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A unified formulation of wave phenomena in electromagnetics and elasticity

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Contents

1 Solution of Maxwell's Equations

We give the two common forms for the solution of Maxwell's equations for the electric field due to the presence of a real volume current density, $\boldsymbol{j}$ and a real volume charge density, ρ that are related by the equation of continuity. The aim is to express the electric field, $\boldsymbol{E}$ in terms of the current density, $\boldsymbol{j}$ [1]. This formalism is then used to show how scattering problems by perfect electric conductors are solved currently.

In the frequency domain, the electromagnetic fields due to a real volume current density, $\boldsymbol{j}$ and a real volume charge density, ρ with harmonic time dependence $e^{-i\omega t}$ are governed by Maxwell's equations:

$$\nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H} \tag{1.1}$$

$$\nabla \times \boldsymbol{H} = -i\omega\varepsilon\boldsymbol{E} + \boldsymbol{j} \tag{1.2}$$

$$\nabla \cdot \boldsymbol{D} = \rho \tag{1.3}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.4}$$

In a linear homogeneous medium we have the material constitutive equations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} \equiv \varepsilon_r\varepsilon_0\,\boldsymbol{E} \tag{1.5}$$

$$\boldsymbol{B} = \mu\boldsymbol{H} \equiv \mu_r\mu_0\,\boldsymbol{H} \tag{1.6}$$

The continuity equation relates the current density, $\boldsymbol{j}$ and the charge density, ρ

$$\nabla \cdot \boldsymbol{j} = i\omega\rho \tag{1.7}$$

1.1 *Electric field in terms of the current density*

We can express the electric field $\boldsymbol{E}$ in terms of the current density by taking the curl of eq (1.1) together with the vector identity: $\nabla \times \nabla \times \boldsymbol{V} = \nabla(\nabla \cdot \boldsymbol{V}) - \nabla^2\boldsymbol{V}$, and using eqs (1.2), (1.3), (1.5) and (1.7) to eliminate $\boldsymbol{H}$ and ρ to give

$$\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = -i\omega\mu\left(\boldsymbol{j} + \frac{1}{k^2}\nabla(\nabla \cdot \boldsymbol{j})\right) \tag{1.8}$$

where $k^2 = \omega^2\varepsilon\mu$. The solution of eq (1.8) can be written as

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{i\omega\mu}{4\pi}\iiint_V G(\boldsymbol{x},\boldsymbol{x}')\left(\boldsymbol{j}(\boldsymbol{x}') + \frac{1}{k^2}\nabla'(\nabla' \cdot \boldsymbol{j}(\boldsymbol{x}'))\right)\,d\boldsymbol{x}' \tag{1.9}$$

where the Green's function

$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{\exp\,(ik|\boldsymbol{x}-\boldsymbol{x}'|)}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{1.10}$$

satisfies

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x}') + k^2 G(\boldsymbol{x},\boldsymbol{x}') = -4\pi\,\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{1.11}$$

Note that the solution in eq (1.8) requires derivatives of the current density and the integrand has a weak, but integrable singularity at $\boldsymbol{x}' = \boldsymbol{x}$ due to the Green's function $G(\boldsymbol{x}, \boldsymbol{x}')$. In some problems, it is not possible or convenient to calculate derivatives of the current density, $\boldsymbol{j}$ so an alternate formulation is used.

1.2 *Electric field via the vector potential*

We begin with the vector potential $\boldsymbol{A}$ defined by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.12}$$

and using this in eq (1.1) means that the most general form for the electric field can be expressed in terms of $\boldsymbol{A}$ and a scalar potential ϕ

$$\boldsymbol{E} = i\omega\boldsymbol{A} - \nabla\phi \tag{1.13}$$

Then taking the curl of eq (1.12) and using eqs (1.2) and (1.13) gives

$$\nabla^2\boldsymbol{A} + k^2\boldsymbol{A} = -\mu\boldsymbol{j} + \nabla(\nabla \cdot \boldsymbol{A} - i\omega\mu\varepsilon\,\phi) \tag{1.14}$$

The choice

$$\nabla \cdot \boldsymbol{A} = i\omega\mu\varepsilon\,\phi \tag{1.15}$$

means that eq (1.14) simplifies to

$$\nabla^2\boldsymbol{A} + k^2\boldsymbol{A} = -\mu\boldsymbol{j} \tag{1.16}$$

This has solution

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi}\,\mu \iiint_V G(\boldsymbol{x}, \boldsymbol{x}')\,\boldsymbol{j}(\boldsymbol{x}')\,d\boldsymbol{x}' \tag{1.17}$$

and the electric field then follows from eqs (1.13) and (1.15)

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x}) &= i\omega\boldsymbol{A} - \nabla\phi = i\omega\boldsymbol{A} + \frac{i}{\omega\mu\varepsilon}\nabla(\nabla \cdot \boldsymbol{A}) \\ &= \frac{i\omega\mu}{4\pi}\iiint_V G(\boldsymbol{x}, \boldsymbol{x}')\,\boldsymbol{j}(\boldsymbol{x}')\,d\boldsymbol{x}' + \frac{i}{4\pi\omega\varepsilon}\,\nabla\nabla \cdot \left(\iiint_V G(\boldsymbol{x}, \boldsymbol{x}')\,\boldsymbol{j}(\boldsymbol{x}')\,d\boldsymbol{x}'\right)\end{aligned} \tag{1.18}$$

In the solution represented by eqs (1.17) and (1.18), the integral for $\boldsymbol{A}$ in eq (1.17) contains a weak, but integrable singularity due to the Green's function $G(\boldsymbol{x}, \boldsymbol{x}')$ and in eq (1.18) derivatives of $\boldsymbol{A}$ are required to obtain $\boldsymbol{E}$. Such derivatives are well defined.

However, in many implementations, the two derivatives in eq (1.18) are applied to the integral for $\boldsymbol{A}$ in eq (1.17) and then a change of the order of differentiation and integration is made to take the two derivatives with respect to $\boldsymbol{x}$ *inside* the integral. This results in the dyadic Green's function $\nabla\nabla G(\boldsymbol{x}, \boldsymbol{x}')$ that gives rise to a hypersingular integrand because it diverges as $|\boldsymbol{x} - \boldsymbol{x}'|^{-3}$. This divergent behaviour is then interpreted mathematically as the Hadamard finite part and provides additional challenges in numerical implementation.

This mathematical divergence has no physical basis. The weak but integrable singularity of $G(\boldsymbol{x}, \boldsymbol{x}')$ in the integrand of eq (1.17) means that the convergent properties of integral does not permit interchange the order of differentiation and integration when eq (1.18) is applied to calculate $\boldsymbol{E}$, without the use of generalized functions to interpret the resulting divergence.

2 Scattering by perfect electric conductors (PEC)

The scattering of a given incident electric field, $\boldsymbol{E}^{inc}$ by perfect electric conductors is treated by finding the induced 2D surface current density, $\boldsymbol{J}$ on the surface of conductor S that would extinguish all fields in the interior of the conductor. Thus the scattered electric field, $\boldsymbol{E}^{scat}$ that originates from the induced surface current density, $\boldsymbol{J}$ can be obtained using eq (1.9)

$$\boldsymbol{E}^{scat}(\boldsymbol{x}) = \frac{i\omega\mu}{4\pi} \iint_S G(\boldsymbol{x}, \boldsymbol{x}') \left(\boldsymbol{J}(\boldsymbol{x}') + \frac{1}{k^2} \nabla'(\nabla' \cdot \boldsymbol{J}(\boldsymbol{x}')) \right) d\boldsymbol{x}' \qquad 2.1$$

The boundary condition on the surface of the conductor requires the tangential component of the total field, that is, the sum of the incident field and the scattered field, to vanish

$$\boldsymbol{E}_{||} = [\boldsymbol{E}^{inc} + \boldsymbol{E}^{scat}]_{||} = \boldsymbol{0} \qquad 2.2$$

From eq (1.19), this can be written as

$$-\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{inc}(\boldsymbol{x}) = \frac{i\omega\mu}{4\pi} \boldsymbol{n}(\boldsymbol{x}) \times \iint_S G(\boldsymbol{x}, \boldsymbol{x}') \left(\boldsymbol{J}(\boldsymbol{x}') + \frac{1}{k^2} \nabla'(\nabla' \cdot \boldsymbol{J}(\boldsymbol{x}')) \right) d\boldsymbol{x}' \qquad 2.3$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector at position $\boldsymbol{x}$ on the surface S. Thus for a given incident field, $\boldsymbol{E}^{inc}$, the scattering problem then involves solving eq (2.3) for the induced surface current density, $\boldsymbol{J}$ and then eq (2.1) can be used to compute the scattered field as a post-processing task.

The most successful and common practical implementation of eq (2.3) is to use planar triangular elements to represent the surfaces, S of general 3D conducting bodies and the Rao-Wilton-Glisson (RWG) basis functions are used to represent the induced surface current density, $\boldsymbol{J}$. The coefficients of the basis functions are found by inverting the linear matrix problem that results from a discretised integration of the surface integral in eq (2.3).

3 Boundary regularised integral equation formulation of the Helmholtz equation

The solution of the scalar Helmholtz equation $\nabla^2 p + k^2 p = 0$ can be found by solving the boundary integral equation [2]

$$c_0\, p(\boldsymbol{x}_0) + \iint_S p(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}_0, \boldsymbol{x})}{\partial n}\, d\boldsymbol{x} = \iint_S G(\boldsymbol{x}_0, \boldsymbol{x}) \frac{\partial p(\boldsymbol{x})}{\partial n}\, d\boldsymbol{x} \tag{3.1}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}_0$ are on the boundary S and c_0 is related to the solid angle at $\boldsymbol{x}_0$. The singularities associated with G and $\partial G/\partial n$ can be eliminated by noting that the function $\psi(\boldsymbol{x}) \equiv p(\boldsymbol{x_0})g(\boldsymbol{x}) + (\partial p/\partial n)_0 f(\boldsymbol{x})$ satisfies the Helmholtz equation and hence the boundary integral equation (3.1) if the functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ also satisfy the Helmholtz.

By subtracting the boundary integral equation for $\psi(\boldsymbol{x})$ from that for $p(\boldsymbol{x})$ we obtain the following boundary integral equation

$$\iint_S \left[p(\boldsymbol{x}) - p(\boldsymbol{x_0})g(\boldsymbol{x}) - \frac{\partial p(\boldsymbol{x}_0)}{\partial n} f(\boldsymbol{x})\right] \frac{\partial G(\boldsymbol{x}_0, \boldsymbol{x})}{\partial n}\, d\boldsymbol{x}$$
$$= \iint_S G(\boldsymbol{x}_0, \boldsymbol{x}) \left[\frac{\partial p(\boldsymbol{x})}{\partial n} - p(\boldsymbol{x_0})\, \boldsymbol{n} \cdot \nabla g(\boldsymbol{x}) - \frac{\partial p(\boldsymbol{x}_0)}{\partial n}\, \boldsymbol{n} \cdot \nabla f(\boldsymbol{x})\right] d\boldsymbol{x} \tag{3.2}$$

that will be free of singularities if we choose the functions f and g to have the properties [3]

$$\begin{aligned} &\nabla^2 f + k^2 f = 0, \quad f(\boldsymbol{x}_0) = 0, \quad \nabla f(\boldsymbol{x}_0) \cdot \boldsymbol{n}_0 = 1 \\ &\nabla^2 g + k^2 g = 0, \quad g(\boldsymbol{x}_0) = 1, \quad \nabla g(\boldsymbol{x}_0) \cdot \boldsymbol{n}_0 = 0 \end{aligned} \tag{3.3}$$

This desingularized formulation also provides a robust way to evaluate p near the boundary that does not involve singularities, see [3] for details.

The absence of singularities makes it straightforward to use quadrature to evaluate the surface integrals and to use higher order elements to represent the surface S since the solid angle related term c_0 in the standard form of the boundary integral equation, eq (3.1) has also been eliminated. Such flexibility can improve the accuracy by orders of magnitude for the same number of unknowns [3].

In electromagnetic scattering applications, the limit $k \to 0$ is known to pose numerical challenges because of the $1/k^2$ term in eq (2.3). To handle this limit in our implementation, we write

$$G(\boldsymbol{x}_0, \boldsymbol{x}) \equiv \frac{\exp(ik|\boldsymbol{x}-\boldsymbol{x}_0|)}{|\boldsymbol{x}-\boldsymbol{x}_0|} = \frac{\exp(ik|\boldsymbol{x}-\boldsymbol{x}_0|)-1}{|\boldsymbol{x}-\boldsymbol{x}_0|} + \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|} \qquad 3.4$$

where the first term on the right hand side of eq (3.4) is regular as $\boldsymbol{x} \to \boldsymbol{x}_0$ and vanishes as $k \to 0$. The second term has a $1/|\,\boldsymbol{x}-\boldsymbol{x}_0|$ singularity. Similarly, we write

$$\begin{aligned}\frac{\partial G(\boldsymbol{x}_0, \boldsymbol{x})}{\partial n} &\equiv \left(ik - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}\right)\frac{\exp(ik|\boldsymbol{x}-\boldsymbol{x}_0|)}{|\boldsymbol{x}-\boldsymbol{x}_0|^2}(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{n} \\ &= \left[ik\frac{\exp(ik|\boldsymbol{x}-\boldsymbol{x}_0|)}{|\boldsymbol{x}-\boldsymbol{x}_0|^2} - \frac{\exp(ik|\boldsymbol{x}-\boldsymbol{x}_0|)-1}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}\right](\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{n} + \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}(\boldsymbol{x}-\boldsymbol{x}_0) \\ &\quad \cdot\boldsymbol{n}\end{aligned} \qquad 3.5$$

where again the first term on the right hand side is regular as $\boldsymbol{x} \to \boldsymbol{x}_0$ and vanishes in the limit $k \to 0$. The second term has a $1/|\,\boldsymbol{x}-\boldsymbol{x}_0|$ singularity. This leads us to just focus on removing the $1/|\,\boldsymbol{x}-\boldsymbol{x}_0|$ singularity.

To achieve this, we construct a function, $\psi_0(\boldsymbol{x}) \equiv p(\boldsymbol{x}_0)g_0(\boldsymbol{x}) + (\partial p/\partial n)_0 f_0(\boldsymbol{x})$, which satisfies the Laplace equation: $\nabla^2\psi_0(\boldsymbol{x}) = 0$ [4,5]. The boundary integral equation that $\psi_0(\boldsymbol{x})$ satisfies is

$$c_0\,\psi_0(\boldsymbol{x}_0) + \iint_S \psi_0(\boldsymbol{x})\frac{\partial G_0(\boldsymbol{x}_0, \boldsymbol{x})}{\partial n}\,d\boldsymbol{x} = \iint_S G_0(\boldsymbol{x}_0, \boldsymbol{x})\frac{\partial \psi_0(\boldsymbol{x})}{\partial n}\,d\boldsymbol{x} \qquad 3.6$$

where $G_0(\boldsymbol{x}_0, \boldsymbol{x}) = 1/|\boldsymbol{x}_0 - \boldsymbol{x}|$.

Now subtracting the above equation (3.6) from eq (3.1), we obtain the new boundary integral equation

$$\begin{aligned}&\iint_S \left\{p(\boldsymbol{x})\frac{\partial G(\boldsymbol{x}_0, \boldsymbol{x})}{\partial n} - \left[p(\boldsymbol{x_0})g_0(\boldsymbol{x}) + \frac{\partial p(\boldsymbol{x}_0)}{\partial n}f_0(\boldsymbol{x})\right]\frac{\partial G_0(\boldsymbol{x}_0, \boldsymbol{x})}{\partial n}\right\}\,d\boldsymbol{x} \\ &= \iint_S \left\{\frac{\partial p(\boldsymbol{x})}{\partial n}G(\boldsymbol{x}_0, \boldsymbol{x}) - \left[p(\boldsymbol{x_0})\,\boldsymbol{n}\cdot\nabla g_0(\boldsymbol{x}) + \frac{\partial p(\boldsymbol{x}_0)}{\partial n}\,\boldsymbol{n}\cdot\nabla f_0(\boldsymbol{x})\right]G_0(\boldsymbol{x}_0, \boldsymbol{x})\right\}\,d\boldsymbol{x}\end{aligned} \qquad 3.7$$

that will be fully regular if we choose the functions f_0 and g_0 to have the properties [4,5]

$$\nabla^2 f_0 = 0, \quad f_0(\boldsymbol{x}_0) = 0, \quad \nabla f_0(\boldsymbol{x}_0) \cdot \boldsymbol{n}_0 = 1$$
$$\nabla^2 g_0 = 0, \quad g_0(\boldsymbol{x}_0) = 1, \quad \nabla g_0(\boldsymbol{x}_0) \cdot \boldsymbol{n}_0 = 0 \qquad 3.8$$

By using this idea, the numerical implementation will remain robust even in the limit $k = 0$.

4 Identities for propagating vector fields: $\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0$

4.1 $\quad \nabla \cdot \boldsymbol{E} = 0 \quad \Leftrightarrow \quad \nabla^2(\boldsymbol{x} \cdot \boldsymbol{E}) + k^2(\boldsymbol{x} \cdot \boldsymbol{E}) = 0$

Using Cartesian tensor notation, we have

$$\nabla^2(\boldsymbol{x} \cdot \boldsymbol{E}) = \frac{\partial^2}{\partial x_k^2}[x_i E_i] = \frac{\partial}{\partial x_k}\left[\delta_{ik} E_i + x_i \frac{\partial E_i}{\partial x_k}\right] = \frac{\partial E_k}{\partial x_k} + \delta_{ik}\frac{\partial E_i}{\partial x_k} + x_i \frac{\partial^2 E_i}{\partial x_k^2} \qquad 4.1$$

that, in vector notation becomes

$$\nabla^2(\boldsymbol{x} \cdot \boldsymbol{E}) = 2\nabla \cdot \boldsymbol{E} + \boldsymbol{x} \cdot \nabla^2 \boldsymbol{E} \qquad 4.2$$

If $\boldsymbol{E}$ obeys the Helmholtz equation

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \qquad 4.3$$

then eq (4.2) gives

$$2\nabla \cdot \boldsymbol{E} = \nabla^2(\boldsymbol{x} \cdot \boldsymbol{E}) + k^2(\boldsymbol{x} \cdot \boldsymbol{E}) \qquad 4.4$$

In other words, if $\boldsymbol{E}$ satisfies the Helmholtz equation: $\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0$, and $\nabla \cdot \boldsymbol{E} = 0$, then $(\boldsymbol{x} \cdot \boldsymbol{E})$ also satisfies the Helmholtz equation: $\nabla^2(\boldsymbol{x} \cdot \boldsymbol{E}) + k^2(\boldsymbol{x} \cdot \boldsymbol{E}) = 0$, a result that has been derived by Wilcox [6] via a different method.

4.2 $\quad \nabla \times \boldsymbol{u} = \boldsymbol{0} \quad \Leftrightarrow \quad \nabla^2(\boldsymbol{x} \times \boldsymbol{u}) + k^2(\boldsymbol{x} \times \boldsymbol{u}) = \boldsymbol{0}$

Using Cartesian tensor notations, we have

$$\nabla^2(\boldsymbol{x} \times \boldsymbol{u}) = \frac{\partial^2}{\partial x_n^2}\left[\epsilon_{ijk}\, x_j\, u_k\right] = \epsilon_{ijk}\frac{\partial}{\partial x_n}\left[\delta_{nj} u_k + x_j \frac{\partial u_k}{\partial x_n}\right] = \epsilon_{ijk}\left[\frac{\partial u_k}{\partial x_j} + \delta_{nj}\frac{\partial u_k}{\partial x_n} + x_j \frac{\partial^2 u_k}{\partial x_n^2}\right] \qquad 4.5$$

that is,

$$\nabla^2(\boldsymbol{x} \times \boldsymbol{u}) = 2\nabla \times \boldsymbol{u} + \boldsymbol{x} \times \nabla^2 \boldsymbol{u} \qquad 4.6$$

If $\boldsymbol{u}$ obeys the Helmholtz equation

$$\nabla^2\boldsymbol{u} + k^2\boldsymbol{u} = \boldsymbol{0} \tag{4.7}$$

then we have the result

$$2\nabla \times \boldsymbol{u} = \nabla^2(\boldsymbol{x} \times \boldsymbol{u}) + k^2(\boldsymbol{x} \times \boldsymbol{u}) \tag{4.8}$$

In other words, if $\nabla^2\boldsymbol{u} + k^2\boldsymbol{u} = \boldsymbol{0}$ then $\nabla^2(\boldsymbol{x} \times \boldsymbol{u}) + k^2(\boldsymbol{x} \times \boldsymbol{u}) = \boldsymbol{0}$ is equivalent to $\nabla \times \boldsymbol{u} = \boldsymbol{0}$.

5 Solution for a PEC sphere at $k = 0$

In the limit $k = 0$, the scattering by a perfect electric conductor (PEC) is a standard textbook problem of the Laplace equation: $\nabla^2 V = 0$ for the electrostatic potential V subject to an incident field: $\boldsymbol{E}^{inc} = E_0\,\mathbf{k}$ so that the far field boundary condition means $V = -E_0\, z = -E_0\, r \cos\theta$, as $r \to \infty$, and that the tangential components of the total field are zero on the surface of the conducting sphere. The general solution of $\nabla^2 V = 0$ is

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n^m(\cos\theta)(S_n \sin(m\varphi) + C_n \cos(m\varphi)) \tag{5.1}$$

The above far field and boundary conditions determine the solution

$$V = -E_0 \left(1 - \frac{a^3}{r^3}\right) r \cos\theta \tag{5.2}$$

The corresponding non-zero components of $\boldsymbol{E}$ are

$$E_r = -\frac{\partial V}{\partial r} = E_0 \left(1 + \frac{2a^3}{r^3}\right) \cos\theta \tag{5.3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_0 \left(1 - \frac{a^3}{r^3}\right) \sin\theta \tag{5.4}$$

It is then straightforward to show that the Cartesian components

$$E_z = E_r \cos\theta - E_\theta \sin\theta = -E_0 \left(1 + \frac{2a^3}{r^3} P_2\ (\cos\theta)\right) \tag{5.5}$$

$$E_x = (E_r \sin\theta + E_\theta \cos\theta) \cos\varphi = -E_0\, \frac{a^3}{r^3} P_2^1(\cos\theta) \cos\varphi \tag{5.6}$$

$$E_y = (E_r \sin\theta + E_\theta \cos\theta)\sin\varphi = -\,E_0\,\frac{a^3}{r^3}P_2^1(\cos\theta)\sin\varphi \qquad 5.7$$

all satisfy the Laplace equation in view of eq (5.1).

Finally we find

$$\boldsymbol{x}\cdot\boldsymbol{E} = xE_x + yE_y + zE_z = -\,E_0\,\frac{a^3}{r^3}P_2^1(\cos\theta)r\,\sin\theta + E_0\left(1+\frac{2a^3}{r^3}P_2\,(\cos\theta)\right)r\,\cos\theta \qquad 5.8$$

$$= E_0\,r\cos\theta + 2E_0\,\frac{a^3}{r^3}\,P_1\,(\cos\theta)$$

also satisfies the Laplace equation because of eq (5.1).

Note: $P_1\,(\cos\theta) = \cos\theta$, $P_2\,(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$, $P_2^1(\cos\theta) = -3\cos\theta\sin\theta$

6 Linear system for the discretised boundary integral equations

In our numerical implementation, we solve for the scattered field, $\boldsymbol{E}$ that satisfies the system of 4 scalar equations

$$\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = 0 \qquad 6.1$$

$$\nabla^2(\boldsymbol{x}\cdot\boldsymbol{E}) + k^2(\boldsymbol{x}\cdot\boldsymbol{E}) = 0 \qquad 6.2$$

by the boundary integral method. On the surface of a boundary, it is convenient to work in terms of the normal and tangential components $\boldsymbol{E} = \boldsymbol{E}_n + \boldsymbol{E}_t$ that is related to the Cartesian *x*-component and the surface unit normal vector $\boldsymbol{n}$ by

$$E_x = \boldsymbol{E}_n\cdot\boldsymbol{i} + \boldsymbol{E}_t\cdot\boldsymbol{i} = E_n(\boldsymbol{n}\cdot\boldsymbol{i}) + \boldsymbol{E}_t\cdot\boldsymbol{i} = E_n\,n_x - E_{t,x}^{inc} \qquad 6.3$$

where the tangential component of the scattered field, $E_{t,x} = \boldsymbol{E}_t\cdot\boldsymbol{i}$ cancels the corresponding component of the incident field $E_{t,x}^{inc}$, on the surface of the perfect conductor. Similarly, we have for the *y* and *z* components

$$E_y = E_n\,n_y - E_{t,y}^{inc} \qquad 6.4$$

$$E_z = E_n\,n_z - E_{t,z}^{inc} \qquad 6.5$$

We discretize the surface with N nodes and let H and G be the discrete representations of the integrals involving the p_i and $\partial p_i/\partial n$ terms in eq (6) of the main text where p_i represents (E_x, E_y, E_z, $\boldsymbol{x}\cdot\boldsymbol{E}$), we obtain

$$\mathsf{H}\cdot E_x = \mathsf{G}\cdot(\partial E_x/\partial n) \tag{6.6}$$

$$\mathsf{H}\cdot E_y = \mathsf{G}\cdot(\partial E_y/\partial n) \tag{6.7}$$

$$\mathsf{H}\cdot E_z = \mathsf{G}\cdot(\partial E_z/\partial n) \tag{6.8}$$

$$\mathsf{H}\cdot(\boldsymbol{x}\cdot\boldsymbol{E}) = \mathsf{G}\cdot\partial(\boldsymbol{x}\cdot\boldsymbol{E})/\partial n \tag{6.9}$$

For the left hand side of eqs (6.6) – (6.8), we use eqs (6.3) – (6.5) to eliminate the Cartesian components: E_x, E_y and E_z in terms of the normal component, E_n, and the tangential component of the incident field, E_t^{inc}. For eq (6.9) we use eqs (6.3) – (6.5) to write

$$\begin{aligned}(\boldsymbol{x}\cdot\boldsymbol{E}) &= x\big(E_n\, n_x - E_{t,x}^{inc}\big) + y(E_y = E_n\, n_y - E_{t,y}^{inc}) + z(E_n\, n_z - E_{t,z}^{inc}) \\ &= (\boldsymbol{x}\cdot\boldsymbol{n})\, E_n - \big(x\, E_{t,x}^{inc} + y\, E_{t,y}^{inc} + z\, E_{t,z}^{inc}\big)\end{aligned} \tag{6.10}$$

and

$$\frac{\partial(\boldsymbol{x}\cdot\boldsymbol{E})}{\partial n} = E_n + \boldsymbol{x}\cdot\frac{\partial\boldsymbol{E}}{\partial n} \tag{6.11}$$

Thus eqs (6.6) – (6.9) can be expressed in terms of the normal component E_n and the normal derivative $\partial\boldsymbol{E}/\partial n$ of the scattered field as

$$\mathsf{H}\cdot(n_x E_n) - \mathsf{H}\cdot E_{t,x}^{inc} = \mathsf{G}\cdot(\partial E_x/\partial n) \tag{6.12}$$

$$\mathsf{H}\cdot\big(n_y E_n\big) - \mathsf{H}\cdot E_{t,y}^{inc} = \mathsf{G}\cdot(\partial E_y/\partial n) \tag{6.13}$$

$$\mathsf{H}\cdot(n_z E_n) - \mathsf{H}\cdot E_{t,z}^{inc} = \mathsf{G}\cdot(\partial E_z/\partial n) \tag{6.14}$$

$$\mathsf{H}\cdot(\boldsymbol{x}\cdot\boldsymbol{n})\, E_n = \mathsf{G}\cdot[E_n + \boldsymbol{x}\cdot\partial\boldsymbol{E}/\partial n] \tag{6.15}$$

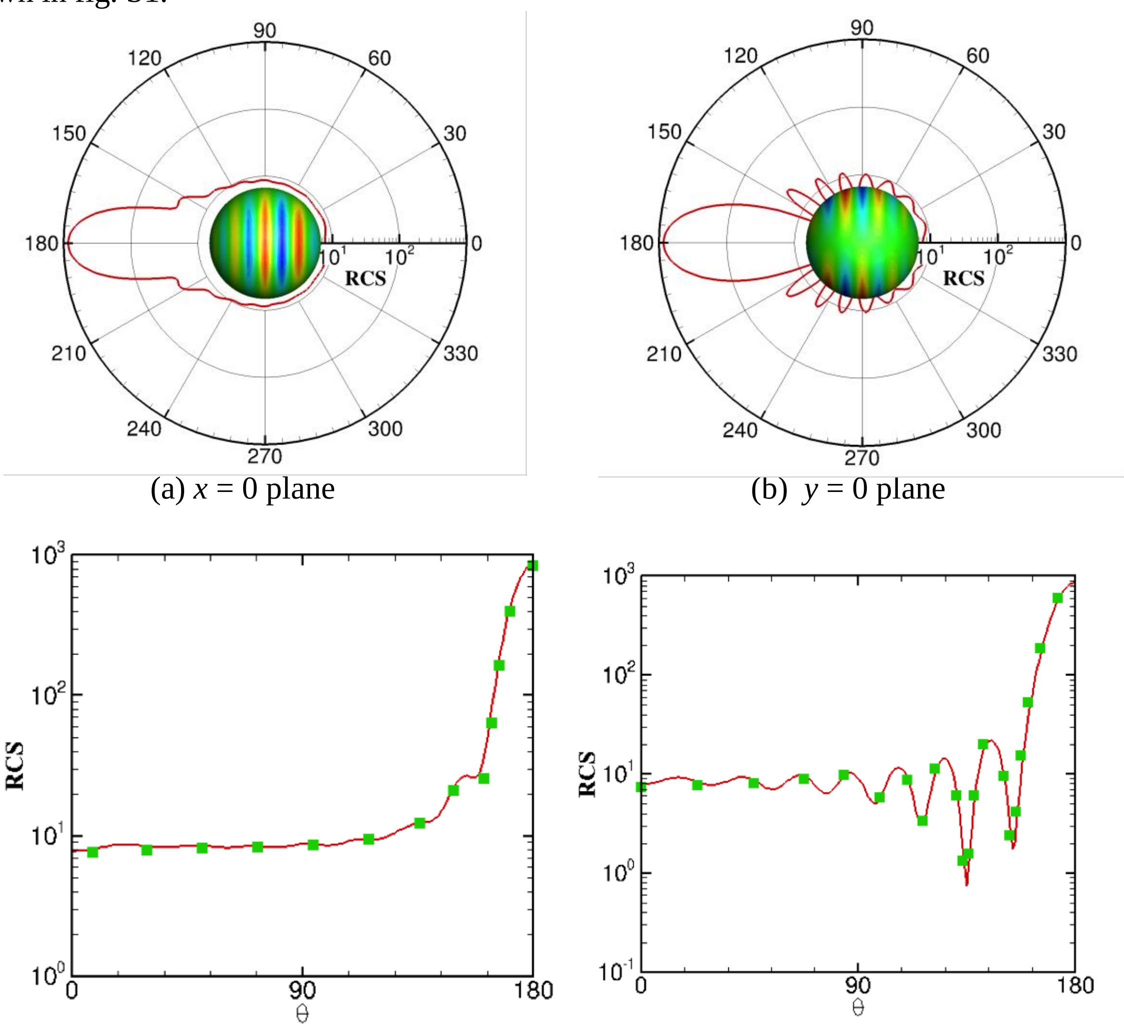
By discretising the surface as N nodes, eqs (6.12) – (6.15) is a $4N \times 4N$ linear system for the 4 unknowns N-vectors: {E_n, $\partial E_x/\partial n$, $\partial E_y/\partial n$, $\partial E_z/\partial n$} on the surface

$$\begin{bmatrix} -\mathsf{G} & 0 & 0 & \mathsf{H}\, n_x \\ 0 & -\mathsf{G} & 0 & \mathsf{H}\, n_y \\ 0 & 0 & -\mathsf{G} & \mathsf{H}\, n_z \\ -\mathsf{G}x & -\mathsf{G}y & -\mathsf{G}z & -\mathsf{G}+\mathsf{H}(\boldsymbol{x}\cdot\boldsymbol{n}) \end{bmatrix} \begin{bmatrix} \partial E_x/\partial n \\ \partial E_y/\partial n \\ \partial E_z/\partial n \\ E_n \end{bmatrix} = \begin{bmatrix} \mathsf{H}\cdot E_{t,x}^{inc} \\ \mathsf{H}\cdot E_{t,y}^{inc} \\ \mathsf{H}\cdot E_{t,z}^{inc} \\ \mathsf{H}\cdot(\boldsymbol{x}\cdot\boldsymbol{E}_t^{inc}) \end{bmatrix} \tag{6.16}$$

7 Induced surface charge and radar cross-section of a perfect conducting sphere

7.1 *Radar cross sections*

We give results computed by our method for the scattering of a linearly polarized plane wave by a perfect conducting sphere of radius *a* – the Mie problem. We show the induced surface charge density that is proportional to the normal component of the total electric field and radar cross section computed from the far field values. The incident wave has electric field is polarized in the z-direction: $\boldsymbol{E}^{inc} = (0, 0, E_0)$ and propagates in the *x* direction: $\boldsymbol{k} = (k, 0, 0)$. The results for $ka = 10$ and $E_0 = 1$ are shown in fig. S1.



(a) $x = 0$ plane (b) $y = 0$ plane

(c) $x = 0$ plane (d) $y = 0$ plane

Fig. S1. Induced surface charge density (color sphere) and radar cross-section (RCS) in the planes (a) $x = 0$ and (b) $y = 0$ represented as polar plots and as functions of the angle θ, for $\boldsymbol{E}^{inc} = (1, 0, 0)$ and $\boldsymbol{k} = (0, 0, k)$ calculated using 1442 nodes on the sphere (lines). Symbols (squares) in (c) and (d) are results from an implementation of the infinite series Mie solution [7].

The radar cross section (RCS), scaled by the incident wavelength, λ, is the ratio of the magnitudes of the scattered far field to the incident field:

$$\text{RCS} = \lim_{r\to\infty} 4\pi \frac{r^2}{\lambda^2} \frac{|\boldsymbol{E}^{scat}|^2}{|\boldsymbol{E}^{inc}|^2} \quad 7.1$$

7.2 *The k = 0 limit*

In the conventional formulation of scattering by perfect electric conductors, the long wavelength limit ($k \to 0$) is singular, see eq (2.3). However, our formulation does not suffer from this limitation. In fig S2 we compare our numerical results for $k = 0$ with the analytic results derived in eqs (5.6) and (5.7) for the scattered fields E_x and E_z on the surface of a unit sphere in response to an incident field $\boldsymbol{E}^{inc}= (0, 0, 1)$.

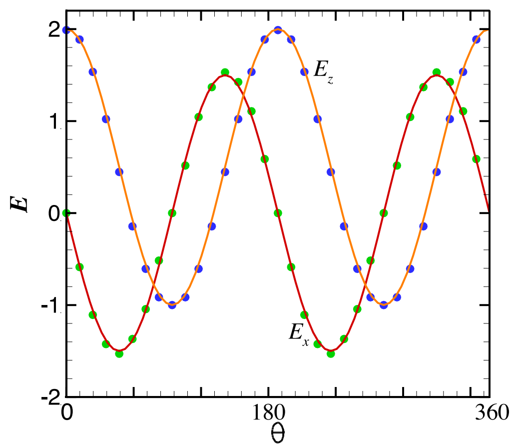


Fig. S2. Scattered fields E_x and E_z on the surface of a unit sphere as functions of the polar angle, θ in response to an incident field $\boldsymbol{E}^{inc}= (0, 0, 1)$ at $k = 0$ computed by our method (points) using 386 nodes and the analytic solutions (lines) given by eqs (5.6) and (5.7).

8 Needle and ellipsoid shape function

We give the equation for the axisymmetric needle shape function with fore-aft symmetry [8]. In the body coordinate system, the axisymmetric surface is defined by rotating the closed curve (ξ, ρ) with semi-major axis, a and semi-minor axis, b about the ξ-axis. The equation for a point on the curve or the axisymmetric surface (ξ, ρ) is given by

$$\frac{1}{\rho^2}\left[\frac{\xi + c}{R_+} - \frac{\xi - c}{R_-}\right] = \frac{1}{R^2} \quad 8.1$$

where $R_+(\xi, \rho)$ and $R_-(\xi, \rho)$ are defined by

$$R_{\pm} \equiv [\ (\xi \pm c)^2 + \rho^2\ ]^{\frac{1}{2}}, \tag{8.2}$$

the constant R is given in terms of the semi-axes a and b; and the position of the focus, c ($< a$) on the ξ-axis, is calculated via two equivalent expressions

$$\frac{2ac}{(a^2 - c^2)^2} = \frac{1}{R^2} = \frac{2c}{b^2(b^2 + c^2)^{1/2}} \tag{8.3}$$

The shape varies from a sphere at $b/a = 1$ to a long thin needle as $b/a \to 0$, approaching a circular cylinder with spherical cap ends.

Thus given the aspect ratio, b/a, eq (8.3) can be used to determine c/a and R/a, and the rotation of the curve given by eq (8.1) about the ξ-axis generates the axisymmetric surface.

The 3D ellipsoid is represented by the equation

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1 \tag{8.4}$$

9 Animations - phase variations of scattered field and the induced surface charge and boundary integral parameters

One_sphere.mov: Animations of the variation of the induced surface charge density and the scattered components of the electric field in a single perfect electric conducting sphere of radius, a over one cycle for $ka = 10$ - see Fig 1 in the main text.

Three_sphere.mov: Animations of the variation of the induced surface charge density and the scattered components of the electric field in three single perfect electric conducting sphere of radius, a over one cycle for $ka = 1$ – see Fig 2 in the main text.

Needle.mov: Animations of the variation of the induced surface charge density and the scattered components of the electric field in a perfect conducting needle of aspect ratio 10 over one cycle for $ka = 5$, $ba = 0.5$ – see Fig 3 in the main text.

Ellipsoid.mov: Animations of varying views of the induced surface charge density and the scattered components of the electric field in a perfect conducting ellipsoid of aspect ratio 1:3:9 for $ka = 1$, $kb = 3$, $kc = 9$ – see Fig 3 in the main text.

10 Boundary integral parameters

All surfaces in the examples in the main text are represented by quadratic elements. The number of nodes, that is, the number of degrees of freedom, and the number of elements per surface are listed in Table 1. The ease with which quadratic elements can be used in our non-singular formulation of the boundary integral equation means that good accuracy can be achieved using only a very modest number of nodes or elements to represent surfaces that can have high or disparate geometric aspect ratios.

Table 1 Numbers of nodes and quadratic elements used for the examples in Figs. 1 to 3 in the main text

Example	**Number of nodes**	**Number of quadratic elements**
Fig.1 *One sphere*	1442	720
Fig. 2 *Three spheres* (on each sphere)	362	180
Fig. 3a *Needle* (aspect ratio 10:1)	1442	720
Fig. 3b *Ellipsoid* (aspect ratio 1:3:9)	2562	1280

References

1 W. C. Gibson, The method of moments in electromagnetics, Chapman & Hall/CRC, New York (2008).

2 A.A. Becker, The boundary element method in engineering. McGraw-Hill, London (1992).

3 Q. Sun, E. Klaseboer, B. C. Khoo, D. Y. C. Chan, Roy. Soc. Open Sci. 2, 140520 (2015).

4 E. Klaseboer, Q. Sun, D. Y. C. Chan, J. Fluid Mech. 696, 469 (2012).

5 Q. Sun, E. Klaseboer, B. C. Khoo, D. Y. C. Chan, Engin. Anal. Bound. Elem. 43, 117 (2014).

6 C. H. Wilcox, Comm. Pure Applied Maths 9, 115 (1956).

7 C. Matzler, Matlab Functions for Mie Scattering and Absorption. Research Report No. 2002-08, Institut für Angewandte Physik (2002). http://omlc.org/software/mie/

8 A. T. Chwang, T. Y.-T. Wu, J. Fluid Mech. 63, 607 (1974).