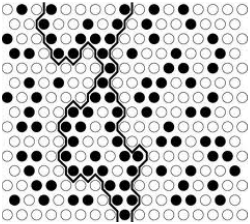
What's New in Critical Lattice Phenomena?



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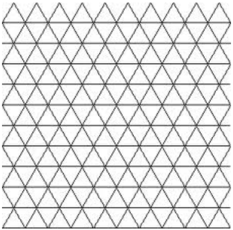
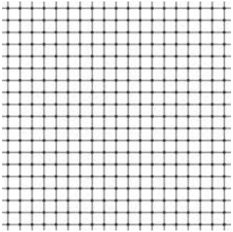
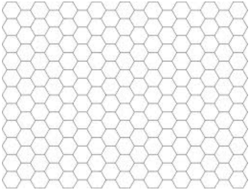
Outline of Talk

What is a critical lattice phenomenon?

By lattice, we mean a regular grid of some fixed shape and size, *eg*.



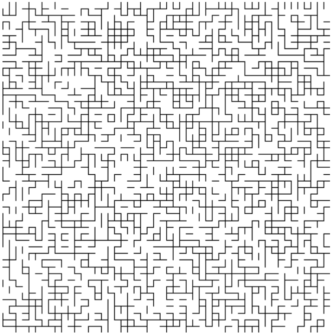




What is a critical lattice phenomenon?

To have a phenomenon, we first need to impose some set of rules on the chosen lattice (specifying our model). *eg.*

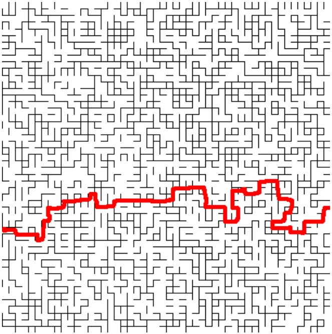
> *Consider each edge of the lattice in turn, removing it if a generated random number in $[0,1]$ exceeds p, for some given p.*



What is a critical lattice phenomenon?

A phenomenon would then be something that can be observed about a given random configuration from our lattice model. *eg*.

> *Does the configuration still contain a path from the west wall to the east wall?*



In this case, yes!

What is a critical lattice phenomenon?

An obvious question one can ask about our chosen lattice phenomenon is:

> *What is the probability π that a randomly generated lattice configuration will contain a path from the west to the east wall?*

This is a very very hard question to answer, unless the number of configurations is very very small or very very big!

In the first case, one simulates the model on a computer. In the second case, one approximates the lattice by a continuum.

In the first case, one needs to know some programming. In the second case, one needs to know some mathematics.

What is a critical lattice phenomenon?

In the continuum limit, where the size of the lattice spacing goes to zero, but the grid shape is left invariant, one can prove (this is basically a whole book by Kesten) that

$$\pi = \begin{cases} 0 & \text{if } p < p_c, \\ 1 & \text{if } p > p_c, \end{cases}$$

where p_c is the critical probability.

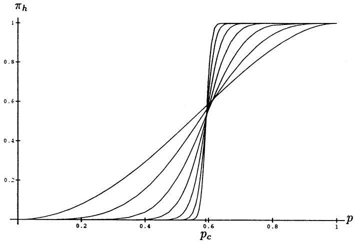


FIGURE 2.1c. The curves $\pi_h^n(p)$ for $n = 2, 4, 8, 16, 32, 64$, and 128. Larger slopes around p_c correspond to larger values of n.

Langlands, Pouliot and Saint-Aubin, Bull. Am. Math. 30 (1994), 1–61.

What is a critical lattice phenomenon?

This justifies the term critical in "critical lattice phenomenon" — questions tend to be more interesting (and accessible) in the continuum limit when the model parameters are tuned just right.

eg. the critical question for our example is:

What is π when $p = p_c$ in the continuum limit?

A mathematician would instead ask:

Does $\pi(p_c)$ even exist in the continuum limit?

Kesten can only tell us that if this limit does exist, then it is not 0 and not 1. Cardy answered the critical question in 1991. Mathematicians had to wait until Lawler-Schramm-Werner and Smirnov both published rigorous proofs of Cardy's result in 2001.

Why do we care?

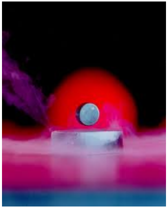
Theoretical physicists love to play with toy models.

- They're easier to study than the real world.
- They often contain enough "truth" to guide real-world studies.

The lattice models we have discussed are toy models which exhibit phase transitions in the continuum limit.

Specifically, they model continuous (or second-order) phase transitions (no latent heat):

First-Order



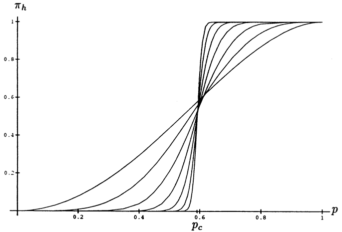
Second-Order

Why do we care?

Toy lattice models allow us to compute critical exponents. These quantities are largely independent of the microscopic model structure (*ie*. the lattice), depending rather on the dimension and the range of the interactions.

eg. We saw before that the probability π undergoes a "phase transition" at $p = p_c$. It seems that $\frac{d\pi}{dp}$ *diverges* at $p = p_c$ as the lattice spacing Δ goes to zero. In fact,

$$\frac{d\pi}{dp} \sim \Delta^{-\mu}, \qquad \mu = \frac{3}{4}.$$



μ is a critical exponent!

Why do we care?

More generally, models typically assign data σ_i to the lattice components and the correlations of this data typically decay exponentially with the distance:

$$\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \sim e^{-|i-j|/\xi}.$$

This defines the correlation length ξ.

As a model parameter p approaches its critical value p_c, the correlation length diverges as

$$\xi \sim \frac{1}{|p - p_c|^{\nu}}.$$

Moreover, when $p = p_c$, correlators decay algebraically:

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{|i - j|^{\eta}}.$$

Both ν and η are examples of critical exponents.

Why do we care?

So, we may predict these exponents in real world (continuous) phase transitions by replacing the system with a toy model whose interaction ranges are similar.

This observation is referred to as universality, and is generally explained as a consequence of Wilson's renormalisation group.

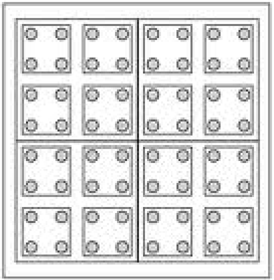
> *"The theory of continuous phase transitions provides a bridge between probabilistic mechanics and continuous field theory, using the renormalisation group to filter out relevant operators and interactions."*
>
> Itzykson and Drouffe, *Statistical Field Theory I*, 1989.

Let's turn to the continuous field theory approach to critical exponents.

The Renormalisation Group

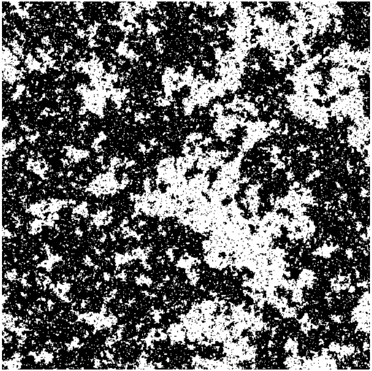
The interpretation of the correlation length ξ is that it gives a distance scale above which one does not expect fluctuations in the correlation between model quantities.

Renormalisation may be thought of as replacing the model by one in which the quantities have been smoothed over a length scale less than ξ. Macroscopic properties should be unaffected.



The Renormalisation Group

When ξ diverges (at the critical point), we expect fluctuations at all length scales, hence a fractal structure:



The continuum limit should then be described by a scale-invariant (in addition to Poincaré-invariant) theory.

Conformal Field Theory (CFT)

Scale-invariant theories are almost always conformally invariant.
This means that they respect symmetries that preserve angles.

Conformal Field Theory (CFT)

In dimensions $d > 2$, the (infinitesimal) angle-preserving symmetries (of *eg*. Euclidean space) constitute the Lie algebra $\mathfrak{so}(d+1,1)$.

When $d = 2$, the conformal algebra is the infinite-dimensional Virasoro algebra, and this gives us a massive amount of control over the continuum theory.

In either case, the representation theory of the conformal algebra fixes the form of certain correlation functions:

$$\langle \sigma_i \sigma_j \rangle \sim \langle 0 | \sigma(z_i) \sigma(z_j) | 0 \rangle = \frac{1}{|z_i - z_j|^{4h}}.$$

The conformal dimension h is the continuum analogue of a critical exponent, and has a representation-theoretic meaning!

Conformal Field Theory (CFT)

In $d = 2$, Belavin, Polyakov and Zamolodchikov showed that there is a set of conformal dimensions h for which the CFT is exactly solvable. These are arranged into the Kac table.

Because one can identify which CFT should describe the continuum limit of a given lattice model, this predicts the critical exponents of many lattice models:

eg. the Ising model assigns a spin $\sigma_i \in \{+, -\}$ to each lattice site i and an interaction energy $\varepsilon_i \sim \sigma_i \sigma_{i+1}$. Onsager's exact solution gives

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{|i-j|^{1/4}} \quad \text{and} \quad \langle \varepsilon_i \varepsilon_j \rangle \sim \frac{1}{|i-j|^2}$$

at the critical interaction strength.

This agrees with the CFT whose Kac table contains the conformal dimensions $\frac{1}{16}$ and $\frac{1}{2}$.

Logarithmic Conformal Field Theory

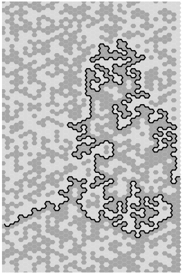
While all this is regarded as a triumph of modern physics, life isn't all tea and biscuits!

It is easy to describe observable quantities in lattice models whose "critical exponents" are not in the appropriate Kac table.

eg. for a random spin configuration of the Ising model, we can ask:

What is the probability that there is a path of positive spins which crosses from west to east?

What is the expected fractal dimension of a cluster of positive spins?



The numerically measured "exponents" require $h = \frac{1}{6}$ and $\frac{5}{3}$.

Logarithmic Conformal Field Theory

These conformal dimensions don't appear in the Kac table, but they do appear on the boundary of the Kac table:

$\frac{-1}{48}$	$\frac{1}{6}$	$\frac{35}{48}$	$\frac{5}{3}$	$\frac{143}{48}$
$\frac{5}{16}$	0	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{21}{16}$
$\frac{21}{16}$	$\frac{1}{2}$	$\frac{1}{16}$	0	$\frac{5}{16}$
$\frac{143}{48}$	$\frac{5}{3}$	$\frac{35}{48}$	$\frac{1}{6}$	$\frac{-1}{48}$

When we consider non-local observables, the CFT of Belavin, Polyakov and Zamolodchikov is replaced by a logarithmic CFT.

This is a technical generalisation of standard CFT in which some fields have correlators with logarithmic singularities. Logarithmic CFTs are relevant to string theory, the AdS/CFT correspondence and black hole holography.

Enter the Mathematicians

So, modern physics is in good shape. But, the mathematical community wasn't particularly happy.

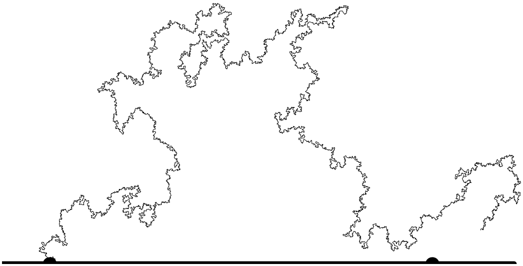
- What are these renormalised fields that are supposed to describe the lattice model observables in the continuum limit?
- What does it mean to say that the "theory" is conformally invariant?
- Why should quantum field theory be necessary to answer fundamental *probabilistic* questions about these lattice models?

While mathematicians are frequently impressed with the answers that physicists obtain, they are generally stubborn when it comes to method. Thus, it is no surprise that they continued to look for rigorous probabilistic justifications of the answers obtained through CFT. And in many cases, they found them!

Schramm-Loewner Evolution

The mathematical theory that aims to provide a rigorous basis for CFT "predictions" was originally called stochastic Loewner evolution (SLE). The "S" is now taken to refer to Schramm.

It is not a theory of discrete lattice models, but rather a theory of random curves in the half-plane.



The idea is that such curves should model, *eg*. cluster boundaries of Ising spins, in the continuum limit.

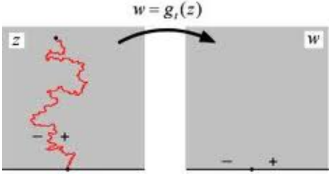
Schramm-Loewner Evolution

This is referred to as an "evolution" because the random curve is generated as a solution of a stochastic differential equation:

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}.$$

Here, B_t is Brownian motion and $\kappa > 0$ parametrises the model that the SLE curve describes. Brownian motion is the unique probability law which is preserved by conformal transformations.

However, g_t is not the coordinate of the curve at time t — it is the conformal transformation that maps to half-plane minus the piece of the curve from time 0 to time t back to the half-plane!



Schramm-Loewner Evolution

These random curves may be shown to be "typical" continuum behaviour for certain critical lattice models:

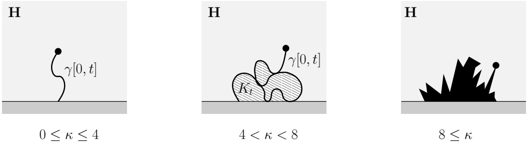
SLE_κ	Lattice Model
$\kappa = 2$	loop-erased random walk
$\kappa = 3$	Ising spin cluster boundary
$\kappa = 4$	Gaussian free field level lines
$\kappa = \frac{16}{3}$	Ising FK cluster boundary
$\kappa = 6$	Percolation cluster boundary
$\kappa = 8$	Uniform spanning tree

This only details what mathematicians can prove. They also suspect that the self-avoiding walk is $SLE_{8/3}$ and have conjectures for general Q-state Potts and $O(N)$ models.

Schramm-Loewner Evolution

The properties of these random curves can be studied probabilistically. The following are almost surely true:

- When $0 \leqslant \kappa \leqslant 4$, the curve does not intersect itself.
- When $4 < \kappa < 8$, the curve does intersect itself, but without crossing.
- When $\kappa \geqslant 8$, the curve fills space!



Beffara has shown that the Hausdorff dimension of the SLE_κ random curve is almost surely $1 + \frac{\kappa}{8}$. For the Ising FK cluster boundaries ($\kappa = 16/3$), this gives $\frac{5}{3}$ as mentioned earlier.

SLE vs CFT

So, SLE is allowing mathematicians to recover many of the "predictions" that physicists have made using CFT.

Can CFT answer questions that SLE cannot? Almost certainly!

But, there are many variants of SLE to explore:

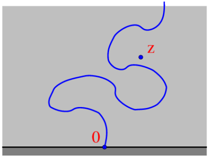
- $\mathrm{SLE}_\kappa(\rho)$: Brownian motion with a drift term.
- Multiple SLE's
- CLE: Conformal Loop Ensembles.

Can SLE (and variants) answer questions that (logarithmic) CFT cannot? We don't know!

Could it be that there is some sort of partial equivalence between logarithmic CFT and SLE and its variants?

SLE vs CFT

One of the first SLE calculations (Schramm) involves the probability $P(z)$ that a random SLE curve starting at 0 passes to the left of a given point z.



A mathematician would say that $P(z)$ satisfies a second-order differential equation because this probability defines a martingale.

A physicist would say that $P(z)$ satisfies a second-order differential equation because it is given by a correlation function involving a field with a null-vector at grade 2.

Both obtain (with $t = 4/\kappa$):

$$P(z) = \frac{1}{2} - \frac{i\Gamma(t)}{\sqrt{\pi}\,\Gamma\left(t-\frac{1}{2}\right)} \frac{z+\bar{z}}{z-\bar{z}}\, {}_2F_1\left(\frac{1}{2}, t, \frac{3}{2}; \left(\frac{z+\bar{z}}{z-\bar{z}}\right)^2\right).$$

When $\kappa = 4$, this becomes $P(z) = \frac{1}{2\pi i} \log \frac{z}{\bar{z}}$.

SLE vs CFT

Based on examples such as that just given, Bernard and Bauer have developed a theory connecting the probabilistic martingales of SLE with the conformal algebra representations of CFT.

Recently, Kytölä has refined their theory to demonstrate that the CFT representations that correspond to SLE (with a drift term) are those of logarithmic CFT.

There have also been tentative suggestions for special cases of an SLE/(log)CFT correspondence (Rasmussen; Moghimi-Araghi, Rajabpour and Rouhani; Mathieu and Ridout; Saint-Aubin, Pearce and Rasmussen).

The big problem is that the methods used by each camp are completely foreign to the other: CFT uses field theory and algebra, SLE uses probability theory and analysis.

Conclusions and Outlook

- We've recalled the physical approach to understanding critical lattice phenomena through renormalisation and conformal field theory.
- We've noted that the consideration of non-local observables requires one to consider logarithmic CFTs.
- We've discussed the mathematical approach to understanding critical lattice phenomena through the statistical properties of random fractals and Schramm-Loewner evolution.
- We've also compared the two approaches, leading to a brief discussion of a possible SLE/(log)CFT correspondence.
- Building such a correspondence is an exciting task for the future, and one that promises to teach us more about our respective specialities.
- One can also ask what survives if one drops the criticality requirement (assuming *eg*. integrability).

THANKYOU!!

... and don't forget:

The 2^{nd} Asia-Pacific Summer School in Mathematical Physics:

CFT, AdS/CFT and Integrability

The Australian National University

12–16 December 2011

See http://cmtp.anu.edu.au/ss2011/ for details!