A monotonicity property for once reinforced biased random walk on $\mathbb{Z}^d$.

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Dedicated to Chuck Newman after 70 years of life and 50 years of massive contributions to probability.
Bring on NYU Waiheke!

Abstract

We study once-reinforced biased random walk on $\mathbb{Z}^d$. We prove that for sufficiently large bias, the speed $v(\beta)$ is monotone decreasing in the reinforcement parameter β in the region $[0, \beta_0]$, where β_0 is a small parameter depending on the underlying bias. This result is analagous to results on Galton-Watson trees obtained by Collevecchio and the authors.

1 Introduction

Reinforced random walks have been studied extensively since the introduction of the (linearly)-reinforced random walk of Coppersmith and Diaconis [9]. In this paper we study (a biased version of) once-reinforced random walk, which was introduced by Davis [10] as a possible simpler model of reinforcement to understand. While there have been recent major advances in the understanding of linearly reinforced walks on $\mathbb{Z}^d$ (see e.g. [1, 29, 11, 30]

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and the references therein), rather less is known about once-reinforced walks on $\mathbb{Z}^d$.

When the underlying random walk is biased we expect the once-reinforced random walk to be ballistic in the direction of the bias. On regular trees with d offspring per vertex, the underlying walk has a drift away from the root when $d \geq 2$ and the ballisticity of the once-reinforced walk is a well known result due to Durrett, Kesten and Limic [12] (see [6] for a softer proof, and [25, 8] for further results). When one introduces an additional bias on the tree (letting children have initial weight α), one must first clarify what one means by once-reinforcement. Indeed, ballisticity depends on the sign of $d\alpha - 1$ in the setting of *additive* once-reinforcement, but a different criterion reveals itself in the setting of *multiplicative* once-reinforcement (see [7], and also [25]). It is also shown in [7], for sufficiently large d and sufficiently small $\beta_0 > 0$, depending on d, that the speed of the walk away from the root is monotone in the reinforcement $\beta \in [0, \beta_0]$. In this paper we prove analygous monotonicity results on $\mathbb{Z}^d$, when the underlying bias is sufficiently large.

1.1 The model

Fix $d \geq 2$. Let $\mathcal{E}_+ = (e_i)_{i=1,\dots,d}$ denote the canonical basis on $\mathbb{Z}^d$, and $\mathcal{E} = \{e \in \mathbb{Z}^d : |e| = 1\}$ denote the set of neighbours of the origin in $\mathbb{Z}^d$. Let $\mathcal{E}_- = \mathcal{E} \setminus \mathcal{E}_+$.

Given $\boldsymbol{\alpha} = (\alpha_e)_{e\in\mathcal{E}} \in \mathbb{R}_+^{\mathcal{E}}$ and $\beta > 0$, we define a once-edge-reinforced random walk $\boldsymbol{X}$ on $\mathbb{Z}^d$ with natural filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ (i.e. $\mathcal{F}_n = \sigma(X_k : k \leq n)$) as follows. Set $X_0 = 0$ almost surely. For any $n \geq 0$, let $E_n = \{[X_{i-1}, X_i] : 1 \leq i \leq n\}$ denote the set of non-oriented edges crossed by $\boldsymbol{X}$ up to time n. Define

$$W_e(n) := \alpha_e(1 + \beta \mathbb{1}_{\{[X_n, X_n+e] \in E_n\}}). \tag{1}$$

The walk jumps to a neighbor $X_n + e$ with conditional probability given by

$$\mathbb{P}_\beta\left(X_{n+1} = X_n + e \middle| \mathcal{F}_n\right) = \frac{W_e(n)}{\sum_{e'\in\mathcal{E}} W_{e'}(n)}. \tag{2}$$

Together, (1) and (2) correspond to *multiplicative* once-edge reinforcement, with reinforcement parameter β. See Section 1.2 for other related models.

Without loss of generality we may assume that the direction of bias (if any) is in the positive coordinate direction for every coordinate. Moreover, we assume that every direction has positive weight. Thus, up to a rescaling of parameters, without loss of generality $\alpha_e \geq 1$ for every $e \in \mathcal{E}$.

Condition D. *For every $e \in \mathcal{E}_+$, $\alpha_e \geq \alpha_{-e} \geq 1$.*

Let $\alpha_+ = \sum_{e\in\mathcal{E}_+} \alpha_e$ and $\alpha_- = \sum_{e\in\mathcal{E}_-} \alpha_e$, and let $\alpha = \alpha_+ + \alpha_- = \sum_{e\in\mathcal{E}} \alpha_e$. Our results depend upon a modification of an argument of [2] involving a coupling with a 1-dimensional biased random walk. This general coupling approach has also been utilised on $\mathbb{Z}^d$ for biased random walk on random conductances in [3] (see [7] as well).
For this reason, we will assume the following with $\kappa \gg 1$.

Condition κ. *The parameters $\boldsymbol{\alpha}$ and β_0 are such that $\beta_0 \leq 1/\alpha_+$ and*

$$\frac{\alpha_+}{(1+\beta_0)^2\alpha_-^2} > \kappa. \tag{3}$$

If Condition κ holds with $\kappa = 1$ then a simple comparison with 1-dimensional biased random walk shows that the walker is *ballistic* in direction $\ell_+ := \sum_{e\in\mathcal{E}_+} e$ (i.e. $\liminf_{n\to\infty} n^{-1}X_n \cdot \ell_+ > 0$). In particular the walk is *transient* in direction ℓ_+ (i.e. $\liminf_{n\to\infty} X_n \cdot \ell_+ = \infty$). Combined with regeneration arguments (see e.g. [32, 31]), this allows one to prove that there exists $v \in \mathbb{R}^d$ with $v \cdot \ell_+ > 0$ such that $\mathbb{P}(\lim_{n\to\infty} n^{-1}X_n = v) = 1$. Since the norm $\|x\| := \sum_{i=1}^d |x_i|$ is a continuous function on $\mathbb{R}^d$ we also have that

$$\|v\| = \lim_{n\to\infty} n^{-1}\|X_n\|, \qquad \text{almost surely.}$$

Note that if $\boldsymbol{X}$ is a nearest neighbour walk on $\mathbb{Z}^d$ then $\|\boldsymbol{X}\|$ is a nearest neighbour walk on $\mathbb{Z}_+$. Therefore if $v\cdot e \geq 0$ for each $e \in \mathcal{E}_+$ then $v\cdot\ell_+ = \|v\|$. Assuming Condition D, it is *intuitively obvious* that if v exists then $v \cdot e \geq 0$ for each $e \in \mathcal{E}_+$. We conjecture that this is true however it does not seem easy to prove.

Conjecture 1.1. *Assume Condition D. Then for each $\beta > 0$ there exists $v = v_\beta \in \mathbb{R}^d$ with $v \cdot e \geq 0$ for each $e \in \mathcal{E}_+$ such that $\mathbb{P}(n^{-1}X_n \to v) = 1$.*

Note that it is not at all obvious that $v \cdot e$ should be strictly positive for all β when the underlying random walk (i.e. $\beta = 0$) has a positive speed in direction e. In particular on regular trees there are settings where the underlying random walk is ballistic but the multiplicative-once-reinforced walk is recurrent [7]. On the other hand, we believe that on $\mathbb{Z} \times F$ where F is a finite graph $v_\beta \cdot e_1 > 0$ whenever $v_0 \cdot e_1 > 0$.

We make the following conjecture about the behaviour of v_β as β varies.

Conjecture 1.2. *Assume Condition D. Then for β_0 such that Condition κ holds with $\kappa \geq 1$, $v_\beta \cdot \ell_+$ is strictly decreasing in $\beta \leq \beta_0$.*

There are at least 3 different strategies for proving monotonicity of the speed for random walks on $\mathbb{Z}^d$: coupling, expansion, and Girsanov transformation methods, with the former usually being the weapon of choice, where possible. When $d = 1$ there is a rather general coupling method [22, 18] for proving monotonicity, however this argument completely breaks down when $d \geq 2$. We are not aware of any current technology that lets one resolve Conjecture 1.2 for all $\kappa \geq 1$.

If the bias in direction ℓ_+ is sufficiently large then for small reinforcements the reinforced walker still has a large bias in direction ℓ_+ (e.g. when Condition κ holds for large κ). Thus, what the walker sees locally almost all of the time is a single reinforced edge in some direction $-e \in \mathcal{E}_-$, and no reinforced edges in directions in $\mathcal{E}_+$. In this case it is again *intuitively obvious* that the speed of the reinforced version of the walk in direction ℓ_+ is decreasing in β for small β. Actually proving this is non-trivial.

We will prove a version of this result assuming one of the following:

Condition S. *The parameters $\boldsymbol{\alpha}$ satisfy $\alpha_e = \alpha_+/d$ and $\alpha_{-e} = \alpha_-/d$ for each $e \in \mathcal{E}_+$.*

Note that Condition S (which is a symmetry condition) together with Condition κ (for $\kappa \geq 1$) implies Condition D. Condition S implies that the true direction of bias of the underlying random walk is ℓ_+. At the other extreme, the following condition (with $\kappa \gg 1$) implies that the true direction of bias of the underlying random walk is almost in direction e_1.

Condition e_1. *The parameters $\boldsymbol{\alpha}$ and β_0 are such that $\beta_0 \leq 1/\alpha_+$ and satisfy*

$$\frac{\alpha_{e_1}}{(1+\beta_0)^2(\alpha - \alpha_{e_1})^2} > \kappa.$$

Note that this implies Condition κ.

Curiously, with the technique that we employ, it seems considerably harder to prove our results for parameters $\boldsymbol{\alpha}$ between the two extremes given by Conditions S and e_1.

Our main results are the following Theorems, which verify that if either Condition S or Condition e_1 (for small ε) hold for large drifts and small reinforcement, increasing the reinforcement slows the walker down.

Theorem 1.3. *There exists $\kappa_0 < \infty$ such that if Condition S holds and if Condition κ holds for κ_0, $\boldsymbol{\alpha}$ and β_0, then for all $\beta \leq \beta' \leq \beta_0$,*

$$(v_\beta - v_{\beta'}) \cdot \ell_+ > 0 \quad \text{and} \quad \|v_\beta\| > \|v_{\beta'}\|.$$

(Note that the second conclusion in Theorem 1.3 is immediate from the first and Condition S.)

Theorem 1.4. *Suppose that Condition D holds. There exists $\kappa_0 < \infty$ such that if Condition e_1 holds for κ_0, $\boldsymbol{\alpha}$ and β_0 then for all $\beta \leq \beta' \leq \beta_0$,*

$$(v_\beta - v_{\beta'}) \cdot \ell_+ > 0.$$

Note that if $\alpha_- = 0$ then there is nothing to prove in either case. Otherwise $\alpha_e > 0$ for some $e \in \mathcal{E}_-$ in which case by Condition D we have that

$$\alpha_- \geq 1. \tag{4}$$

1.2 Discussion

We have restricted ourselves to multiplicative-once-edge-reinforced random walk. In the general $\boldsymbol{\alpha}$ setting one can describe a rather general once-reinforcement scheme as follows. Recall that E_n is the set of edges crossed by the walk up to time n and let $V_n = \{X_0, \dots, X_n\}$ denote the set of vertices visited up to time n. Then, given a parameter set $(\boldsymbol{\alpha}, \boldsymbol{\alpha}^V, \boldsymbol{\alpha}^E) \in (0,\infty)^{2d\times 3}$, define the law of a walk $\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\alpha}^V,\boldsymbol{\alpha}^E}$ via (2) and the edge weights:

$$W_e(n) = \alpha_e \mathbb{1}_{\{X_n+e\notin V_n\}} + \alpha_e^V \mathbb{1}_{\{X_n+e\in V_n, [X_n,X_n+e]\notin E_n\}} + \alpha_e^E \mathbb{1}_{\{[X_n,X_n+e]\in E_n\}}. \tag{5}$$

This general setting includes multiplicative-once-edge-reinforced random walk (this is the choice $\alpha_e^V = \alpha_e$ and $\alpha_e^E = (1+\beta)\alpha_e$ for each $e \in \mathcal{E}$), additive-once-edge-reinforced random walk (the choice $\alpha_e^V = \alpha_e$ and $\alpha_e^E = \alpha_e + \beta$ for each $e \in \mathcal{E}$), and vertex-reinforced versions of these models (by setting $\alpha_e^V = \alpha_e^E$ for each $e \in \mathcal{E}$). It also includes hybrid models where the weight of a directed edge depends on whether the undirected edge has been traversed, and otherwise whether the other endvertex of the edge has been visited before.

The increased level of difficulty (for the coupling technique) that we encounter in studying models with $\boldsymbol{\alpha}$ that are not at the "extremes" given by either Conditions S or e_1, seems (at first glance) to persist for expansion methods (see e.g. [16, 17, 23, 15]). We have not investigated the possibility of proving such results using Girsanov transformation methods (see e.g. [27, 28]).

In the setting of reinforcement for an unbiased walk, the velocity $\vec{v}$ is zero, but one imagines that (e.g. in the case of once-reinforcement), monotonicity in β still holds for various quantities such as $\mathbb{E}[|X_n|^2]$, $\mathbb{E}[\|X_n\|]$ and the expected number of visits to 0 up to time n. Results of this kind hold in the elementary setting where one only keeps track of the most recently traversed edge (but can in fact fail in this setting with more general reinforcement schemes than once-reinforcement), see e.g. [20, 19].

There is also a substantial literature on monotonicity (or lack thereof) for random walks in random graphs, where one is often interested in the monotonicity (or lack thereof) of the speed of the walk in some parameter defining the bias of the walk or the structure of the underlying graph (see e.g. [26, 2, 4, 13, 5, 21, 7, 3]).

Open problem: How much wood would Chuck chuck, if Chuck would chuck wood? We conjecture that the correct answer is 42.

2 Preliminary results

We will need the concept of regeneration times. Let $\boldsymbol{Y}$ denote a nearest neighbour simple random walk on $\mathbb{Z}$ with probability $p > 1/2$ of stepping to the right. By the law of large numbers $n^{-1}Y_n \to 2p-1 > 0$ almost surely. Moreover it is easily computed that

$$\mathbb{P}(\inf_{n\geq 0} Y_n \geq Y_0) = p^{-1}(2p-1). \tag{6}$$

We say that $N \in \mathbb{Z}_+$ is a *regeneration time* of Y if $\sup_{n<N} Y_n < Y_N \leq \inf_{n\geq N} Y_n$. Letting $\mathcal{D}_Y$ denote the set of regeneration times for Y, and $D_0 = \{0 \in \mathcal{D}_Y\}$ we see from (6) that $\mathbb{P}(D_0) = p^{-1}(2p-1) > 0$. In fact $|\mathcal{D}_Y| = \infty$ almost surely and we write $(\tau_i)_{i\in\mathbb{N}} = \mathcal{D}_Y \cap \mathbb{N}$ (with $\tau_i < \tau_{i+1}$ for each i) for the ordered strictly positive elements of $\mathcal{D}_Y$. Set $\tau_0 = 0$ (this may or may not be a regeneration time).

More generally, for a walk $\boldsymbol{X}$ on $\mathbb{Z}^d$ we say that $N \in \mathbb{Z}_+$ is a *regeneration time* of $\boldsymbol{X}$ in the direction of $x \in \mathbb{R}^d \setminus \{o\}$ if $\sup_{n<N} X_n \cdot x < X_N \cdot x \leq \inf_{n\geq N} Y_n \cdot x$. For $n \geq 1$ let $\Delta_n^{\boldsymbol{X}} = X_n - X_{n-1}$.

The following is Lemma 3.1 of [7].

Lemma 2.1 (Lemma 3.1 of [7]). *Suppose that* $\mathbf{Z}'$ *and* $\mathbf{Z}$ *are nearest neighbour walks on* $\mathbb{Z}$ *and that* $\boldsymbol{Y}$ *is a nearest neighbour simple random walk on* $\mathbb{Z}$ *with* $\mathbb{P}(Y_1 = 1) = p > 1/2$*, all on the same probability space such that:*

(i) $\Delta_n^{\mathbf{Z}'} = \Delta_n^{\mathbf{Z}} = 1$ *whenever* $\Delta_n^{\boldsymbol{Y}} = 1$*,*

then the regeneration times τ_i *of* $\boldsymbol{Y}$ *are also regeneration times for* $\mathbf{Z}$ *and* $\mathbf{Z}'$*. Moreover if*

(ii) $(Z'_{\tau_{i+1}} - Z'_{\tau_i})_{i\in\mathbb{N}}$ *are i.i.d. random variables and* $(Z_{\tau_{i+1}} - Z_{\tau_i})_{i\in\mathbb{N}}$ *are i.i.d. random variables, with* $Z'_{\tau_{i+1}} - Z'_{\tau_i}$ *and* $Z_{\tau_{i+1}} - Z_{\tau_i}$ *being independent of* $(\tau_k : k \leq i)$ *for each* i*, and*

(iii) $\mathbb{E}\left[Z_{\tau_1} - Z'_{\tau_1} \middle| D_0\right] > 0$*.*

Then there exist $v > v' > 0$ *such that* $\mathbb{P}(n^{-1}Z_n \to v, n^{-1}Z'_n \to v') = 1$*.*

Let $\overline{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|D_0)$. Let $\mathfrak{B} = \big\{1 \leq i < \tau_1 : \Delta_i^{\boldsymbol{Y}} = -1\big\}$ denote the set of times before τ_1 when $\boldsymbol{Y}$ takes a step back. The following statement (and its proof) is a trivial modification of Lemma 3.2 of [7].

Lemma 2.2. *Let* $\mathbf{Z}, \mathbf{Z}'$ *be nearest neighbour walks on* $\mathbb{Z}$*, and* $\boldsymbol{Y}$ *a biased random walk on* $\mathbb{Z}$ *satisfying assumption Lemma 2.1(i). Suppose also that* $\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} < 0\big) = 0$ *and*

$$\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} \geq 1\big) > \sum_{k=2}^{\infty} 2k\overline{\mathbb{P}}(|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0). \tag{7}$$

Then (iii) of Lemma 2.1 holds. Therefore if the assumption of Lemma 2.1(ii) also holds then Lemma 2.1 holds.

To prove Theorem 1.3 it therefore suffices to prove the following theorem.

Theorem 2.3. *There exists* $\kappa_0 < \infty$ *such that if Condition* κ *holds for* κ_0*,* $\boldsymbol{\alpha}$ *and* β_0*, and Condition S holds, then for all* $\beta \leq \beta' \leq \beta_0$*, there exists a probability space on which the conditions of Lemma 2.1 hold for* $\mathbf{Z} = \boldsymbol{X}(\beta)\cdot\ell_+$ *and* $\mathbf{Z}' = \boldsymbol{X}(\beta')\cdot\ell_+$*.*

Similarly, to prove Theorem 1.4 it suffices to prove the following.

Theorem 2.4. *Suppose that Condition D holds. There exists* $\kappa_0 < \infty$ *such that if Condition* e_1 *holds for* κ_0*,* $\boldsymbol{\alpha}$ *and* β_0 *then for all* $\beta \leq \beta' \leq \beta_0$*, there exists a probability space on which the conditions of Lemma 2.1 hold for* $\mathbf{Z} = \boldsymbol{X}(\beta)\cdot\ell_+$ *and* $\mathbf{Z}' = \boldsymbol{X}(\beta')\cdot\ell_+$*.*

In the next section we construct the probability spaces relevant to Theorems 2.3 and 2.4.

3 The coupling

This section adapts an argument of Ben Arous, Fribergh and Sidoravicius [2].

In the construction of the coupling and verification of its properties we will use various jargon as follows.

We say that a walk $\boldsymbol{Y}$ on $\mathbb{Z}$ jumps *forward* (at time $n+1$) if $\Delta^{\boldsymbol{Y}}_{n+1} = 1$. Otherwise, we say that Y jumps *backwards*.

For a walk $\boldsymbol{X}$ on $\mathbb{Z}^d$ (which will have the law of a once reinforced biased random walk on Z^d), we will say that $\boldsymbol{X}$ jumps *forward* (at time $n+1$) if $\Delta^{\boldsymbol{X}}_{n+1} \in \mathcal{E}_+$ and *backwards* otherwise. We will write that a (non-oriented) edge e of $\mathbb{Z}^d$ is *reinforced* at a given time n if it has already been crossed by X, i.e. if $e \in E_n$. We will use the notation $\mathcal{I}_n := \{e \in \mathcal{E} : [X_n, X_n + e] \in E_n\}$, and say that $\boldsymbol{X}$ jumps *on its trace* (at time $n+1$) if $\Delta^{\boldsymbol{X}}_{n+1} \in \mathcal{I}_n$ (i.e. it jumps through an edge which is already reinforced), otherwise we say that it jumps *out of its trace*. We call *local environment* of $\boldsymbol{X}$ at time n the collection of weights of the edges adjacent to X_n.

Recall that we have defined $\mathcal{E}_+ = (e_i)_{i=1,\dots,d}$. We extend the notation to $e_{i+d} = -e_i \in \mathcal{E}_-$ for any $i \in \{1, \dots, d\}$. Moreover, we use the shorthand $\alpha_i = \alpha_{e_i}$ for any $i \in \{1, \dots, 2d\}$. For each $I \subset [2d]$, for any $i \in [2d]$ and for each $\beta \geq 0$, define

$$p^{(\beta)}_{i,I} := \frac{\alpha_i(1 + \mathbb{1}_{\{i \in I\}}\beta)}{\sum_{j=1}^{2d} \alpha_j(1 + \mathbb{1}_{\{j \in I\}}\beta)}.$$

Note that if $\boldsymbol{X}$ is a biased ORRW on $\mathbb{Z}^d$ with reinforcement parameter β, then $\mathbb{P}(\Delta^{\mathbf{X}^{(\beta)}}_{n+1} = e_i | \mathcal{F}_n) = p^{(\beta)}_{i,\mathcal{I}_n(\beta)}$ a.s.

Note that if $\beta' > \beta$ then for any $I \subset [2d]$ we have

$$\begin{aligned} p^{(\beta')}_{i,I} &\geq p^{(\beta)}_{i,I}, \text{ for any } i \in I, \\ p^{(\beta')}_{i,I} &\leq p^{(\beta)}_{i,I}, \text{ for any } i \notin I, \\ p^{(\beta')}_{i,I} \wedge p^{(\beta)}_{i,I} &\geq p^Y_i, \text{ for any } i \in [d], \end{aligned}$$

where

$$p^Y_i := \frac{\alpha_i}{\alpha + \beta'(\alpha - \alpha_i)}. \tag{8}$$

Let us also define

$$p^Y := \sum_{i=1}^{d} \alpha_i,\ q^Y := 1 - p^Y,\ \text{and}\ p^Y_{i+d} := 0,\ \forall i \in [d]. \tag{9}$$

Note that from summing (8) and using Condition κ and (4) we have $\beta_0 \leq 1/\alpha_+$ and $\alpha_+/\alpha_- > \kappa$, and therefore

$$p^Y > \frac{\alpha_+}{(1+\beta_0)\alpha} = 1 - \frac{\beta_0\alpha_+}{(1+\beta_0)\alpha} - \frac{\alpha_-}{\alpha} \geq 1 - \frac{2}{\kappa}. \tag{10}$$

In particular, p^Y can be taken arbitrarily close to 1 under Condition κ for large κ.

The main idea is to couple three walks, $\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{Y}$ satisfying the conditions of Theorems 2.3 and 2.4:

(1) $\boldsymbol{X} = \mathbf{X}^{(\beta)}$ a biased ORRW on $\mathbb{Z}^d$ with reinforcement parameter β;

(2) $\boldsymbol{X}' = \mathbf{X}^{(\beta')}$ a biased ORRW on $\mathbb{Z}^d$ with reinforcement parameter $\beta' > \beta$;

(3) $\mathbf{Y}$ a biased random walk on $\mathbb{Z}$.

In particular (see (7)) we must be able to control how each walk can make gains on the other in direction ℓ_+. To this end, we will say that $\boldsymbol{X}$ and $\boldsymbol{X}'$ are *still coupled at time* n if $(X_k)_{k\leq n} = (X'_k)_{k\leq n}$, and that they decouple at time $n+1$ if also $X_{n+1} \neq X'_{n+1}$. We will write $\delta = \inf\{n : X_n \neq X'_n\}$ to denote the decoupling time. We call *discrepancy* (at time n) the difference $X_n - X'_n$. We say that the decoupling creates a *negative discrepancy* if $(X_\delta - X'_\delta) \cdot \ell_+ < 0$.

3.1 The dynamics

Fix $\beta' > \beta \geq 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space on which $(U_i)_{i\geq 1}$ is an i.d.d. collection of $U[0,1]$ random variables. We will set $X_0 = X'_0 = o \in \mathbb{Z}^d$ and $Y_0 = 0$, and (X_n, X'_n, Y_n) will be $\mathcal{G}_n = \sigma(U_k : k \leq n)$-measureable. We will use $'$ notation to denote quantities depending on $\boldsymbol{X}'$ e.g. $E'_n = \{[X'_{k-1}, X'_k] : k \leq n\}$ and $\mathcal{I}'_n := \{e \in \mathcal{E} : [X'_n, X'_n + e] \in E'_n\}$.

The coupling is given by the following rules, that we will explain in Section 3.2. Firstly,

(0_Y) for any $n \in \mathbb{N}$, $Y_n = \sum_{i=1}^n \left(\mathbb{1}_{\{U_i > q^Y\}} - \mathbb{1}_{\{U_i \leq q^Y\}}\right)$.

The above takes care of the marginal distribution of $\boldsymbol{Y}$. Next, regardless of the environment at time n,

(0_X) If $U_{n+1} \in \left(1 - \sum_{j=1}^{i} p_j^Y, 1 - \sum_{j=1}^{i-1} p_j^Y\right]$ for $i \in [d]$, then $\Delta_{n+1}^{\boldsymbol{X}} = \Delta_{n+1}^{\boldsymbol{X}'} = e_i$;

Otherwise we define the joint increments inductively, considering separately the cases when the two walks $\boldsymbol{X}$, $\boldsymbol{X}'$ have ($*$) the same local environments or ($**$) different local environments.

($=$) Suppose that $\mathcal{I}_n = \mathcal{I}_n'$, and let $\mathcal{I} = \mathcal{I}_n$. Denote $k = |\mathcal{I}|$, write $r_1 < \cdots < r_k$ for the elements of $\mathcal{I}$ listed in increasing order and $\bar{r}_1 < \cdots < \bar{r}_{2d-k}$ for the elements of $[2d] \setminus \mathcal{I}$ in increasing order. Then,

($=_{\mathcal{I}}$) If $U_{n+1} \in \left(q^Y - \sum_{j=1}^{i} \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right), q^Y - \sum_{j=1}^{i-1} \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right)\right]$ for $i \in [k]$, then $\Delta_{n+1}^{\boldsymbol{X}} = \Delta_{n+1}^{\boldsymbol{X}'} = e_{r_i}$;

($=_{\mathcal{I}^c}$) If

$$U_{n+1} \in \left(q^Y - \sum_{j=1}^{k} \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right) - \sum_{j=1}^{i} \left(p_{\bar{r}_j,\mathcal{I}}^{(\beta')} - p_{\bar{r}_j}^Y\right), \right.$$
$$\left. q^Y - \sum_{j=1}^{k} \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right) - \sum_{j=1}^{i-1} \left(p_{\bar{r}_j,\mathcal{I}}^{(\beta')} - p_{\bar{r}_j}^Y\right)\right],$$

then $\Delta_{n+1}^{\boldsymbol{X}} = \Delta_{n+1}^{\boldsymbol{X}'} = e_{\bar{r}_i}$;

($=_{\mathcal{I},\mathcal{I}^c}$) If $U_{n+1} \in \left(0, 1 - \sum_{j=1}^{k} p_{r_j,\mathcal{I}}^{(\beta)} - \sum_{j=1}^{2d-k} p_{\bar{r}_j,\mathcal{I}}^{(\beta')}\right]$ then

(i) if $U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p_{r_j,\mathcal{I}}^{(\beta')} - p_{r_j,\mathcal{I}}^{(\beta)}\right), \sum_{j=1}^{i} \left(p_{r_j,\mathcal{I}}^{(\beta')} - p_{r_j,\mathcal{I}}^{(\beta)}\right)\right]$ for $i \in \{1, \ldots, k\}$, then $\Delta_{n+1}^{\boldsymbol{X}'} = e_{r_i}$;

(ii) if $U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p_{\bar{r}_j,\mathcal{I}}^{(\beta)} - p_{\bar{r}_j,\mathcal{I}}^{(\beta')}\right), \sum_{j=1}^{i} \left(p_{\bar{r}_j,\mathcal{I}}^{(\beta)} - p_{\bar{r}_j,\mathcal{I}}^{(\beta')}\right)\right]$ for $i \in \{1, \ldots, 2d-k\}$, then $\Delta_{n+1}^{\boldsymbol{X}} = e_{\bar{r}_j}$.

($\neq$) Now, if $\mathcal{I}_n' \neq \mathcal{I}_n$ then we follow:

($\neq_{X'}$) if $U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p_{j,\mathcal{I}}^{(\beta')} - p_j^Y\right), \sum_{j=1}^{i} \left(p_{j,\mathcal{I}}^{(\beta')} - p_j^Y\right)\right]$ for $i \in [2d]$ then $\Delta_{n+1}^{\boldsymbol{X}'} = e_i$;

($\neq_X$) if $U_{n+1} \in \left(\sum_{j=1}^{i-1}\left(p_{j,\mathcal{I}}^{(\beta)} - p_j^Y\right), \sum_{j=1}^{i}\left(p_{j,\mathcal{I}}^{(\beta)} - p_j^Y\right)\right]$ for $i \in [2d]$ then $\Delta_{n+1}^{\boldsymbol{X}} = e_i$.

From now on, we denote $(\mathcal{G}_n)$ the natural filtration generated by the sequence (U_n) and note that the three walks are measurable with respect to this filtration.

3.2 Properties of the coupling

Let us explain the coupling defined in Section 3.1. The proof that the marginals of $\mathbf{Y}$, $\mathbf{X}^{(\beta)}$ and $\mathbf{X}^{(\beta')}$ have the correct distributions is left to the reader. Let us simply emphasize that

$$
\begin{aligned}
0 &\leq q^Y - \sum_{j=1}^{k}\left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right) - \sum_{j=1}^{d-k}\left(p_{\bar{r}_j,\mathcal{I}}^{(\beta')} - p_{\bar{r}_j}^Y\right) \\
&= 1 - \sum_{j=1}^{k} p_{r_j,\mathcal{I}}^{(\beta)} - \sum_{j=1}^{d-k} p_{\bar{r}_j,\mathcal{I}}^{(\beta')} \\
&= \sum_{j=1}^{d-k}\left(p_{\bar{r}_j,\mathcal{I}}^{(\beta)} - p_{\bar{r}_j,\mathcal{I}}^{(\beta')}\right) = \sum_{j=1}^{k}\left(p_{r_j,\mathcal{I}}^{(\beta')} - p_{r_j,\mathcal{I}}^{(\beta)}\right).
\end{aligned}
$$

For $I \subset [2d]$ let $\alpha_I = \sum_{i \in I} \alpha_i$ (so e.g. $\alpha_+ = \alpha_{[d]}$). We note that

$$
\sum_{j=1}^{k} p_{r_j,\mathcal{I}}^{(\beta)} = \frac{\alpha_{\mathcal{I}}(1+\beta)}{\alpha_{\mathcal{I}}(1+\beta) + \alpha_{\mathcal{I}^c}} \tag{11}
$$

$$
\sum_{j=1}^{d-k} p_{\bar{r}_j,\mathcal{I}}^{(\beta')} = \frac{\alpha_{\mathcal{I}^c}}{\alpha_{\mathcal{I}}(1+\beta') + \alpha_{\mathcal{I}^c}}, \tag{12}
$$

and therefore that the quantity in the first interval of $(=_{\mathcal{I},\mathcal{I}^c})$ is

$$
1 - \sum_{j=1}^{k} p_{r_j,\mathcal{I}}^{(\beta)} - \sum_{j=1}^{d-k} p_{\bar{r}_j,\mathcal{I}}^{(\beta')} = \frac{\alpha_{\mathcal{I}}\alpha_{\mathcal{I}^c}(\beta' - \beta)}{(\alpha_{\mathcal{I}}(1+\beta) + \alpha_{\mathcal{I}^c})(\alpha_{\mathcal{I}}(1+\beta') + \alpha_{\mathcal{I}^c})}. \tag{13}
$$

So the first discrepancy by the walk will give us a (small) factor of $(\beta' - \beta)$.

Similarly, letting $\mathcal{I}_+ = \mathcal{I} \cap [d]$ and $\mathcal{I}_+^c = \mathcal{I}^c \cap [d]$ we see that the union of the intervals in ($=_{\mathcal{I},\mathcal{I}^c}$) (i) over $i \le |\mathcal{I}_+|$ gives the interval

$$\left(0, \frac{\alpha_{\mathcal{I}_+}\alpha_{\mathcal{I}^c}(\beta' - \beta)}{(\alpha_{\mathcal{I}}(1+\beta) + \alpha_{\mathcal{I}^c})(\alpha_{\mathcal{I}}(1+\beta') + \alpha_{\mathcal{I}^c})}\right]. \tag{14}$$

Similarly, the union over $i \le |\mathcal{I}_+^c|$ of the intervals in ($=_{\mathcal{I},\mathcal{I}^c}$) (ii) gives

$$\left(0, \frac{\alpha_{\mathcal{I}_+^c}\alpha_{\mathcal{I}}(\beta' - \beta)}{(\alpha_{\mathcal{I}}(1+\beta) + \alpha_{\mathcal{I}^c})(\alpha_{\mathcal{I}}(1+\beta') + \alpha_{\mathcal{I}^c})}\right]. \tag{15}$$

Here are the main properties satisfied under the coupling:

(P1) Whenever $\mathbf{Y}$ jumps forward, so do $\boldsymbol{X}$ and $\boldsymbol{X}'$, and they take the same step that is independent of the local environment (this holds by (0_Y) and (0_X) of the coupling);

(P2) When $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment, if $\boldsymbol{X}$ jumps on its trace then $\boldsymbol{X}'$ also jumps on its trace and takes the same step (this holds by ($\neq_{\mathcal{I}}$) (and (0_X)) of the coupling);

(P3) When $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment, if $\boldsymbol{X}'$ jumps out of its trace then $\boldsymbol{X}$ also jumps out of its trace and takes the same step (this holds by ($\neq_{\mathcal{I}^c}$) (and (0_X)) of the coupling);

(P4) When $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment and if the walks decouple, then $\boldsymbol{Y}$ jumps backwards, $\boldsymbol{X}$ jumps out of its trace and $\boldsymbol{X}'$ jumps on its trace (this holds by ($=_{\mathcal{I},\mathcal{I}^c}$) (and ($0_Y$)) of the coupling).

Items ($\neq_X$) and ($\neq_{X'}$) in the coupling are dealing with the case when $\boldsymbol{X}$ and $\boldsymbol{X}'$ do not have the same local environment and $\mathbf{Y}$ jumps backwards. For this case, we only define simple rules in order for the marginals to have the good distributions.

3.2.1 Common regeneration structure

Let $\mathbf{Z} = \boldsymbol{X} \cdot \ell_+$ and $\mathbf{Z}' = \boldsymbol{X}' \cdot \ell_+$. By property (P1) our coupling satisfies Lemma 2.1(i). This implies that if t is a regeneration time for the walk $\mathbf{Y}$, then it is also a regeneration time for $\boldsymbol{X}$ and $\boldsymbol{X}'$. Recall that $\tau_1, \tau_2, ...,$ denotes the sequence of positive regeneration times of $\mathbf{Y}$. Recall that D_0 is the event on which 0 is a regeneration time, and we defined $\overline{\mathbb{P}}[\cdot] := \mathbb{P}[\cdot | D_0]$.

Using that $\mathbf{Y}$ is a biased random walk on $\mathbb{Z}$ with probability to jump on the right equal to p^Y, these regeneration times are well defined as soon as $p^Y > 1/2$ and (6) shows that $\mathbb{P}(D_0) = (2p^Y - 1)/p^Y =: p_\infty$.

Using classical arguments on regeneration times and taking advantage of the common regeneration structure, we obtain the following result.

Proposition 3.1. *For any $\beta' > \beta > 0$ and $(\alpha_i)_{i=1,\dots,2d}$ such that $p^Y > 1/2$, we have that, under $\mathbb{P}$, $\Big(Y_{\tau_{k+1}} - Y_{\tau_k}, X_{\tau_{k+1}} - X_{\tau_k}, X'_{\tau_{k+1}} - X'_{\tau_k}, \tau_{k+1} - \tau_k\Big)$, $k \geq 0$ are independent and (except for $k = 0$) have the same distribution as $\big(Y_{\tau_1}, X_{\tau_1}, X'_{\tau_1}, \tau_1\big)$ under $\overline{\mathbb{P}}$.*

Since this result is classical (see e.g. [32, 26], or [14]) and intuitively clear, we only give a sketch proof.

Sketch proof of Proposition 3.1. Suppose that $t \in \mathcal{D}_Y$, i.e. t is a regeneration time for $\boldsymbol{Y}$. Then $\Delta^{\boldsymbol{Y}}_{t+1} = 1$ so both $\boldsymbol{X}$ and $\boldsymbol{X'}$ take a forward step which is chosen independent of the environment. Moreover, whenever $(X_{t+n} - X_t) \cdot \ell_+ = 0$ or $(X'_{t+n} - X'_t) \cdot \ell_+ = 0$ we must have that $Y_{t+n} = Y_t$ and therefore again $\Delta^{\boldsymbol{Y}}_{t+n+1} = 1$ and both $\boldsymbol{X}$ and $\boldsymbol{X'}$ take a forward step independent of the environment. On the other hand, whenever both $(X_{t+n} - X_t) \cdot \ell_+ > 0$ and $(X_{t+n} - X_t) \cdot \ell_+ > 0$, we have that neither $\boldsymbol{X}$ nor $\boldsymbol{X'}$ are incident to an edge reinforced before time t. This shows that for any possible paths $\vec{y}_t, \vec{x}_t, \vec{x'}_t$, the conditional distribution of $(Y_{t+n} - Y_t, X_{t+n} - X_t, X'_{t+n} - X'_t)$ given $((Y_k)_{k\leq t}, (X_k)_{k\leq t}, (X'_k)_{k\leq t}) = (\vec{y}_t, \vec{x}_t, \vec{x'}_t)$ and $t \in \mathcal{D}_Y$ does not depend on t or $(\vec{y}_t, \vec{x}_t, \vec{x'}_t)$ and the result follows. ■

4 Proofs of Theorem 2.3 and Theorem 2.4

Proposition 3.1 implies that item (ii) of Lemma 2.1 is satisfied by $\mathbf{Y}$, $\mathbf{Z'} = \boldsymbol{X'} \cdot \ell_+$ and $\mathbf{Z} = \boldsymbol{X} \cdot \ell_+$. Hence, to prove Theorem 2.3 and Theorem 2.4, we only need to check the requirements of Lemma 2.2.

Define the first time before τ_1 the walks $\boldsymbol{X}$ and $\boldsymbol{X'}$ decouple as $\delta_1 = \inf\{i \leq \tau_1 : \ X_i \neq X'_i\}$. Note that $\delta_1 = \infty$ if the walks do not decouple before τ_1.

Proposition 4.1. *For any $\beta' > \beta > 0$, and α such that $p^Y > 1/2$, we have that*

$$\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} < 0\big) = 0.$$

Proof. The fact that $p^Y > 1/2$ guarantees that τ_1 and $\overline{\mathbb{P}}$ are well defined. Note that, by property (P1), at any time $n \in \mathbb{N}$ such that $\Delta_n^{\mathbf{Y}} = 1$, we have that $\Delta_n^{\mathbf{Z}} = \Delta_n^{\mathbf{Z}'} = 1$ and thus $Z_{n+1} - Z'_{n+1} = Z_n - Z'_n$.
On the event $\{|\mathfrak{B}| = 1\}$, there exists only one time $0 < n_b < \tau_1$ such that $\Delta_{n_b}^{\mathbf{Y}} = -1$ and thus $Z_{\tau_1} - Z'_{\tau_1} = (\Delta_{n_b}^{\mathbf{Z}} - \Delta_{n_b}^{\mathbf{Z}'}) \cdot \ell_+$. At time $n_b - 1$, $\boldsymbol{X}$ and $\boldsymbol{X}'$ are still coupled and thus have the same local environment, which is such that $\mathcal{I}_{n_b-1} = \mathcal{I}_{n_b-1} = \{e_{d+i}\}$ for some $i \in [d]$. Thus by (P4) in order for the walks to decouple on this step we must have $\delta_{n_b}^{\boldsymbol{X}'} = e_{d+i}$, which cannot create a negative discrepancy. ∎

The last result together with the two following Propositions respectively imply Theorem 2.3 and Theorem 2.4 by Lemma 2.2 and Lemma 2.1.

Proposition 4.2. *There exists $\kappa_0 < \infty$ such that if Condition κ holds for κ_0, $\boldsymbol{\alpha}$ and β_0, and Condition S holds, then for all $0 < \beta < \beta' < \beta_0$, inequality* (7) *is satisfied.*

Proposition 4.3. *Suppose that Condition D holds. There exist $\kappa_0 < \infty$ and $\beta_0 > 0$ such that if Condition e_1 holds for κ_0, $\boldsymbol{\alpha}$ and β_0, then for all $0 < \beta < \beta' < \beta_0$, inequality* (7) *is satisfied.*

4.1 Bounds on the decoupling events

Lemma 4.4. *Assume that $\alpha_{i+d} > 0$ for any $i \in [d]$. There exists $C_0 > 0$ such that: There exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \leq \beta < \beta' < \beta_0$,*

$$\overline{\mathbb{P}}\left(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} = 2\right) \geq C_0 \frac{\beta' - \beta}{\alpha_+}.$$

Proof. Let $A = \left\{|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} = 2\right\}$ and note that

$$\overline{\mathbb{P}}(A) = \frac{1}{p_\infty} \mathbb{P}\left(A, D_0\right).$$

Now, let us describe the following scenario (which is in fact the only possible event on which the walks decouple) such that $A \cap D_0$ holds:

(i) $(Y_1, Y_2, Y_3, Y_4) = (1, 0, 1, 2)$ and $\tau_1 = 4$, so also D_0 occurs;

(ii) $\Delta_1^{\boldsymbol{X}} = \Delta_1^{\boldsymbol{X}'} \in \mathcal{E}_+$, then $\boldsymbol{X}'$ steps back onto its trace ($\Delta_2^{X'} = -\Delta_1^{X'}$), while $\boldsymbol{X}$ steps forward ($\Delta_2^{\boldsymbol{X}} \in \mathcal{E}_+$).

Now, following this scenario and conditional on $\Delta_1^{\boldsymbol{X}} = \Delta_1^{\boldsymbol{X}'} = e_i$, we have (from $(=_{\mathcal{I},\mathcal{I}^c})$) that

$$\mathbb{P}\left(\Delta_2^{\boldsymbol{X}} \in \mathcal{E}_+, \Delta_2^{\boldsymbol{X}'} = e_{i+d} \middle| \mathcal{F}_2\right) = \frac{\alpha_+}{\alpha + \beta\alpha_{i+d}} - \frac{\alpha_+}{\alpha + \beta'\alpha_{i+d}} \tag{16}$$

$$= \frac{(\beta' - \beta)\alpha_+\alpha_{i+d}}{(\alpha + \beta\alpha_{i+d})(\alpha + \beta'\alpha_{i+d})} \tag{17}$$

$$\geq \frac{(\beta' - \beta)\alpha_+}{(\alpha_+ + (1+\beta')\alpha_-)^2} \tag{18}$$

$$= \frac{(\beta' - \beta)}{\alpha_+}\left(\frac{\alpha_+}{(\alpha_+ + (1+\beta')\alpha_-)}\right)^2 \tag{19}$$

$$\geq \frac{(\beta' - \beta)\kappa_0^2}{\alpha_+(\kappa_0 + 1)^2} \geq c\frac{\beta' - \beta}{\alpha_+}. \tag{20}$$

Hence, summing over $i \in [d]$ and using Condition D we obtain

$$\overline{\mathbb{P}}(A) \geq \frac{1}{p_\infty} \times p^Y \times cd\frac{\beta' - \beta}{\alpha_+} \times \left(p^Y\right)^2 \times p_\infty.$$

We conclude noting that for $\kappa_0 > 4$, $p^Y > 1/2$ (by (10)). ∎

Lemma 4.5. *There exists $C_1 > 0$ such that: If Condition S holds then there exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \leq \beta < \beta' < \beta_0$,*

$$\overline{\mathbb{P}}\left[|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \leq \frac{C_1(\beta' - \beta)}{\alpha_+\kappa_0}.$$

Proof. Assume that, at a given time n, the walks are still coupled. Let $J(\mathcal{I})$ denote the right hand side of (13) and $J_1(\mathcal{I})$ and $J_2(\mathcal{I})$ denote the right hand sides of the intervals in (14) and (15) respectively. Under Condition S these quantities can be written as $J(k_+, k_-)$, $J_1(k_+, k_-)$ and $J_2(k_+, k_-)$, corresponding to the values above for given k_+ and k_-.

Then we can write item $(=_{\mathcal{I},\mathcal{I}^c})$ as:

$(=_{\mathcal{I},\mathcal{I}^c,S})$ If $U_{n+1} \in (0, J(k_+, k_-)]$ then

(i) if $U_{n+1} \in (0, J_1(k_+, k_-)]$ then $\Delta_{n+1}^{\boldsymbol{X}'} \in \mathcal{E}_+$ and $\Delta_{n+1}^{\boldsymbol{X}'} \in \mathcal{E}_-$ otherwise;

(ii) if $U_{n+1} \in (0, J_2(k_+, k_-)]$ then $\Delta^{\boldsymbol{X}}_{n+1} \in \mathcal{E}_+$ and $\Delta^{\boldsymbol{X}}_{n+1} \in \mathcal{E}_-$ otherwise.

On the event $\{|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0, D_0\}$, the walk $\mathbf{Y}$ can only do one of the following:

1. $E_1 = \{\{Y_0, \ldots, Y_7\} = \{0, 1, 2, 1, 0, 1, 2, 3\}, \tau_1 = 7\}$;
2. $E_2 = \{\{Y_0, \ldots, Y_6\} = \{0, 1, 0, 1, 0, 1, 2\}, \tau_1 = 6\}$;
3. $E_3 = \{\{Y_0, \ldots, Y_7\} = \{0, 1, 0, 1, 2, 1, 2, 3\}, \tau_1 = 7\}$.

Recall the remarks from Section 3.2. In particular, recall that, by (P1), when $\boldsymbol{Y}$ steps forward, $\boldsymbol{X}$ and $\boldsymbol{X'}$ take the same step. Moreover, if $\boldsymbol{X}$ and $\boldsymbol{X'}$ have the same local environment with only one reinforced edge and if this edge is in one of the directions of $\mathcal{E}_-$, then, by (P4), $\boldsymbol{X}$ and $\boldsymbol{X'}$ cannot decouple creating a negative discrepancy. It follows that no negative discrepancy can be created the first time $\boldsymbol{Y}$ steps back, so the magnitude of any negative discrepancy can only be 2. It also follows that on $(E_2 \cap \{\delta = 4\}) \cup E_3$, $\boldsymbol{X}$ and $\boldsymbol{X'}$ cannot create any negative discrepancy.

It therefore remains to consider the cases $E_1 \cap \{\delta = 3\}$, $E_2 \cap \{\delta = 2\}$, and $E_1 \cap \{\delta = 4\}$.

On $(E_1 \cap \{\delta = 3\}) \cup (E_2 \cap \{\delta = 2\})$, $\boldsymbol{X'}$ jumps backwards on its trace at time δ and one of the following happens:

- At time δ, $\boldsymbol{X}$ jumps forward, hence $Z_\delta - Z'_\delta = 2$, thus $Z_{\tau_1} - Z'_{\tau_1} \geq 0$ and the walks cannot create any negative discrepancy before the regeneration time.
- At time δ, the $\boldsymbol{X}$ jumps backwards out of the trace, hence $Z_\delta - Z'_\delta = 0$ but the two walks do not have the same local environment anymore. Then, the second time $\mathbf{Y}$ jumps backwards, hence the two walks can create a negative discrepancy.

Using the item $(=_{\mathcal{I},\mathcal{I}^c,S})$-(ii) above (and (13),(15)) in the case $k_+ = 0$ and $k_- = 1$, we have (by bounding only the conditional probability of the step taken at time δ) that

$$\begin{aligned}
&\mathbb{P}\left((E_1 \cap \{\delta = 3\}) \cup (E_2 \cap \{\delta = 2\}), Z_{\tau_1} - Z'_{\tau_1} < 0\right) \\
&\leq J(0,1) - J_2(0,1) = \frac{(d-1)(\alpha_-)^2(\beta' - \beta)}{(\alpha_-\beta + d\alpha)(\alpha_-\beta' + d\alpha)} \\
&\leq (\beta' - \beta)\left(\frac{\alpha_-}{\alpha}\right)^2 \leq \frac{(\beta' - \beta)\alpha_+}{\alpha^2\kappa_0} \leq \frac{(\beta' - \beta)}{\alpha_+\kappa_0},
\end{aligned} \tag{21}$$

where we have used Condition κ for the penultimate inequality.

Now, we want to study the probability of the event $\{E_1, \delta = 4\}$ and consider the cases when the discrepancy can be negative or not. One of the following happens:

- If $\Delta_3^{\boldsymbol{X}} = \Delta_3^{\boldsymbol{X}'} \in \mathcal{E}_+$ then, the local environment $\mathcal{I}_3$ is made of one single reinforced edge, in the direction $\mathcal{E}_-$, hence $\boldsymbol{X}$ and $\boldsymbol{X}'$ cannot decouple creating a negative discrepancy;

- At time 3, both $\boldsymbol{X}$ and $\boldsymbol{X}'$ can jump backward on their trace. This corresponds to the event $E_{1,1} := \{E_1, \delta = 4, \Delta_3^{\boldsymbol{X}} = -\Delta_2^{\boldsymbol{X}}\}$, which is treated below;

- At time 3, both $\boldsymbol{X}$ and $\boldsymbol{X}'$ can jump backward out of their trace. This corresponds to the event $E_{1,2} := \{E_1, \delta = 4, \Delta_3^{\boldsymbol{X}} \in \mathcal{E}_- \setminus \{-\Delta_2^{\boldsymbol{X}}\}\}$, which is treated below.

On the event $E_{1,1}$, the local environment $\mathcal{I}_3$ of the walks is made of one reinforced edge in some direction in $\mathcal{E}_+$ and one reinforced edge in some direction in $\mathcal{E}_-$. By property (P4), in order to decouple, $\boldsymbol{X}'$ has to jump on its trace and $\boldsymbol{X}$ out of its trace. More precisely, in order to create a negative discrepancy, $\boldsymbol{X}'$ has to jump forward on its trace and $\boldsymbol{X}$ backwards out of its trace. Now, note that, in the case $k_+ = k_- = 1$, $J_1(k_+, k_-) = J_2(k_+, k_-)$ so the two intervals of item ($=_{\mathcal{I},\mathcal{I}^c,S}$)-(i) and item ($=_{\mathcal{I},\mathcal{I}^c,S}$)-(ii) are equal. Hence, on $E_{1,1}$, we have that $\Delta_\delta^{\boldsymbol{X}} \cdot \ell_+ = \Delta_\delta^{\boldsymbol{X}'} \cdot \ell_+$. This implies that Therefore, we have that

$$\mathbb{P}\left(E_{1,1}, Z_{\tau_1} - Z'_{\tau_1}\right) = 0. \tag{22}$$

Note that this is not necessarily true if we do not assume Condition S, with the prescribed ordering of $\mathcal{I}$ and $\mathcal{I}^c$.

On the event $E_{1,2}$, the local environment $\mathcal{I}_3$ of the walks is made of one single reinforced edge in some direction of $\mathcal{E}_+$. As the walks decouple, we have that $\boldsymbol{X}'$ jumps forward on the trace and $\boldsymbol{X}$ jumps out of the trace. The walks create a negative discrepancy only if, furthermore, $\boldsymbol{X}$ jumps backwards: the probability of this step is given by item $(=_{\mathcal{I},\mathcal{I}^c,S})$ above. Recalling that, at time 3, $\mathbf{Y}$ steps backwards and using the item $(=_{\mathcal{I},\mathcal{I}^c,S})$-(ii) above (and (13),(15)) in the case $k_+ = 1$ and $k_- = 0$, we have that

$$\begin{aligned}\mathbb{P}\left[E_{1,2}, Z_{\tau_1} - Z'_{\tau_1} < 0\right] &\le q^Y \times (J(1,0) - J_2(1,0)) \\ &\le q^Y \frac{d\alpha_-\alpha_+(\beta'-\beta)}{(\alpha_+\beta + d\alpha)(\alpha_+\beta' + d\alpha)} \\ &\le C\frac{(1+\beta_0)\alpha_-}{\alpha_+} \times (\beta'-\beta) \times \frac{\alpha_-}{\alpha_+} \\ &\le C(\beta'-\beta)\left(\frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^2 \le C\frac{(\beta'-\beta)}{\alpha_+\kappa_0},\end{aligned}$$

where we used that, under Condition κ ,

$$q^Y \le 2\frac{(1+\beta_0)\alpha_-}{\alpha_+}.$$

This, together with (21) and (22), implies the conclusion. ■

Lemma 4.6. *There exist $C_2, C_2' > 0$ such that: There exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \le \beta < \beta' < \beta_0$, and all $k \ge 2$,*

$$\overline{\mathbb{P}}\left(|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right) \le C_2' k(\beta'-\beta)\left(\frac{\alpha-\alpha_1}{\alpha} \wedge 1\right)\left(C_2\frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^{k-1}.$$

Proof. Denote $A_k = \left\{|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right\}$.
Recall that δ is the time of decoupling. Note that $\mathbf{Y}$ necessarily starts by jumping forward and that the two last steps before τ_1 are also necessarily forward (otherwise this contradicts the definition of τ_1).
Following [2, Lemma 4.1] and adjusting it to the definition of the regeneration times that we use here, one can prove (see e.g. Lemma 3.3 of [7]) that $|\mathfrak{B}| = k \Rightarrow \tau_1 \le 3k+1$, almost surely, i.e. $\overline{\mathbb{P}}\big(\{|\mathfrak{B}| = k\} \setminus \{\tau_1 \le 3k+1\}\big) = 0$. Using this fact, we have that

$$\overline{\mathbb{P}}(A_k) \quad\le\quad p_\infty^{-1}\mathbb{P}\big(2 \le \delta \le 3k, |\{1 \le n \le 3k-2 : \Delta^{\boldsymbol{Y}}_{n+1} = -1\}| \ge k\big). \tag{23}$$

On the event $\{\delta = n+1\}$ the walks are still coupled at time n and thus, in particular, $\mathcal{I} := \mathcal{I}_n = \mathcal{I}'_n$.
Denote $k = |\mathcal{I}|$, $\mathcal{I} = \{r_1, \dots, r_k\}$ and $\{1, \dots, 2d\} \setminus \mathcal{I} = \{\bar{r}_1, \dots, \bar{r}_{2d-k}\}$. Then the event of decoupling at time $n+1$ is controlled as follows, according to the item $(=_{\mathcal{I},\mathcal{I}^c})$ of the coupling,

$$
\begin{aligned}
\{\delta = n+1\} = \{\delta > n\} \cap \left\{ U_{n+1} \le 1 - \sum_{j=1}^{k} p^{(\beta)}_{r_j,\mathcal{I}} - \sum_{j=1}^{d-k} p^{(\beta')}_{\bar{r}_j,\mathcal{I}} \right\} \\
= \{\delta > n\} \cap \left\{ U_{n+1} \le 1 - \frac{(1+\beta)\alpha_{\mathcal{I}}}{\alpha + \beta\alpha_{\mathcal{I}}} - \frac{\alpha_{\mathcal{I}^c}}{\alpha + \beta'\alpha_{\mathcal{I}}} \right\} \\
= \{\delta > n\} \cap \left\{ U_{n+1} \le \frac{\alpha_{\mathcal{I}^c}\alpha_{\mathcal{I}}(\beta' - \beta)}{(\alpha + \beta\alpha_{\mathcal{I}})(\alpha + \beta'\alpha_{\mathcal{I}})} \right\} \\
\subset \left\{ U_{n+1} \le (\beta' - \beta) \left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1 \right) \right\},
\end{aligned}
$$

where we have used the fact that either $1 \in \mathcal{I}$ or $1 \in \mathcal{I}^c$ to obtain the last relation.

Thus, we have

$$
\begin{aligned}
\overline{\mathbb{P}}(A_k) &\le p_\infty^{-1} \sum_{n=2}^{3k-1} \mathbb{P}\Big(U_n \le (\beta' - \beta)\big(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\big), \\
&\qquad\qquad |\{j \in \{1, \dots, 3k-1\} \setminus \{n\} : U_j \le q^Y\}| \ge k-1 \Big) \\
&\le ck(\beta' - \beta)\Big(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\Big) \\
&\qquad\qquad \times \mathbb{P}\big(\exists B \subset [3k-2] : |B| = k-1, U_j \le q^Y \forall j \in B\big) \\
&\le ck(\beta' - \beta)\Big(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\Big)\binom{3k-2}{k-1}(q^Y)^{k-1} \\
&\le c'k(\beta' - \beta)\left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right)\left(c'' q^Y\right)^{k-1} \qquad (24) \\
&\le c'k(\beta' - \beta)\left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right)\left(\frac{c'''(1+\beta_0)\alpha_-}{\alpha_+}\right)^{k-1}, \qquad (25)
\end{aligned}
$$

where we used the inequality $e^{11/12}(n/e)^n \le n! \le e(n/e)^n$, which holds for any $n \ge 1$, see [24], and we also used that

$$
q^Y \le C\frac{(1+\beta_0)\alpha_-}{\alpha_+} \text{ and } p_\infty \ge 1/2.
$$

$\blacksquare$

Lemma 4.7. *There exist $C_3 > 0$ such that: There exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \le \beta < \beta' < \beta_0$, we have that*

$$\sum_{k=3}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \le \frac{C_3(\beta' - \beta)}{\alpha_+\kappa_0}.$$

Proof. Using Lemma 4.6, we have that

$$\begin{aligned}\sum_{k=3}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] &\le C(\beta' - \beta)\sum_{k=3}^{\infty} k^2\left(C_2\frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^{k-1}\\ &\le C'(\beta' - \beta)\left(\frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^2\left(1 - \frac{C_2}{\kappa_0}\right)^{-3}\\ &\le C'\frac{\beta' - \beta}{\alpha_+\kappa_0},\end{aligned}$$

where we have used Condition κ for large κ_0. $\blacksquare$

Lemma 4.8. *There exist $C_4 > 0$ such that: There exists κ_0 such that for α, β_0 satisfying Condition e_1 and Condition D then, for all $0 < \beta < \beta' < \beta_0$, we have that*

$$\overline{\mathbb{P}}\left(|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0\right) \le C_4\frac{\beta' - \beta}{\alpha_+\kappa_0}.$$

Proof. This is a direct consequence of Condition e_1 and Lemma 4.6 for $k = 2$. $\blacksquare$

4.2 Proofs of Proposition 4.2 and Proposition 4.3

Recall that Propositions 4.2 and 4.3 respectively imply Theorem 2.3 and Theorem 2.4.

Proof of Proposition 4.2. By Lemma 4.4, Lemma 4.5 and Lemma 4.7, there exists $\kappa_0 < \infty$ such that if Condition κ holds for κ_0, $\boldsymbol{\alpha}$ and β_0 and if Condition

S holds, then, for all $0 < \beta < \beta' < \beta_0$, we have that

$$
\begin{aligned}
&\sum_{k=2}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \\
&\leq 4\overline{\mathbb{P}}\left[|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0\right] + \sum_{k=3}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \\
&\leq C\frac{(\beta' - \beta)}{\alpha_+\kappa_0} \leq \frac{C'}{\kappa_0}\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} \geq 1\big).
\end{aligned}
$$

Hence, we can conclude that inequality (7) is satisfied as soon as κ_0 is large enough, which proves the Proposition. ■

Proof of Proposition 4.3. Suppose Condition D holds. There exist $\kappa_0 < \infty$ and $\beta_0 > 0$ such that if Condition e_1 holds for κ_0, $\boldsymbol{\alpha}$ and β_0, then, for all $0 < \beta < \beta' < \beta_0$, we conclude exactly as in the previous proof that (7) holds by Lemma 4.4, Lemma 4.8 and Lemma 4.7. ■

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