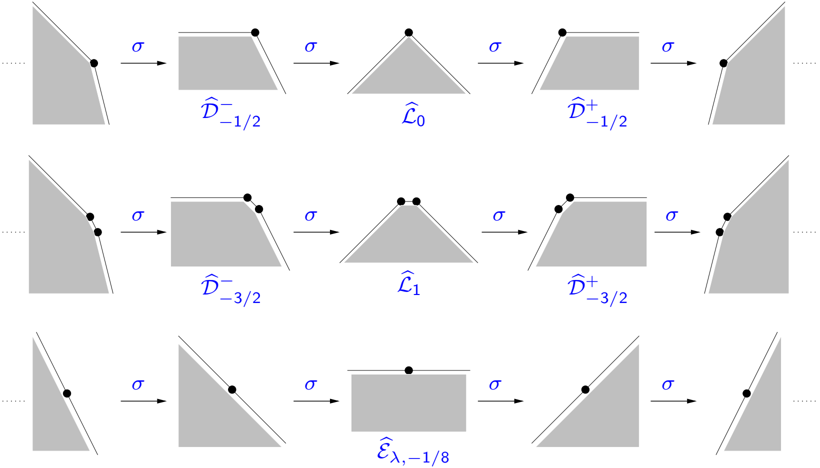
The set of allowed modules is closed under conjugation, unlike the set of admissible modules.

This set of allowed modules does not close under spectral flow! But,

$$\widehat{\mathcal{D}}^-_{1/2} \xrightarrow{\sigma} \widehat{\mathcal{L}}_0 \xrightarrow{\sigma} \widehat{\mathcal{D}}^+_{-1/2} \quad \text{and} \quad \widehat{\mathcal{D}}^-_{3/2} \xrightarrow{\sigma} \widehat{\mathcal{L}}_1 \xrightarrow{\sigma} \widehat{\mathcal{D}}^+_{-3/2},$$

suggesting that their twisted spectral flow images should also be allowed. *ie.* we should work in the category of twisted relaxed highest weight modules.

The following picture summarises the spectrum (of irreducibles):



# A Minimal Theory

We can try to construct a minimal CFT whose spectrum is generated by the admissibles and their conjugates.

We test for closure under fusion using the algorithm of Nahm and Gaberdiel-Kausch. We compute (carefully) that

$$\widehat{\mathcal{L}}_0 \times \widehat{\mathcal{L}}_0 = \widehat{\mathcal{L}}_0, \qquad \widehat{\mathcal{L}}_0 \times \widehat{\mathcal{L}}_1 = \widehat{\mathcal{L}}_1, \qquad \widehat{\mathcal{L}}_1 \times \widehat{\mathcal{L}}_1 = \widehat{\mathcal{L}}_0.$$

Since spectral flow is expected to satisfy

$$\sigma^{\ell_1}(\mathcal{M}) \times \sigma^{\ell_2}(\mathcal{N}) = \sigma^{\ell_1+\ell_2}(\mathcal{M} \times \mathcal{N})$$

(and we can check this for small $\ell_1$, $\ell_2$), we conclude that fusion closes on the twisted $\widehat{\mathcal{L}}_\lambda$.

This includes all the allowed modules except the (twisted) $\widehat{\mathcal{E}}_{\lambda,-1/8}$.

# Ghosts!

The only candidates we have for the $\beta\gamma$ ghost fields are those corresponding to the dimension $\frac{1}{2}$ states of $\widehat{\mathcal{L}}_1$.

The fusion rules indicate that $\widehat{\mathcal{L}}_1$ is a simple current, so there is an algorithmic procedure to compute the extension of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ by these candidate ghost fields.

The result is, of course, the $\beta\gamma$ ghost algebra. However, this is only strictly true if we give $\widehat{\mathfrak{sl}}(2)_{-1/2}$ the adjoint derived from the real form $\mathfrak{sl}(2;\mathbb{R})$.

If we use instead the adjoint coming from the real form $\mathfrak{su}(2)$, then the extended algebra fails to be associative.

The $\beta\gamma$ irreducibles are composed of the orbits of $\widehat{\mathcal{L}}_1$ under fusion (there are no fixed points — the orbits always have length two).

# Modular Properties

The characters of the admissibles have the following form:

$$\chi_{\widehat{\mathcal{L}}_0} = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4\left(z;q\right)} + \frac{\eta(q)}{\vartheta_3\left(z;q\right)}\right] \qquad \chi_{\widehat{\mathcal{L}}_1} = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4\left(z;q\right)} - \frac{\eta(q)}{\vartheta_3\left(z;q\right)}\right]$$

$$\chi_{\widehat{\mathcal{D}}^+_{-1/2}} = \frac{1}{2}\left[\frac{-\mathrm{i}\eta(q)}{\vartheta_1\left(z;q\right)} + \frac{\eta(q)}{\vartheta_2\left(z;q\right)}\right] \quad \chi_{\widehat{\mathcal{D}}^+_{-3/2}} = \frac{1}{2}\left[\frac{-\mathrm{i}\eta(q)}{\vartheta_1\left(z;q\right)} - \frac{\eta(q)}{\vartheta_2\left(z;q\right)}\right].$$

They span a (reducible) representation of $\mathsf{SL}\left(2;\mathbb{Z}\right)$:

$$S = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & \mathrm{i} & \mathrm{i} \\ -1 & -1 & \mathrm{i} & \mathrm{i} \end{pmatrix} \qquad T = \begin{pmatrix} e^{\mathrm{i}\pi/12} & 0 & 0 & 0 \\ 0 & -e^{\mathrm{i}\pi/12} & 0 & 0 \\ 0 & 0 & e^{-\mathrm{i}\pi/6} & 0 \\ 0 & 0 & 0 & e^{-\mathrm{i}\pi/6} \end{pmatrix}.$$

Both $S$ and $T$ are symmetric and unitary.

The characters of the other twisted modules are obtained by spectral flow. It turns out that we have a periodicity of the form

$$\cdots \xrightarrow{\sigma} -\chi_{\widehat{\mathcal{L}}_1} \xrightarrow{\sigma} -\chi_{\widehat{\mathcal{D}}^+_{-3/2}} \xrightarrow{\sigma} \chi_{\widehat{\mathcal{L}}_0} \xrightarrow{\sigma} \chi_{\widehat{\mathcal{D}}^+_{-1/2}} \xrightarrow{\sigma} -\chi_{\widehat{\mathcal{L}}_1} \xrightarrow{\sigma} \cdots$$

$$\cdots \xrightarrow{\sigma} -\chi_{\widehat{\mathcal{L}}_0} \xrightarrow{\sigma} -\chi_{\widehat{\mathcal{D}}^+_{-1/2}} \xrightarrow{\sigma} \chi_{\widehat{\mathcal{L}}_1} \xrightarrow{\sigma} \chi_{\widehat{\mathcal{D}}^+_{-3/2}} \xrightarrow{\sigma} -\chi_{\widehat{\mathcal{L}}_0} \xrightarrow{\sigma} \cdots$$

There are only four linearly independent characters!

Soyez Prudent! For this claim, we must analytically extend the characters to meromorphic functions of $z$.

As power series, $\chi_{\sigma^\ell(\widehat{\mathcal{L}}_\lambda)}(z;q)$ converges for

$$|q| < 1 \quad \text{and} \quad |q|^{(-\ell+1)/2} < |z| < |q|^{(-\ell-1)/2}.$$

These annuli of convergence are not conserved under modular transformations, hence the above meromorphic continuation.

The map from the modules to the characters is not 1–1. Its kernel is spanned by the modules $\sigma^{\ell\pm1}(\widehat{\mathcal{L}}_0) \oplus \sigma^{\ell\mp1}(\widehat{\mathcal{L}}_1)$ and these form an ideal of the fusion ring.

*ie*. fusion induces a well-defined product $\boxtimes$ on the span of the characters. The result is the Grothendieck ring of characters.

| $\boxtimes$ | $\chi_{\widehat{\mathcal{L}}_0}$ | $\chi_{\widehat{\mathcal{L}}_1}$ | $\chi_{\widehat{\mathcal{D}}^+_{-1/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-3/2}}$ |
|---|---|---|---|---|
| $\chi_{\widehat{\mathcal{L}}_0}$ | $\chi_{\widehat{\mathcal{L}}_0}$ | $\chi_{\widehat{\mathcal{L}}_1}$ | $\chi_{\widehat{\mathcal{D}}^+_{-1/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-3/2}}$ |
| $\chi_{\widehat{\mathcal{L}}_1}$ | $\chi_{\widehat{\mathcal{L}}_1}$ | $\chi_{\widehat{\mathcal{L}}_0}$ | $\chi_{\widehat{\mathcal{D}}^+_{-3/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-1/2}}$ |
| $\chi_{\widehat{\mathcal{D}}^+_{-1/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-1/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-3/2}}$ | $-\chi_{\widehat{\mathcal{L}}_1}$ | $-\chi_{\widehat{\mathcal{L}}_0}$ |
| $\chi_{\widehat{\mathcal{D}}^+_{-3/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-3/2}}$ | $\chi_{\widehat{\mathcal{D}}^+_{-1/2}}$ | $-\chi_{\widehat{\mathcal{L}}_0}$ | $-\chi_{\widehat{\mathcal{L}}_1}$ |

Now recall that $S^2$ is supposed to be conjugation:

$$S^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} \chi_{\mathsf{w}\left(\widehat{\mathcal{L}}_0\right)} = \chi_{\widehat{\mathcal{L}}_0} \quad \chi_{\mathsf{w}\left(\widehat{\mathcal{L}}_1\right)} = \chi_{\widehat{\mathcal{L}}_1} \\ \chi_{\mathsf{w}\left(\widehat{\mathcal{D}}^+_{-1/2}\right)} = \chi_{\widehat{\mathcal{D}}^-_{1/2}} = -\chi_{\widehat{\mathcal{D}}^+_{-3/2}} \\ \chi_{\mathsf{w}\left(\widehat{\mathcal{D}}^+_{-3/2}\right)} = \chi_{\widehat{\mathcal{D}}^-_{3/2}} = -\chi_{\widehat{\mathcal{D}}^+_{-1/2}}. \end{cases}$$

The negative signs simply reflect the fact that $S$ sees only the characters, not the modules. The negative "fusion coefficients" resulting from applying the Verlinde formula are explained the same way: They are the structure constants of the Grothendieck ring of characters!

*eg*. $\chi_{\widehat{\mathcal{D}}^+_{-3/2}} \boxtimes \chi_{\widehat{\mathcal{D}}^+_{-3/2}} = -\chi_{\widehat{\mathcal{L}}_1}$ and

$$\mathbf{N}_{\widehat{\mathcal{D}}^+_{-3/2}\,\widehat{\mathcal{D}}^+_{-3/2}}{}^{\widehat{\mathcal{L}}_1} = \sum_{\mathcal{M}} \frac{S_{\widehat{\mathcal{D}}^+_{-3/2}\,\mathcal{M}} S_{\widehat{\mathcal{D}}^+_{-3/2}\,\mathcal{M}} S^*_{\widehat{\mathcal{L}}_1\,\mathcal{M}}}{S_{\widehat{\mathcal{L}}_0\,\mathcal{M}}} = -1.$$

# Where are the Logs?

The field $h(z)$ generates a $\widehat{\mathfrak{u}}(1)$-subalgebra, so the coset

$$\frac{\widehat{\mathfrak{sl}}(2)_{-1/2}}{\widehat{\mathfrak{u}}(1)}$$

has $c = -2$. The coset algebra is the $W(1,2)$ triplet algebra of Gaberdiel and Kausch (and the coset algebra for the ghosts is symplectic fermions).

The correspondence at the level of modules is extremely simple:

$$\left.\begin{array}{lcr} \sigma^{\ell}(\widehat{\mathcal{L}}_0) \longrightarrow \mathcal{W}_0 & \text{singlet} \\ \sigma^{\ell}(\widehat{\mathcal{L}}_1) \longrightarrow \mathcal{W}_1 & \text{doublet} \end{array}\right\} \quad W(1,2)\text{-irreducibles.}$$

But where are the singlet $\mathcal{W}_{-1/8}$ and the doublet $\mathcal{W}_{3/8}$?

To obtain the remaining $W(1,2)$-irreducibles, we must look beyond the admissibles and their twisted images. Indeed,

$$\begin{aligned} \sigma^{\ell}\big(\widehat{\mathcal{E}}_{0,-1/8}\big) &\longrightarrow \mathcal{W}_{-1/8} && \text{singlet} \\ \sigma^{\ell}\big(\widehat{\mathcal{E}}_{1,-1/8}\big) &\longrightarrow \mathcal{W}_{3/8} && \text{doublet.} \end{aligned}$$

We are therefore required to augment our spectrum!

The remaining allowed modules (the $\sigma^{\ell}\big(\widehat{\mathcal{E}}_{\lambda,-1/8}\big)$ with $\lambda \notin \mathbb{Z}$) give $W(1,2)$-modules with non-integral monodromy.

Just as the triplet model may be viewed as a $\mathbb{Z}_2$-orbifold of symplectic fermions, it seems that we should take our $\widehat{\mathfrak{sl}}(2)_{-1/2}$ spectrum to correspond to a $\mathbb{Z}_2$-orbifold of the $\beta\gamma$ ghosts.

# Logarithmic Structure at Last

We compute the remaining fusion rules again using NGK:

$$\widehat{\mathcal{L}}_0 \times \widehat{\mathcal{E}}_{0,-1/8} = \widehat{\mathcal{E}}_{0,-1/8} \qquad \widehat{\mathcal{L}}_1 \times \widehat{\mathcal{E}}_{0,-1/8} = \widehat{\mathcal{E}}_{1,-1/8}$$
$$\widehat{\mathcal{L}}_0 \times \widehat{\mathcal{E}}_{1,-1/8} = \widehat{\mathcal{E}}_{1,-1/8} \qquad \widehat{\mathcal{L}}_1 \times \widehat{\mathcal{E}}_{1,-1/8} = \widehat{\mathcal{E}}_{0,-1/8}.$$

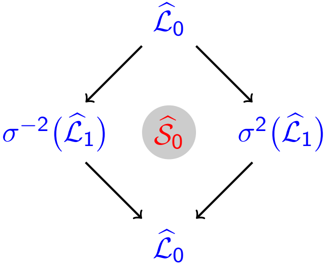
Indecomposables appear as expected (since they do for $W(1,2)$):

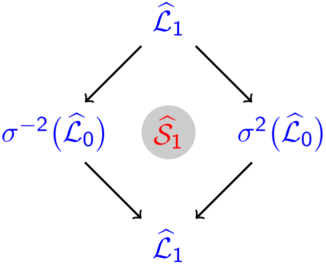
$$\widehat{\mathcal{E}}_{0,-1/8} \times \widehat{\mathcal{E}}_{0,-1/8} = \widehat{\mathcal{S}}_0 \qquad \widehat{\mathcal{E}}_{1,-1/8} \times \widehat{\mathcal{E}}_{1,-1/8} = \widehat{\mathcal{S}}_0$$
$$\widehat{\mathcal{E}}_{0,-1/8} \times \widehat{\mathcal{E}}_{1,-1/8} = \widehat{\mathcal{S}}_1.$$

Indeed, the $\widehat{\mathcal{S}}_\lambda$ are staggered, meaning that they may be realised as extensions of (twisted relaxed) highest weight modules on which $L_0$ acts non-diagonalisably.

$$0 \longrightarrow \sigma^{-1}\left(\widehat{\mathcal{E}}^{+}_{-1/2,-1/8}\right) \longrightarrow \widehat{\mathcal{S}}_0 \longrightarrow \sigma\left(\widehat{\mathcal{E}}^{+}_{-3/2,-1/8}\right) \longrightarrow 0$$

$$0 \longrightarrow \sigma^{-1}\left(\widehat{\mathcal{E}}^{+}_{-3/2,-1/8}\right) \longrightarrow \widehat{\mathcal{S}}_1 \longrightarrow \sigma\left(\widehat{\mathcal{E}}^{+}_{-1/2,-1/8}\right) \longrightarrow 0$$



# Logarithmic Couplings

One should ask if these staggered modules are completely specified by their structure diagrams, or whether further information (beta-invariants/logarithmic couplings) are required.

For $\widehat{\mathcal{S}}_0$, inspection shows that one must determine the action of $e_1$ and $f_1$ on a certain state. By symmetry, this reduces to the evaluation of a single scalar. However, the normalisation of the Jordan cell fixes it — $\widehat{\mathcal{S}}_0$ has no logarithmic couplings.

For $\widehat{\mathcal{S}}_1$, one must determine the action of $e_0$ and $f_1$ on the corresponding state. This requires evaluating six scalars. Algebraic consistency and normalisation reduce this to a single scalar. However, the space of gauge transformations, which describe the non-trivial choices for the reference state, is one — likewise $\widehat{\mathcal{S}}_1$ has no logarithmic couplings.

# Conclusions

We have seen that at $k = -\frac{1}{2}$, there is a subset of the allowed modules which is "modular invariant" and closed under fusion.

This closure requires irreducibles that are not bounded below. However, the problematic negative integers given by conjugation and the Verlinde formula have been explained.

We may have to augment the spectrum, leading to a logarithmic CFT with staggered modules that are not bounded below. The $\beta\gamma$ ghost theory is likewise logarithmic. The parafermionic coset of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ ($\beta\gamma$ ghosts) is $W(1,2)$ (symplectic fermions).

The $\widehat{\mathcal{E}}_{\lambda,-1/8}$ and $\widehat{\mathcal{S}}_{\mu}$ form an ideal of the fusion ring. Quotienting (*ie.* setting their characters to zero) recovers modularity.

One can augment by all the $\widehat{\mathcal{E}}_{\lambda,-1/8}$, $\lambda \notin \mathbb{Z} + \frac{1}{2}$, leading to a continuous spectrum (no new staggered modules are generated).

# Outlook

This leads to many questions, *eg*.

- Can we construct consistent (logarithmic) CFTs in the bulk from these modules? If so, what are the boundary CFTs?
- Do staggered $\widehat{\mathfrak{sl}}(2)$-modules admit a structure theory similar to that of the corresponding Virasoro modules?
- Is the story similar, or more complicated, for the other fractional levels? Do indecomposables beyond the staggered ones naturally appear?
- Can one extend to other affine (super)algebras where admissibility becomes much more troublesome?
- What other interesting CFTs can be constructed from these models via cosets, DS-reduction, *etc*... ?
- Can we use fractional level WZW models to study the logarithmic versions of the minimal models?