Authors' Reply

Mathematical description of isogyre formation in refracting structures

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The cross pattern (isogyres) formed by certain crystals when they are placed between crossed polarizers and illuminated with convergent light results from birefringent properties of the medium. A similar image is seen in refracting structures which exhibit little or no birefringence, such as the eye lens. This paper provides a mathematical explanation of the formation of isogyres by refraction.

Illumination by polarized light is very useful for eliciting information about the internal organization of a solid and is often used to determine the structural properties of crystals. Many crystals exhibit the property of birefringence, the extent and type of which depends on their internal structure. For crystals in which the lattice has cubic symmetry, the refractive index is isotropic and such crystals do not exhibit birefringence. In biaxial crystals, the indices are different in three orthogonal planes. In between these extremes are the uniaxial crystals in which two of the three directional indices are equal. In these, the optic axis runs perpendicular to the plane containing the directions of the equivalent indices.

The type of birefringence in a crystal can be determined from the type of pattern seen within it when it is illuminated between crossed polarizers. When the polarizers are perpendicular to the optic axis of the crystal, and the light beam is convergent, a dark cross (isogyre) pattern can be observed in the plane of a uniaxial crystal. In the quadrants, between the arms of the cross are a series of coloured and dark rings (isochromatics), broken by the cross. In a biaxial crystal the cross is replaced by two hyperbolae.

Isogyres are also observed when the lens of the eye is placed between crossed polarizers but this is not accompanied by isochromatics. Furthermore this occurs regardless of whether or not the beam is convergent. Since the total measurable birefringence is almost negligible[1,2] the isogyres cannot be due to this property. Similarly the cross can be seen in non-birefringent structures, such as glass marbles, projector lenses, and even an oil-filled spheroidal capsule, when illuminated between crossed polarizers. The common property shared by these elements is that they all have a high refractive power and Pierscionek[3] proposed that this alone is responsible for the formation of isogyres in the eye lens and in these other structures. Charman[4] has derived a mathematical explanation based on Fresnel transmittance equations, which supports the hypothesis and shows that intensity of light transmitted through a spherical structure between crossed polarizers, depends on its entry position. Rays which enter along meridians which are parallel to either polarizer are completely extinguished. Charman cites a paper by Wright[5] who showed that the plane of polarization is rotated on refraction and that the amount is dependent on the incident angle. Unaware of Charman's derivation we formulated our own explanation for isogyre formation by tracing rays through a highly refractive structure. The method which is presented here is, like Charman's, based on the Fresnel equations but is more extensive.

As a polarized light ray enters the lens its path alters depending on the curvature at the point of entry as well as the difference in refractive index between the surrounding medium and the lens surface. If the structure has a uniform index the light travels through it along a straight path and is refracted again on exit. In a gradient index structure the path of the ray is curved. On refraction, the vibration direction of the electric component of radiation, **E**, changes since its overall direction must always remain transverse to the direction of the ray path. The transmitted amplitudes of the two components of the electric vector $\mathbf{E}_s$ and $\mathbf{E}_p$ which represent, respectively, vibrations perpendicular and parallel to the plane of incidence (the plane defined by the incident ray and the normal to the surface at the point of incidence) depend on the Fresnel transmission coefficients. These, in turn, depend on the difference in the refractive index across the refracting surface and the angle of incidence. Consequently the entry position of the polarized ray will

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determine the final direction of electric vibration as well as the amplitudes of the component vectors. For certain positions of entry, the ray will reach the analyser polarized in a direction perpendicular to the analyser axis and it will be blocked. It can be shown, by tracing rays and using a mathematical analysis based on Fresnel's refraction equations, that the blocked rays form the cross pattern. In this paper we have chosen as a model a non-birefringent, highly curved lens, to illustrate that isogyres can be formed by refraction.

Mathematical model

In this section we calculate the form of the dark cross isogyre for a spherical lens of radius R and of uniform refractive index located at the origin of a cartesian coordinate system. Plane polarized light is directed along the positive z-axis towards the lens with the electric component of vibration parallel to the y-axis. The vector of the electric vibration of the plane polarized light is:

$$\mathbf{E} = (0,\ E_y,\ 0)$$

Consider then a ray of this beam parallel to the z-axis and located at a distance ρ from this axis. It enters the lens at point A (*Figure 1a*). The entry point coordinates are:

$$\mathrm{A} = (x_\mathrm{A},\ y_\mathrm{A},\ z_\mathrm{A}) = (\rho\cos\phi,\ \rho\sin\phi,\ -(R^2-\rho^2)^{1/2})$$

where ϕ represents the azimuthal angle of the ray measured in the usual way relative to the x-axis (*Figure 1b*). If $\mathbf{n}$ is the unit normal vector pointing into the surface then

$$\mathbf{n} = (1/R)\ (\rho\cos\phi,\ \rho\sin\phi,\ -(R^2-\rho^2)^{1/2}).$$

Upon refraction the polarization of the transmitted beam will change due to the different transmission coefficients for components of the incident field perpendicular and parallel to the plane of incidence. To see this we decompose the incident electric field $\mathbf{E}$ into components that are perpendicular and parallel to the plane of incidence, $\mathbf{E}_\mathrm{s}$ and $\mathbf{E}_\mathrm{p}$ respectively. These can be described by the following equations:

$$\mathbf{E}_\mathrm{s} = (\mathbf{E}\cdot\mathbf{s}_\mathrm{A})\mathbf{s}_\mathrm{A} = E_y\cos\phi\ (-\sin\phi,\ \cos\phi,\ 0)$$
$$= E_y\ (-\cos\phi\sin\phi,\ \cos^2\phi,\ 0) \qquad (1a)$$

$$\mathbf{E}_\mathrm{p} = \mathbf{E} - \mathbf{E}_\mathrm{s} = E_y\ (\sin\phi\cos\phi,\ \sin^2\phi,\ 0) \qquad (1b)$$

where $\mathbf{s}_\mathrm{A}$ is the unit vector normal to the plane of incidence at the entry point A:

$$\mathbf{s}_\mathrm{A} = (-\sin\phi,\ \cos\phi,\ 0)$$

After refraction at the surface of entry, the transmitted amplitudes are described by:

$$E'_\mathrm{s} = t_\mathrm{sA}\ E_\mathrm{s} \qquad (2a)$$

$$E'_\mathrm{p} = t_\mathrm{pA} E_\mathrm{p} \qquad (2b)$$

where t_SA and t_PA are the Fresnel transmission coefficients[6] at the point A for the electric field components perpendicular and parallel to the plane of incidence respectively,

$$t_\mathrm{SA} = 2\sin\alpha_1\cos\alpha_0/\sin(\alpha_0+\alpha_1)$$

$$t_\mathrm{PA} = 2\sin\alpha_1\cos\alpha_0/[\sin(\alpha_0+\alpha_1)\cos(\alpha_0-\alpha_1)]$$

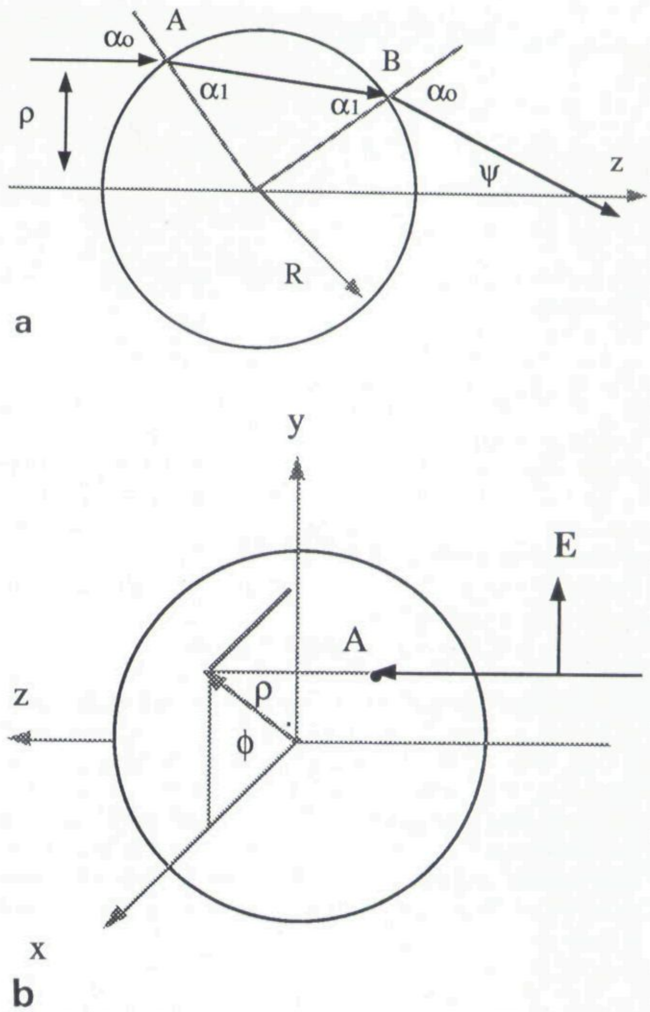

Figure 1 (a) Ray path through a spherical, uniform index lens, for an incident ray parallel to and at distance ρ from the z-axis. The ray enters the lens at point A and exits at point B. α_0 is the incident angle to the surface normal at A, α_1 is the refracted angle to the surface normal at A. R is the radius of the lens and ψ is the angle formed by the emergent ray and the z-axis (b). Ray path diagram showing ray entering lens at A with coordinates ($\rho\cos\phi$, $\rho\sin\phi$, $-(R^2-\rho^2)^{1/2}$) where ρ is the distance from the z-axis, ϕ represents the azimuthal angle of the ray relative to the x-axis and R is the radius of the lens (not shown in this diagram)

where α_0 and α_1 are angles made by the incident and refracted rays and the surface normal at the point of entry, A (*Figure 1a*).

The transmitted s-component of the electric field is

$$\mathbf{E}'_\mathrm{s} = t_\mathrm{SA}\mathbf{E}_\mathrm{s} = t_\mathrm{SA}E_y(-\cos\phi\sin\phi,\ \cos^2\phi,\ 0) \qquad (3a)$$

the transmitted p-component, is

$$\mathbf{E}'_\mathrm{p} = \mathbf{E}'_\mathrm{p}\ (\mathbf{s}_\mathrm{A}\times\mathbf{k}_\mathrm{A})$$

where $\mathbf{k}_\mathrm{A}$ is the direction of the transmitted ray at A and is given by:

$$\mathbf{k}_\mathrm{A} = (-\sin(\alpha_0-\alpha_1)\cos\phi,$$
$$-\sin(\alpha_0-\alpha_1)\sin\phi,\ \cos(\alpha_0-\alpha_1))$$

Hence

$$\mathbf{E}'_\mathrm{p} = t_\mathrm{PA}E_\mathrm{y}\sin\phi\ (\cos(\alpha_0-\alpha_1)\cos\phi,$$
$$\cos(\alpha_0-\alpha_1)\sin\phi,\ \sin(\alpha_0-\alpha_1)) \qquad (3b)$$

and the transmitted field at A is then the sum of the s and p components

$$\mathbf{E}' = \mathbf{E}'_s + \mathbf{E}'_p = E_y(-t_{sA}\sin\phi\cos\phi + t_{pA}\sin\phi\cos\phi\cos(\alpha_0-\alpha_1),\ t_{sA}\cos^2\phi + t_{pA}\sin^2\phi\cos(\alpha_0-\alpha_1), \quad (4a)$$

$$t_{pA}\sin\phi\sin(\alpha_0-\alpha_1)). \quad (4b)$$

In particular, the component of vibration in the x-axis direction,

$$E'_x = E_y\ (-t_{sA}\sin\phi\cos\phi + t_{pA}\sin\phi\cos\phi\cos(\alpha_0-\alpha_1)) \quad (5)$$

becomes 0 when $\phi = 0$ or $\pi/2$, that is, when the entry point of the ray is on the x- or y-axes. This indicates that a dark cross intensity pattern would be seen even after one refraction, that is, if the analyser (with its axis of polarization crossed to that of the polarizer and therefore parallel to the x-axis) was placed immediately behind the front surface. This is essentially the same as Wright's observation[5] mentioned earlier.

At the point B, the exit position from the lens, the incident field is equal to the transmitted field at the point of entry, A, given by Equation (4b). However, the s and p components of the incident field at the exit point B (denoted by $\mathbf{E}'_{sA}$ and $\mathbf{E}'_{pB}$) have to be constructed from the *total* incident field (given by Equation (4b)) and the new surface normal vector at B, analogous to the decomposition given in Equation (1). The transmitted field components at B are then given by applying the Fresnel equation

$$\mathbf{E}''_s = t_{sB}\ \mathbf{E}'_{sA} \quad (6a)$$

$$\mathbf{E}''_p = t_{pB}\ \mathbf{E}_{pB} \quad (6b)$$

where t_{sB} and t_{pB} are the Fresnel transmission coefficients at the point B for the electric field components perpendicular and parallel to the plane of incidence at B:

$$t_{sB} = 2\sin\alpha_0\cos\alpha_1/\sin(\alpha_0+\alpha_1)$$

$$t_{pB} = 2\sin\alpha_0\cos\alpha_1/\ [\sin(\alpha_0+\alpha_1)\cos(\alpha_0-\alpha_1)].$$

After emerging from the lens, the ray meets the optic axis at an angle, ψ, (*Figure 1a*) where

$$\psi = 2\ (\alpha_0-\alpha_1). \quad (7)$$

For computational purposes, it is more convenient to use the relations

$$\sin\alpha_0 = \rho/R, \sin\alpha_1 = \mu\rho/R,$$

where $\mu = n_{surround}/n_{lens}$ is the ratio of refractive indices.

After some algebra, the Equations (6) and (7) simplify to:

$$\mathbf{E}''_s = t_{SB}\ t_{SA}\ E_y\ (-\cos\phi\sin\phi,\ \cos^2\phi,\ 0) \quad (8a)$$

$$\mathbf{E}''_p = t_{PB}\ t_{PA}\ E_y\sin\phi\ (\cos\psi\cos\phi, \cos\psi\sin\phi, \sin\psi) \quad (8b)$$

which give the transmitted field components as a function of entry and exit angles. The combined electric field vector of the exit ray is then

$$\mathbf{E}'' = \mathbf{E}''_s + \mathbf{E}''_p = (E''_x,\ E''_y,\ E''_z) = E_y\ t_{PB}\ t_{PA}\ (-(1/4)\ (1-\cos\psi)\sin 2\phi,\ (1/4)[(1-\cos\psi)\cos 2\phi + (1+3\cos\psi)],\ \sin\phi\sin\psi) \quad (9)$$

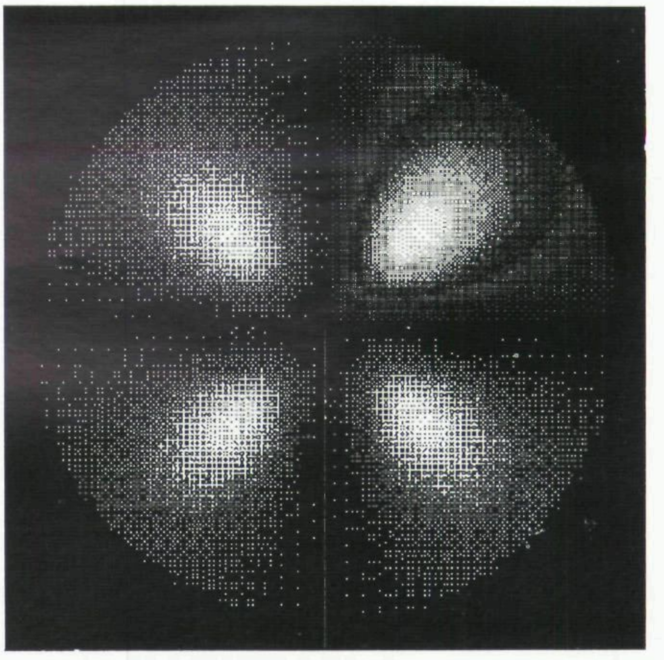

Figure 2 Computer simulated pattern following the described ray tracing procedure for a uniform index, spherical lens with $n_{lens}/n_{surround} = 1.40$

E''_x, E''_y, E''_z are magnitudes of the field in the directions x', y' and z', respectively, which represent the axes of a coordinate system rotated, about the z-axis, through an angle ϕ so that

$$x = x'\cos\phi - y'\sin\phi$$

$$y = x'\sin\phi + y'\cos\phi$$

$$z = z'$$

The intensity of light after passing through two polarizers is

$$I = (E''_y)^2\cos^2\theta + (E''_x)^2\sin^2\theta \quad (10)$$

where θ is the angle between the polarizers. When the polarizers are crossed, that is, $\theta = \pi/2$, only the E''_x component remains and

$$I = (E''_x)^2 = (1/16)\ (E_y\ t_{PB}\ t_{PA}\ (1-\cos\psi)\sin 2\phi))^2$$

At all entry points where $\phi = 0$ or $\pi/2$, $I = 0$, the light is extinguished and the resulting pattern is that of a cross.

A calculation based on equation 10 following the above ray tracing procedure for all rays entering a uniform index, spherical lens is shown in *Figure 2* and this clearly illustrates the pattern of isogyres.

Conclusions

The isogyre pattern seen in non-birefringent, refracting elements placed between crossed polarizers, results from refraction. Refractive power of a lens depends on its shape as well as its refractive index or more specifically index differences in adjacent regions. In the eye lens the refractive index varies, increasing towards the centre.

The major part of the lenticular refraction, *in vitro*, takes place at the lens surfaces where light rays encounter relatively large index differences between the lens edge and the air. A smaller part of the refraction takes place within the lens as the light rays traverse gradual index variations. Consequently the isogyre pattern seen when the lens is placed between crossed polarizers will be largely the result of refraction at the lens surfaces.

In refracting structures, unlike in birefringent crystals, the illuminating source does not need to be convergent to produce the pattern since the structure itself converges the incoming rays. Although in the eye lens the isogyre pattern is the result of refraction, this does not negate the fact that there may be a small degree of birefringence on a cellular level, within in these ocular elements. In the lens, the form birefringence which is caused by arrangement of the fibre cell membranes, and the intrinsic birefingence, which arises from the intercellular organization, are opposite in value and hence almost negligible amounts of total birefringence have been reported[1,2].

Isogyres and isochromatics which have been observed in the corneas of living human and animal eyes[7] have been attributed to birefringence arising from the arrangement of its lamellae[8,9]. There may however be a secondary effect of stress-induced birefringence which occurs because the cornea *in vivo* is stretched by the forces of internal ocular pressure. The effect has been simulated by stretching strips of corneal tissue[10]. Isogyres seen in a cornea, which has been removed from the eye (and is therefore no longer subjected to stretching forces) but retains its shape, are not accompanied by isochromatics and may well result from the corneal curvature; a flattened cornea will not produce the cross. Similarly, a much clearer sharper cross is evident in a bovine lens than in the comparatively less curved human lens.

Birefringence can arise naturally from the structural organization or be induced by application of stretching forces. The cross pattern which is seen in birefringent structures can also result in highly refractive structures with little or no overall birefringence, such as the eye lens. The pattern is therefore not necessarily an indicator of the property of birefringence but a consequence of it.

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