A (gentle) introduction to logarithmic conformal field theory

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Outline

Raison d'être

Conformal field theory (CFT) arose in the early 80s at the confluence of

- Statistical Mechanics: Phase transitions in statistical lattice models may be modelled by conformally invariant quantum field theories.
- String Theory: At the level of the worldsheet, string dynamics defines two-dimensional conformally invariant quantum field theories.
- Pure Mathematics: The "monstrous moonshine" conjectures led to the notion of a vertex algebra which axiomatises (chiral) CFT.

By CFT, we mean a relativistic quantum field theory whose infinitesimal symmetries generate an algebra that contains the conformal algebra.

We will only consider two-dimensional euclidean CFTs. The conformal algebra is then (two commuting copies of) the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n=0}c\,\mathbf{1}, \quad m, n \in \mathbb{Z}.$$

$c \in \mathbb{C}$ is the central charge of the CFT.

Examples

- The free boson CFT describes a (massless spinless) string propagating in a one-dimensional space(time). Its infinitesimal symmetries generate the Heisenberg algebra:

$$[a_m, a_n] = m\delta_{m+n=0}\mathbf{1}, \quad m, n \in \mathbb{Z}.$$

  This algebra has a Virasoro subalgebra of central charge $c = 1$.

- The Ising model CFT describes the critical point of a two-dimensional spin lattice in the limit of zero magnetic field. The algebra is Virasoro with $c = \frac{1}{2}$. This CFT is almost identical to the free fermion CFT that describes a (massless spin-$\frac{1}{2}$) string. The (super)algebra is

$$\{\psi_m, \psi_n\} = \delta_{m+n=0}\mathbf{1}, \quad m, n \in \mathbb{Z} \text{ or } \mathbb{Z} + \tfrac{1}{2}.$$

- The moonshine CFT describes an orbifold of an exotic compactification of 24 free bosons (so $c = 24$). Its algebra is very complicated: the automorphisms form the monster simple group.

The quantum state spaces of these examples are all completely reducible.

- The Ising model state space decomposes into irreducible highest-weight modules:

$$\mathsf{H} = (\mathsf{L}_0 \otimes \mathsf{L}_0) \oplus (\mathsf{L}_{1/16} \otimes \mathsf{L}_{1/16}) \oplus (\mathsf{L}_{1/2} \otimes \mathsf{L}_{1/2}).$$

  Here, the highest-weight state of L_h has L_0-eigenvalue h.

- The moonshine state space is the tensor product of the irreducible vacuum module with itself:

$$\mathsf{H} = \mathsf{L}_0 \otimes \mathsf{L}_0.$$

- The free boson state space decomposes into a direct integral of irreducible Fock spaces:

$$\mathsf{H} = \int_\lambda^{\oplus} (\mathsf{F}_\lambda \otimes \mathsf{F}_\lambda) \, \mathrm{d}\lambda.$$

  Here, λ is the a_0-eigenvalue of the highest-weight state of F_λ.

Rational vs logarithmic CFT

A rational CFT is one for which:

- The quantum state space is *completely reducible*.
- The decomposition of the quantum state space involves *finitely many* irreducible representations of the infinitesimal symmetry algebra.

Along with the free boson, rational CFTs are the ones that are typically encountered in textbooks.

Complete reducibility means that the representation theory is under control and finiteness means that questions of analysis rarely arise.

By contrast, a CFT is said to be logarithmic if

- The quantum state space is *not* completely reducible.

If the quantum state space is nevertheless constructed from finitely many irreducibles, then the CFT is log-rational.

Data for rational CFTs

The business of theoretical physics is to calculate or predict the value of measurable quantities. In field theory, these values are generally held to be obtained from correlation functions.

In applications of rational CFT, such values are typically reduced to

- Eigenvalues of observable operators on highest-weight states.
- Three point constants.

These appear in the primary 2- and 3-point correlation functions:

$$\langle A(z_1)B(z_2)\rangle = \frac{\delta_{A^*=B}}{z_{12}^{h_A+h_B}}, \quad z_{ij} = z_i - z_j,$$

$$\langle A(z_1)B(z_2)C(z_3)\rangle = \frac{\Gamma_{ABC}}{z_{12}^{h_A+h_B-h_C}\, z_{13}^{h_A-h_B+h_C}\, z_{23}^{-h_A+h_B+h_C}}.$$

The eigenvalues constitute the representation-theoretic data, the 3-point constants constitute the algebraic data (operator product expansion).

Complete reducibility

The definition of a logarithmic CFT involves a failure of complete reducibility. What does this mean?

A representation is reducible if it has a non-zero proper subrepresentation.
A representation is decomposable if it is the direct sum of two non-zero proper subrepresentations.
A representation is completely reducible if it may be written as a direct sum (or integral) of irreducible subrepresentations.

For many types of algebra representations, being reducible is equivalent to being decomposable:

- Finite-dim. $\mathbb{C}$-reps of finite group algebras, *eg.* $\mathbb{C}\mathsf{S}_n$;
- Finite-dim. $\mathbb{C}$-reps of semisimple Lie algebras, *eg.* $\mathfrak{sl}_2$;
- Finite-dim. $\mathbb{C}$-reps of compact semisimple Lie groups, *eg.* $\mathsf{SU}(2)$.

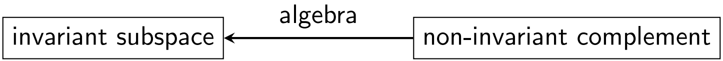
Thus, for these cases, we have complete reducibility.

Reducible but indecomposable

Alas, reducible but indecomposable representations are common, *eg.*:

- $\mathbb{k}$-reps of finite group algebras $\mathbb{k}\mathsf{G}$ when $\operatorname{char}\mathbb{k} \mid \operatorname{card}\mathsf{G}$;
- $\mathbb{C}$-reps of diagram algebras at roots of unity, *eg.* Temperley-Lieb;
- $\mathbb{C}$-reps of quantum groups at roots of unity, *eg.* $\mathscr{U}_q(\mathfrak{sl}_2)$;
- Finite-dim. $\mathbb{C}$-reps of Lie superalgebras, *eg.* $\mathfrak{gl}(1|1)$ (not $\mathfrak{osp}(1|2)$);
- Infinite-dim. $\mathbb{C}$-reps of semisimple Lie (super)algebras and groups;
- $\mathbb{C}$-reps of affine / Virasoro / W- algebras and superalgebras.

Often, one constructs irreducible representations as quotients of indecomposable ones, *cf.* null vectors in Verma modules.



Jordan blocks

Sometimes, reducible but indecomposable representations correspond to non-diagonalisable actions of certain algebra elements.

Example: Any $n \times n$ matrix A defines an n-dimensional representation of $\mathbb{C}[x]$ (or $\mathbb{Z}$ or $\mathfrak{gl}_1$ or ...) by $x \mapsto A$.

- If A is diagonalisable, then this is a direct sum of n irreducibles.
- Otherwise, this is a direct sum of $j < n$ indecomposables, one for each Jordan block in the Jordan canonical form of A.
- Any such indecomposable is irreducible if and only if the rank of the corresponding Jordan block is 1.

$$A \sim \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \quad \mathsf{L}_3 \oplus \begin{array}{c} \mathsf{L}_1 \\ \downarrow \\ \mathsf{L}_1 \end{array} \oplus \mathsf{L}_1$$

In logarithmic CFT, Jordan blocks in the action of L_0 are typical.

Logarithmic CFT

Recall: logarithmic means the failure of complete reducibility which means that the representation theory becomes much more difficult.

But: logarithmic CFTs are non-unitary so maybe they're non-physical.

So why bother?

- Continuum scaling limits of statistical lattice models are frequently logarithmic, eg percolation and polymers. Even the Ising model exhibits logarithmic behaviour if you ask the right questions.
- String theories on non-compact or supersymmetric space(times) are generally logarithmic, as are many of the ghost theories introduced to break the gauge symmetries of the string action.
- We don't want to artificially restrict ourselves to the "easy" cases. The applications of logarithmic CFT alone make it too important to leave to mere mathematicians.

Logarithmic examples

- The symplectic fermions CFT describes two spin-1 fermions:

$$\{J_m^{\pm}, J_n^{\pm}\} = 0, \quad \{J_m^{+}, J_n^{-}\} = m\delta_{m+n=0}\mathbf{1},$$

It's log-rational of central charge $c = -2$.

- The bosonic ghosts CFT commutation rules are

$$[\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0, \quad [\gamma_m, \beta_n] = \delta_{m+n=0}\mathbf{1}.$$

It's logarithmic but not log-rational. The central charge depends on the conformal weights h_β and h_γ of the ghost fields.

(h_β, h_γ)	$(\frac{1}{2}, \frac{1}{2})$	$(1, 0)$	$(\frac{3}{2}, -\frac{1}{2})$	$(2, -1)$
c	-1	2	11	26

- CFTs based on affine Kac-Moody algebras are logarithmic, but not log-rational, at all levels except the non-negative integers. *eg.*, the fractional level $\widehat{\mathfrak{sl}}_2$ models with levels k satisfying

$$k + 2 = \frac{u}{v}, \quad u, v \in \mathbb{Z}_{\geqslant 2}, \ \gcd\{u, v\} = 1.$$

- CFTs based on affine *super*algebras, except $\widehat{\mathfrak{osp}}(1|2n)$, are logarithmic, but not log-rational, at all levels.
- The non-unitary $N = 2$ minimal models and non-unitary parafermion models are all logarithmic, but not log-rational.
- For all central charges, the Virasoro and $N = 1$ superconformal algebras also admit logarithmic, but not log-rational, CFTs called the logarithmic minimal models. These are related to continuum scaling limits of various integrable lattice models.

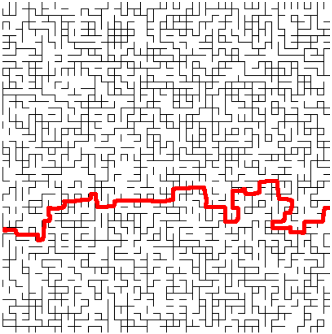
Curiously:

- All known log-rational CFTs have orbifolds that are not log-rational.
- The non-log-rational CFTs listed above all have extensions that are log-rational.
- All known logarithmic CFTs have either a finite number or an uncountably infinite number of irreducible representations.

Example: Percolation

The percolation model is a lattice of bonds (or sites) that are open with probability p and closed otherwise.

A famous question asks for the probability that there is a path through open bonds (sites) connecting one boundary of the lattice to another.



There is a critical value of p for which the answer is interesting.

Using (boundary) CFT, Cardy answered this question for critical p as a four-point correlator

$$\langle \phi_{1,2}(z_1)\phi_{1,2}(z_2)\phi_{1,2}(z_3)\phi_{1,2}(z_4)\rangle,$$

where $\phi_{1,2}$ is primary.

The z_i correspond to the four corners of the lattice, so his result is a function of the aspect ratio. Cardy's answer,

$$P(z) = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} z^{1/3} {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; z), \quad z = \frac{z_{12}z_{34}}{z_{13}z_{24}},$$

is clearly non-constant.

We claim that percolation is described by a logarithmic (boundary) CFT. This follows from P being non-constant and the assertion that $c = 0$.

Why is it so?

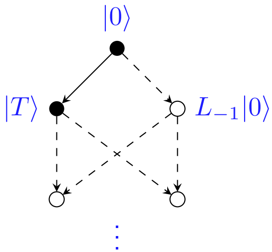
At $c = 0$, there are two choices for the Virasoro vacuum representation. If the representation is irreducible, then we have the singular vector relation

$$|T\rangle = L_{-2}|0\rangle = 0 \quad \Rightarrow \quad T(z) = 0 \quad \Rightarrow \quad L_n = 0.$$

But, this implies that the only state is $|0\rangle$: the theory is trivial and all correlators are constants.

To have a non-trivial correlator, the Virasoro vacuum representation must be reducible but indecomposable so that $|T\rangle \neq 0$.

Percolation must thus be described by a logarithmic (boundary) CFT.



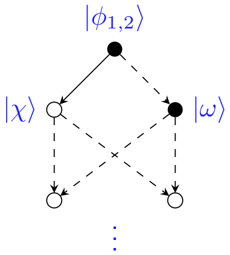
Similarly, there are two possibilities for the Virasoro representation generated by Cardy's primary field because of the vanishing of the grade 2 singular vector $|\chi\rangle$.

If the representation is irreducible, then

$$|\omega\rangle = L_{-1}|\phi_{1,2}\rangle = 0 \quad \Rightarrow \quad \partial\phi_{1,2}(z) = 0$$

and the correlator $\langle\phi_{1,2}(z_1)\phi_{1,2}(z_2)\phi_{1,2}(z_3)\phi_{1,2}(z_4)\rangle$ is constant.

To get a non-constant correlator, the Virasoro representation generated by $\phi_{1,2}$ must be reducible but indecomposable so that $|\omega\rangle \neq 0$.



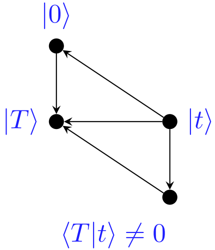
Once again, we conclude that percolation must be a logarithmic (boundary) CFT.

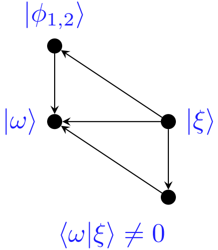
But singular vectors decouple?!

In rational CFT, we are taught that all null vectors must be set to zero because they are orthogonal to every state. Any correlation function involving a null field is then zero.

In percolation, the singular vectors $|T\rangle = L_{-2}|0\rangle$ and $|\omega\rangle = L_{-1}|\phi_{1,2}\rangle$ are null. But, we cannot set them equal to 0 without trivialising the CFT.

The resolution is that each representation is just part of a larger indecomposable representation in which the singular vectors are not null.





These larger representations may be constructed by fusion. This takes us out of the ($c = 0$ minimal model) Kac table and into the ($c = 0$ logarithmic minimal model) extended Kac table.

0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	$\cdots$
$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	$\cdots$
2	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

If we let $\mathsf{K}_{1,1}$ and $\mathsf{K}_{1,2}$ denote the reducible but indecomposable representations generated by $|0\rangle$ and $|\phi_{1,2}\rangle$, respectively, then

$$\mathsf{K}_{1,2} \times \mathsf{K}_{1,2} = \mathsf{K}_{1,1} \oplus \mathsf{K}_{1,3},$$

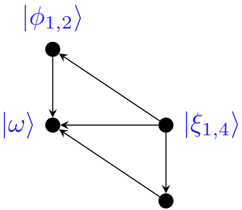
where $\mathsf{K}_{1,3}$ is irreducible of conformal weight $\frac{1}{3}$.

It would be reasonable to expect that $\mathsf{K}_{1,2} \times \mathsf{K}_{1,3} \overset{?}{=} \mathsf{K}_{1,2} \oplus \mathsf{K}_{1,4}$, where $\mathsf{K}_{1,4}$ is some highest-weight representation of conformal weight 1.

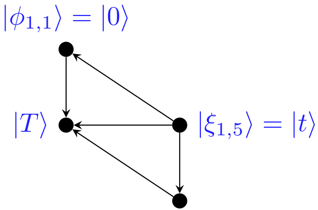
Similarly, we might expect that $\mathsf{K}_{1,3} \times \mathsf{K}_{1,3} \overset{?}{=} \mathsf{K}_{1,1} \oplus \mathsf{K}_{1,3} \oplus \mathsf{K}_{1,5}$, where $\mathsf{K}_{1,5}$ is some highest-weight representation of conformal weight 2.

However, this is wrong and we actually have

$$\mathsf{K}_{1,2} \times \mathsf{K}_{1,3} = \mathsf{P}_{1,4}, \qquad \mathsf{K}_{1,3} \times \mathsf{K}_{1,3} = \mathsf{K}_{1,3} \oplus \mathsf{P}_{1,5}.$$



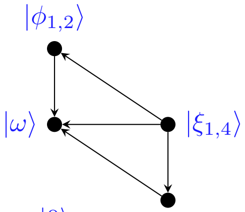
$\mathsf{P}_{1,4}$



$\mathsf{P}_{1,5}$

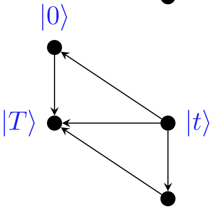
The structure of these larger reducible but indecomposable representations is fixed by a single numerical parameter:

$\mathsf{P}_{1,4}$:



- $|\omega\rangle = L_{-1}|\phi_{1,2}\rangle$.
- $(L_0 - 1)|\xi_{1,4}\rangle = |\omega\rangle$.
- $L_1|\xi_{1,4}\rangle = -\frac{1}{2}|\phi_{1,2}\rangle$.
- $\langle\omega|\xi_{1,4}\rangle = -\frac{1}{2}$.

$\mathsf{P}_{1,5}$:

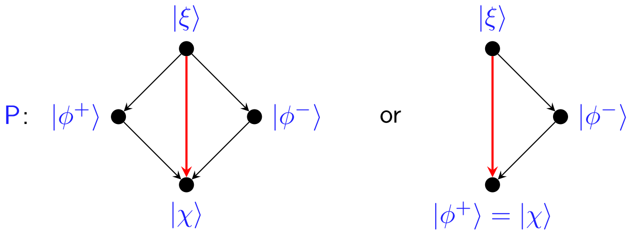


- $|T\rangle = L_{-2}|0\rangle$.
- $(L_0 - 2)|t\rangle = |T\rangle$.
- $L_2|t\rangle = -\frac{5}{8}|0\rangle$.
- $\langle T|t\rangle = -\frac{5}{8}$.

In logarithmic CFT, L_0 acts non-diagonalisably, singular vectors need not be null and generating fields need not be primary!

Staggered modules

The P-type reducible but indecomposable representations encountered in percolation are examples of staggered modules.



These seem to be ubiquitous whenever physics demands reducible but indecomposable representations.

The red vertical arrow represents the non-diagonalisable action of some (almost) central element: L_0 in logarithmic CFT, the quadratic Casimir for Lie superalgebras and quantum groups, the braid transfer matrix for Temperley-Lieb, . . .

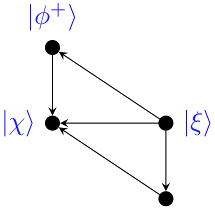
Formally, a staggered module P is an extension of a standard module S_1 by another standard module S_2 (whatever "standard" means):

$$\mathsf{S}_2 \subset \mathsf{P}, \quad \frac{\mathsf{P}}{\mathsf{S}_2} \cong \mathsf{S}_1.$$

For Virasoro logarithmic minimal models, there is a classification of staggered modules if we take "standard" to mean highest-weight with at most one (non-trivial) singular vector.

The classification requires S_1, S_2 and the logarithmic coupling $\beta \in \mathbb{C}$.

- Choose a highest-weight state $|\phi^+\rangle$.
- Choose a singular vector $|\chi\rangle = U|\phi^+\rangle$.
- Pick any $|\xi\rangle$ satisfying $(L_0 - h_\chi)|\xi\rangle = |\chi\rangle$.
- Compute $\beta = \langle\chi|\xi\rangle$ as $U^\dagger|\xi\rangle = \beta|\phi^+\rangle$.



Logarithmic couplings and correlators

Since $\beta = \langle \chi | \xi \rangle$, it is natural that it also appears in correlation functions.

eg., the $c = 0$ $\mathsf{P}_{1,5}$-fields $T(z)$ and $t(z)$ satisfy

$$\langle T(z) T(w) \rangle = \frac{\langle T | T \rangle}{(z-w)^4} = 0,$$

$$\langle T(z) t(w) \rangle = \frac{\langle T | t \rangle}{(z-w)^4} = \frac{\beta}{(z-w)^4},$$

$$\langle t(z) t(w) \rangle = \frac{\alpha - 2\beta \log(z-w)}{(z-w)^4},$$

where α is another (undetermined) constant. The logarithmic singularity in such correlators is the reason for the name "logarithmic" CFT.

While β is an invariant of the representation $\mathsf{P}_{1,5}$, α is not: it may be tuned to any desired value by replacing $|t\rangle$ with $|t\rangle + \gamma |T\rangle$.

Care is needed when discussing logarithmic couplings and correlators.

eg., the $c = 0$ $\mathsf{P}_{1,4}$-singular field $\chi(z)$ and its Jordan partner $\xi_{1,4}(z)$ give

$$\langle \chi(z)\chi(w) \rangle = 0,$$

but

$$\langle \chi(z)\xi_{1,4}(w) \rangle = \frac{-\beta}{(z-w)^2},$$
$$\langle \xi_{1,4}(z)\xi_{1,4}(w) \rangle = \frac{\alpha + 2\beta \log(z-w)}{(z-w)^2}.$$

The unexpected sign here results from $\chi(z)$ being a descendant field:

$$\chi(z) = (L_{-1}\phi_{1,2})(z) = \partial\phi_{1,2}(z).$$

Unfortunately, this sign is frequently ignored in the literature.

Data for logarithmic CFTs

The data of logarithmic CFTs then includes the logarithmic couplings β of any staggered modules.

This is in addition to the conformal weights (and other eigenvalues) of the highest-weight states and the 3-point constants.

One may wish to extend the 3-point constants to cover those involving the Jordan partners $\xi(z)$ of the singular fields $\chi(z)$, but it isn't necessary.

No further invariant couplings generalising β appear in the 3-point functions involving Jordan partner fields.

However, a logarithmic CFT might have reducible but indecomposable representations that are more complicated than staggered modules. One might then need to include further invariants.

Logarithms in percolation?

Recall that Cardy's 4-point function was hypergeometric,

$$\langle \phi_{1,2}(\infty)\phi_{1,2}(1)\phi_{1,2}(z)\phi_{1,2}(0)\rangle = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} z^{1/3}{}_2F_1(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};z),$$

with no logarithmic singularities (since $\frac{4}{3}, \frac{4}{3}-\frac{2}{3}-\frac{1}{3}, \frac{2}{3}-\frac{1}{3} \notin \mathbb{Z}$).

Why not? Because fusion only generates one logarithmic partner field and two are needed to generate logarithmic singularities:

$$(\phi_{1,2}\times\phi_{1,2})\times(\phi_{1,2}\times\phi_{1,2}) \sim \phi_{1,3}\times\phi_{1,3} \sim \xi_{1,5} = t,$$
$$(\phi_{1,2}\times\phi_{1,2}\times\phi_{1,2})\times\phi_{1,2} \sim (\phi_{1,2}\times\phi_{1,3})\times\phi_{1,2} \sim \xi_{1,4}\times\phi_{1,2}.$$

It is now easy to see that the 6-point function of $\phi_{1,2}$ will be the first to have logarithmic singularities at $c=0$.

Flores, Kleban and Simmons have confirmed that crossing probabilities for percolation on a hexagon exhibit logarithmic singularities.

Example: Symplectic fermions

We now to turn to a log-rational CFT that is much better understood than the $c = 0$ Virasoro logarithmic minimal model (percolation).

The symplectic fermions CFT is even described by an action:

$$S = \frac{1}{4\pi} \int \left(\partial\theta^+ \bar{\partial}\theta^- - \partial\theta^- \bar{\partial}\theta^+ \right) \mathrm{d}z \mathrm{d}\bar{z}.$$

The equations of motion give fermionic chiral fields $J^\pm = \partial\theta^\pm$ and $\bar{J}^\pm = \bar{\partial}\theta^\pm$. The (holomorphic) operator product expansions are

$$J^\pm(z) J^\pm(w) \sim 0, \quad J^+(z) J^-(w) \sim \frac{\mathbf{1}}{(z-w)^2}.$$

The energy-momentum tensor is of Sugawara type:

$$T(z) = {:}J^-(z) J^+(z){:}.$$

Finally, the central charge is $c = -2$.

Representations

Being fermionic, representations can be Neveu-Schwarz (the $J_n^\pm$ have $n \in \mathbb{Z}$) or Ramond (the $J_n^\pm$ have $n \in \mathbb{Z} + \frac{1}{2}$). Up to parity, there is:

- A unique Neveu-Schwarz highest-weight representation L_0, the vacuum module.
- A unique Ramond highest-weight representation $\mathsf{L}_{-1/8}$.

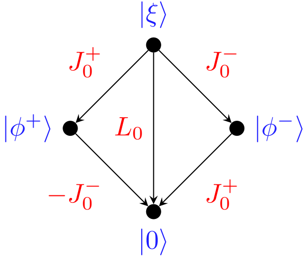
The subscript indicates the conformal weight of the highest-weight state.

Fusing Ramond modules results in a Neveu-Schwarz staggered module:

$$\mathsf{L}_{-1/8} \times \mathsf{L}_{-1/8} = \mathsf{P}_0.$$

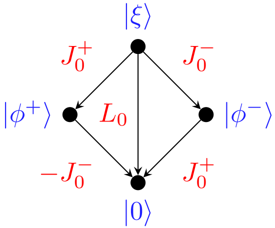
There are no logarithmic couplings.

There are no staggered modules in the Ramond sector.



The states $|\xi\rangle$, $|\phi^{\pm}\rangle$ and $|0\rangle$ all have conformal weight 0.

P_0 is formed by glueing two copies of L_0 and two copies of its parity reversal into an indecomposable.



The scalar products (2-point functions) of the P_0-states are as follows:

- $\langle 0|0\rangle = \langle 0|L_0|\xi\rangle = 0$ (L_0 is still self-adjoint).
- We may choose $\langle 0|\xi\rangle = 1$. Thus, $\langle \mathbf{1}\rangle = 0$ and $\langle \xi(z)\rangle = 1$.
- $\langle \xi|\xi\rangle$ is not defined, but $\langle \xi(z)\xi(w)\rangle = \alpha - 2\log(z-w)$.
- $\langle \phi^{\pm}|\phi^{\pm}\rangle = \langle \phi^{\pm}|J_0^{\pm}|\xi\rangle = \mp\langle 0|\xi\rangle = \mp 1$ and $\langle \phi^{\pm}(z)\phi^{\pm}(w)\rangle = \mp 1$ (since $(J_0^{\pm})^{\dagger} = J_0^{\mp}$).

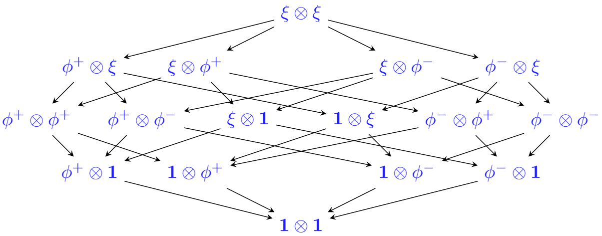
The Gram matrix of 2-point functions is thus non-degenerate! But, this is at the chiral level. . .

Bulk symplectic fermions

In the Ramond sector (no staggered modules), the bulk structure is as expected:

$$\mathsf{H}_{\mathrm{R}} = (\mathsf{L}_{-1/8} \otimes \mathsf{L}_{-1/8}) \oplus \text{(parity-reversed versions)}.$$

However, tensoring the Neveu-Schwarz staggered module P_0 with itself is physically untenable: $\mathsf{e}^{2\pi i(L_0 - \bar{L}_0)}$ does not act as the identity.

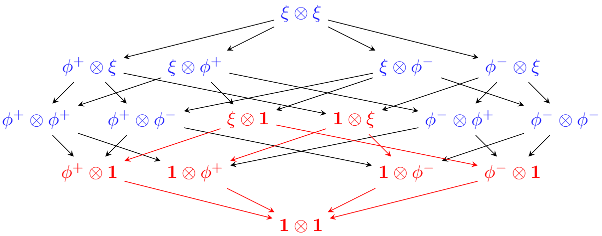


Bulk symplectic fermions

In the Ramond sector (no staggered modules), the bulk structure is as expected:

$$\mathsf{H}_{\mathrm{R}} = (\mathsf{L}_{-1/8} \otimes \mathsf{L}_{-1/8}) \oplus (\text{parity-reversed versions}).$$

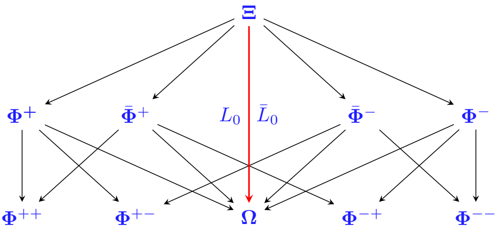
However, tensoring the Neveu-Schwarz staggered module P_0 with itself is physically untenable: $\mathsf{e}^{2\pi\mathsf{i}(L_0-\bar{L}_0)}$ does not act as the identity.



We must quotient by the image of the nilpotent part of $L_0 - \bar{L}_0$.

In this quotient, define:

- $\boldsymbol{\Xi} = [\xi \otimes \xi]$.
- $\boldsymbol{\Omega} = [\xi \otimes \mathbf{1}] = [\mathbf{1} \otimes \xi]$.
- $\boldsymbol{\Phi}^{\pm} = [\phi^{\pm} \otimes \xi]$.
- $\bar{\boldsymbol{\Phi}}^{\pm} = [\xi \otimes \phi^{\pm}]$.
- $\boldsymbol{\Phi}^{ab} = [\phi^{a} \otimes \phi^{b}]$.



It remains to test the non-degeneracy of the 2-point functions. This time, the Gram matrix has a non-trivial kernel (hard yakka):

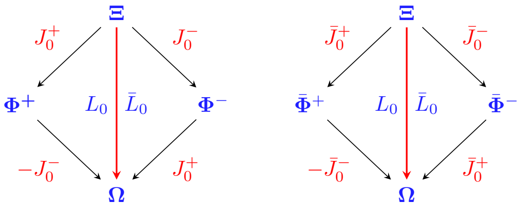
$$\ker\langle\cdot|\cdot\rangle = \operatorname{span}\left\{\boldsymbol{\Phi}^{ab} - \langle\boldsymbol{\Xi}|\boldsymbol{\Phi}^{ab}\rangle\boldsymbol{\Omega}, \bar{\boldsymbol{\Phi}}^{a} + \langle\boldsymbol{\Xi}|\boldsymbol{\Phi}^{-a}\rangle\boldsymbol{\Phi}^{+} - \langle\boldsymbol{\Xi}|\boldsymbol{\Phi}^{+a}\rangle\boldsymbol{\Phi}^{-}\right\}.$$

Quotienting by this kernel gives the Neveu-Schwarz bulk state space

$$\mathsf{H}_{\mathrm{NS}} = \mathbf{P}_0 \oplus \text{(parity-reversed versions)},$$

where $\mathbf{P}_0$ is an indecomposable formed by gluing two copies of $\mathsf{L}_0 \otimes \mathsf{L}_0$ and two copies of its parity-reversed version together.

The bulk quantum state space $\mathbf{P}_0$ can be conveniently visualised in terms of either the holomorphic or antiholomorphic action, but not both.



For a complete description of this bulk staggered module, we must also specify each $\bar{\boldsymbol{\Phi}}^{\pm}$ as a linear combination of the $\boldsymbol{\Phi}^{\pm}$ (or vice versa).

This inconvenience is expected whenever composition factors are repeated at the same Loewy grade.

Here, Π denotes parity reversal.

$$\mathsf{L}_0 \otimes \mathsf{L}_0$$

$$\Pi\mathsf{L}_0 \otimes \Pi\mathsf{L}_0 \quad \oplus \quad \Pi\mathsf{L}_0 \otimes \Pi\mathsf{L}_0$$

$$\mathsf{L}_0 \otimes \mathsf{L}_0$$

Conclusions

- CFT models many interesting (and useful) systems in physics and generates lots of beautiful mathematics. Sometimes this requires reducible but indecomposable representations, *ie.* logarithmic CFT.
- The failure of complete reducibility, combined with the non-degeneracy of 2-point functions, leads to staggered modules on which L_0 acts with rank 2 Jordan blocks.
- These Jordan blocks are directly responsible for the logarithmic singularities observed in correlation functions.
- Singular vectors remain physical by acquiring a Jordan partner to which they are not orthogonal.
- Staggered modules arise naturally when fusing irreducible representations. They are characterised by an invariant called the logarithmic coupling.
- Logarithmic couplings appear in correlators and must be added to the list of fundamental numerical data of a CFT.
- Holomorphic factorisation fails for the bulk quantum state space of a logarithmic CFT. The structure is constrained by the locality and non-degeneracy of 2- and 3-point functions, but the analysis is tough.

What's next?

One thing we have avoided discussing to date is modular invariance.

The invariance of the bulk partition function is a powerful constraint on the quantum state space for rational CFTs.

In the logarithmic case, modularity is much more subtle and there are profound difficulties that are still being addressed.

For example, the symplectic fermions vacuum supercharacter is

$$\mathrm{sch}\left[\mathsf{L}_0\right] = q^{1/12} \prod_{i=1}^{\infty} (1 - q^i)^2 = \eta(\tau)^2, \quad q = \mathsf{e}^{2\pi \mathfrak{i} \tau}.$$

Its modular S-transform therefore has a τ-dependent coefficient:

$$\mathrm{sch}\left[\mathsf{L}_0\right](-1/\tau) = \eta(-1/\tau)^2 = -\mathfrak{i}\tau\eta(\tau)^2.$$

We shall discuss modularity in logarithmic CFT further in my next talk...