Non-C_2-cofinite VOAs and the Verlinde formula

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Rational CFT and the Verlinde formula

Two ingredients of conformal field theory (CFT):

- A vertex operator algebra (VOA) V.
- A physical category $\mathscr{C}$ of V-modules that is
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For rational CFTs, $\mathsf{S}^\top = \mathsf{S}$, $\mathsf{S}^\dagger = \mathsf{S}^{-1}$, $\mathsf{S}^2 = \mathsf{C}$, and S diagonalises the fusion coefficients through the Verlinde formula [Huang]:

$$\mathcal{L}_i \otimes \mathcal{L}_j = \bigoplus_k \begin{bmatrix} & k & \\ i & & j \end{bmatrix} \mathcal{L}_k, \qquad \begin{bmatrix} & k & \\ i & & j \end{bmatrix} = \sum_\ell \frac{\mathsf{S}_{i\ell}\mathsf{S}_{j\ell}\mathsf{S}^*_{k\ell}}{\mathsf{S}_{0\ell}}.$$

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How does the formalism of rational CFT, especially Verlinde, generalise to non-rational and logarithmic CFT?

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If the goal is to decompose fusion products, then a Verlinde formula helps bigtime!

Logarithmic C_2-cofinite CFTs

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But, there is [Fuchs-Hwang-Semikhatov-Tipunin] a $\mathsf{W}(1,p)$ Verlinde-like formula for simple characters (automorphy removes τ-dependence).

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  $\Rightarrow \quad \mathcal{F}_p \otimes \mathcal{F}_q = \int_{-\infty}^{\infty} \begin{bmatrix} & r & \\ p & & q \end{bmatrix} \mathcal{F}_r\,\mathrm{d}r = \mathcal{F}_{p+q}$. Yay! ✓

A logarithmic non-C_2-cofinite CFT

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$$\Rightarrow \quad \begin{gathered} \mathcal{L}_p \otimes \mathcal{L}_q = \mathcal{L}_{p+q}, \quad \mathcal{L}_p \otimes \mathcal{F}_q = \mathcal{F}_{p+q}, \\ [\mathcal{F}_p \otimes \mathcal{F}_q] = [\mathcal{F}_{p+q}] + [\mathcal{F}_{p+q-1}]. \end{gathered}$$

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   $\mathsf{S}\{\mathrm{ch}_m\} = \int_M \mathsf{S}_{mn} \mathrm{ch}_n \, \mathrm{d}\mu(n)$, satisfying

   Symmetry: $\mathsf{S}_{mn} = \mathsf{S}_{nm}$,

   Unitarity: $\int_M \mathsf{S}_{mp} \mathsf{S}^*_{pn} \, \mathrm{d}\mu(p) = \delta(m = n)$,

   Conjugation: S^2 is a permutation of order ≤ 2.

5. $\mathrm{ch}_{\mathcal{M}} = \sum_m a_m \mathrm{ch}_m \quad \Rightarrow \quad \mathsf{S}_{\mathcal{M}n} = \sum_m a_m \mathsf{S}_{mn}$, which converges for all typical n ($n \notin A$).

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7. Define character fusion $\boxtimes$ by standard Verlinde formula:

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"Trivial" example is a rational CFT:

- Standard $=$ simple, so no atypicals ($A = \varnothing$).
- M is finite and μ is counting measure.
- Grothendieck fusion $=$ fusion.

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Conformal field theory	Fusion known?
Virasoro logarithmic minimal models $\mathsf{LM}(p,p')$	Many examples
$N=1$ logarithmic minimal models $\mathsf{LSM}(p,p')$	Some examples
Singlet models $\mathsf{I}(p,p') = \mathsf{W}_{2,(2p-1)(2p'-1)}$	?
Admissible level $\widehat{\mathfrak{sl}}(2)_k$	$k=-\frac{1}{2},\ -\frac{4}{3}$
Bosonic $\beta\gamma$ ghosts	✓
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Singlet model results imply results for triplet models $\mathsf{W}(p, p')$.
Consistent with known triplet fusion results (and conjectures).

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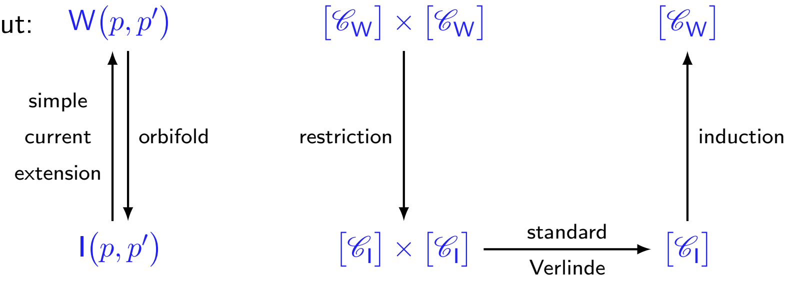

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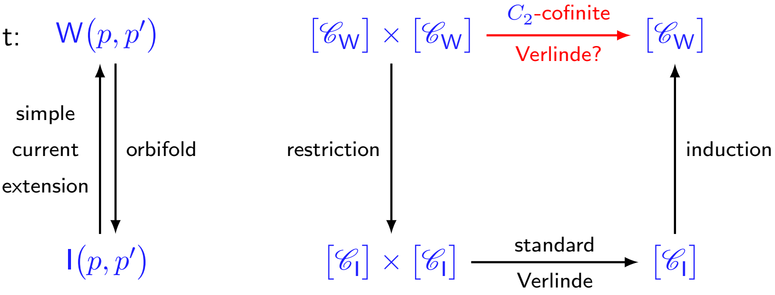

Triplet Verlinde currently being worked out [Melville-DR].

Thank you!

"Only those who attempt the absurd will achieve the impossible."

- M C Escher