The scaling limit of senile reinforced random walk

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Abstract

We prove that the scaling limit of nearest-neighbour senile reinforced random walk in the summable reinforcement regime is a version of Brownian Motion when the time T spent on the first edge has finite expectation. We also show that under suitable conditions, when T has heavy tails the scaling limit is the so-called *fractional kinetics process*, a random time-change of Brownian motion.

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1 Introduction

The senile reinforced random walk is a toy model for a much more mathematically difficult model known as edge-reinforced random walk (for which many basic questions remain open [e.g. see [14]]). It is characterized by a reinforcement function $f : \mathbb{N} \to [-1, \infty)$, such that only the most recently traversed edge is reinforced. As soon as a new edge is traversed, reinforcement begins on that new edge and the reinforcement of the previous edge is forgotten. Such walks may get stuck on a single (random) edge if the reinforcement is strong enough, otherwise (except for one degenerate case) they are recurrent/transient precisely when the corresponding simple random walk is [8].

Formerly, a nearest-neighbour *senile reinforced random walk* is a sequence $\{S_n\}_{n\geq 0}$ of $\mathbb{Z}^d$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_f)$ (with corresponding filtration $\{\mathcal{F}_n = \sigma(S_0, \ldots, S_n)\}_{n\geq 0}$) defined by:

- The walk begins at the origin of $\mathbb{Z}^d$, i.e. $S_0 = o$, $\mathbb{P}_f$-almost surely,
- $\mathbb{P}_f(S_1 = x) = \frac{1}{2d} I_{\{|x|=1\}}$,
- For $n \in \mathbb{N}$, $e_n = (S_{n-1}, S_n)$ is an $\mathcal{F}_n$-measurable *undirected* edge and

$$m_n = \max\{k \geq 1 : e_{n-l+1} = e_n \text{ for all } 1 \leq l \leq k\} \tag{1.1}$$

  is an $\mathcal{F}_n$-measurable, $\mathbb{N}$-valued random variable.
- For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ such that $|x| = 1$,

$$\mathbb{P}_f(S_{n+1} = S_n + x|\mathcal{F}_n) = \begin{cases} \dfrac{1 + f(m_n)}{2d + f(m_n)}, & \text{if } (S_n, S_n + x) = e_n, \\ \dfrac{1}{2d + f(m_n)}, & \text{if } (S_n, S_n + x) \neq e_n. \end{cases} \tag{1.2}$$

Note that the triple (S_n, e_n, m_n) (equivalently (S_n, S_{n-1}, m_n)) is a Markov chain. Hereafter we suppress the f dependence of the probability $\mathbb{P}_f$ in the notation.

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The diffusion constant for a senile random walk $\{S_n\}_{n\geq 0}$ with reinforcement function f is defined as

$$v \equiv \lim_{n\to\infty} v_n, \text{ where } v_n \equiv \frac{1}{n}\mathbb{E}[|S_n|^2], \tag{1.3}$$

whenever this limit exists. For simple random walk $v_n = 1$ for all n. Let T denote the random number of consecutive traversals of the first edge traversed, and $p = \mathbb{P}(T \text{ is odd})$. It was shown in [8] that when $\mathbb{E}[T^{1+\epsilon}] < \infty$ for some $\epsilon > 0$, the diffusion constant is given by

$$v = \frac{dp}{(d-p)\mathbb{E}[T]}, \tag{1.4}$$

which is not monotone in the reinforcement. It is natural to expect that (1.4) holds for all f (in the case $d = 1$ and $f(1) = -1$ this must be interpreted as " $1/0 = \infty$"), and this is verified in [10].

A different but related model, in which the current direction (rather than the current edge) is reinforced according to the function f was studied in [11, 9]. For such a model T is the number of consecutive steps in the same direction before turning. In [9], the authors show that in all dimensions the scaling limit is a version of Brownian motion when $\sigma^2 = \mathrm{Var}(T) < \infty$ and $\sigma^2 + 1 - 1/d > 0$. In the language of this paper, the last condition corresponds to the removal of the special case $d = 1$ and $f(1) = -1$. Moreover when $d = 1$ and T has heavy tails (in the sense of (2.1) below) they show that the scaling limit is an α-stable process when $1 < \alpha < 2$ and a random time change of an α-stable process when $0 < \alpha < 1$. See [9] for more details.

Davis [3] showed that the scaling limit of once-reinforced random walk in one dimension is not Brownian motion (see [14] for further discussion).

The reinforcement regime of most interest is that of linear reinforcement $f(n) = Cn$ for some C. In this case, by the second order mean-value theorem applied to $\log(1 - x)$, $x < 1$ we have

$$\begin{aligned}
\mathbb{P}(T \geq n) \equiv \prod_{j=1}^{n-1} \frac{1 + f(j)}{2d + f(j)} &= \exp\left\{\sum_{j=1}^{n-1} \log\left(1 - \frac{2d-1}{2d+Cj}\right)\right\} \\
&= \exp\left\{-\sum_{j=1}^{n-1} \frac{2d-1}{2d+Cj} - \sum_{j=1}^{n-1} \frac{(2d-1)^2}{2(2d+Cj)^2(1-u_j)^2}\right\} \\
&= \exp\left\{-\sum_{j=1}^{n-1} \frac{2d-1}{2d+Cj} - \sum_{j=1}^{\infty} \frac{(2d-1)^2}{2(2d+Cj)^2(1-u_j)^2} + o(1)\right\} \\
&= \exp\left\{-\frac{2d-1}{C}\log(2d + C(n-1)) + \gamma + o(1)\right\} \sim \frac{\kappa}{n^{\frac{2d-1}{C}}},
\end{aligned} \tag{1.5}$$

where $u_i \in (0, \frac{2d-1}{2d+Cj})$, and γ is a constant arising from the summable infinite series and the approximation of the finite sum by a log. An immediate consequence of (1.5) is that for $f(n) = Cn$, $\mathbb{E}[T]$ is finite if and only if $C < 2d - 1$.

In Section 2 we state and discuss the main result of this paper, which describes the scaling limit of senile reinforced random walk when either $\mathbb{E}[T] < \infty$ or $\mathbb{P}(T \geq n) \sim n^{-\alpha}L(n)$ for some $\alpha > 0$ and L slowly varying at infinity. The scaling limit is not particularly interesting when $\mathbb{P}(T < \infty) < 1$ since the walk gets stuck on some random edge and therefore has finite (but random) range. In this case the random number of times the walk leaves an edge before reaching the edge that it will traverse forever has a geometric distribution. To prove the main result, in Section 3 we first observe the walk at the times that it has just traversed a new edge and describe this as a suitably nice additive functional of a particular Markov chain. In Section 4 we prove the main result assuming the joint convergence of this time-changed walk and the associated time-change process. Finally in Section 5 we prove the convergence of this joint process.

2 Main Result

The assumptions that will be necessary to state the main theorem of this paper are as follows:

(A1) $\mathbb{P}(T < \infty) = 1$, and either $d > 1$ or $\mathbb{P}(T = 1) < 1$.

(A2a) Either $\mathbb{E}[T] < \infty$, or for some $\alpha \in (0, 1]$ and L slowly varying at infinity,

$$\mathbb{P}(T \geq n) \sim L(n)n^{-\alpha}. \tag{2.1}$$

(A2b) If (2.1) holds but $\mathbb{E}[T] = \infty$, then we also assume that

$$\begin{cases} \text{when } \alpha = 1, & \exists \ell(n) \nearrow \infty \text{ such that } (\ell(n))^{-1}L(n\ell(n)) \to 0, \text{ and } (\ell(n))^{-1}\sum_{j=1}^{\lfloor n\ell(n)\rfloor} j^{-1}L(j) \to 1, \\ \text{when } \alpha < 1, & \mathbb{P}(T \geq n, T \text{ odd}) \sim L_o(n)n^{-\alpha_o}, \text{ and } \mathbb{P}(T \geq n, T \text{ even}) \sim L_e(n)n^{-\alpha_e}, \end{cases} \tag{2.2}$$

where ℓ, L_o and L_e are slowly varying at ∞ and L_o and L_e are such that if $\alpha_o = \alpha_e$ then $L_o(n)/L_e(n) \to \beta \in [0, \infty]$ as $n \to \infty$.

By Theorem XIII.6.2 of [5], when $\alpha < 1$ there exists $\ell(\cdot) > 0$ slowly varying such that

$$(\ell(n))^{-\alpha} L\left(n^{\frac{1}{\alpha}}\ell(n)\right) \to (\Gamma(1-\alpha))^{-1}. \tag{2.3}$$

For $\alpha > 0$ let

$$g_\alpha(n) = \begin{cases} \mathbb{E}[T]n & , \text{ if (2.1) is summable} \\ n^{\frac{1}{\alpha}}\ell(n) & , \text{ otherwise} \end{cases} \tag{2.4}$$

By Theorem 1.5.12 of [2], there exists an asymptotic inverse function $g_\alpha^{-1}(\cdot)$ (unique up to asymptotic equivalence) satisfying $g_\alpha(g_\alpha^{-1}(n)) \sim g_\alpha^{-1}(g_\alpha(n)) \sim n$, and by Theorem 1.5.6 of [2] we may assume that g_α and g_α^{-1} are monotone nondecreasing.

A *subordinator* is a real-valued process $V(t)$ with stationary, independent increments such that almost every path is nondecreasing, right continuous and satisfies $V(0) = 0$. Let $B_d(t)$ be a standard d-dimensional Brownian motion. For $\alpha \geq 1$, let $V_\alpha(t) = t$ and for $\alpha \in (0, 1)$, let V_α be an α-stable subordinator (independent of $B_d(t)$) satisfying

$$\mathbb{E}[e^{-\lambda V_\alpha(t)}] = e^{-t\lambda^\alpha}. \tag{2.5}$$

Define the right-continuous inverse of $V_\alpha(t)$ and (when $\alpha < 1$ the fractional-kinetics process) $Z_\alpha(s)$ by

$$V_\alpha^{-1}(s) \equiv \inf\{t : V_\alpha(t) > s\}, \quad Z_\alpha(s) = B(V_\alpha^{-1}(s)). \tag{2.6}$$

Since V_α is strictly increasing, both V_α^{-1} and Z_α are continuous (almost-surely). The main result of this paper is the following Theorem.

Theorem 2.1. *Suppose that f is such that (2.1) holds for some $\alpha > 0$, then for every $K > 0$,*

$$\frac{S_{\lfloor nt\rfloor}}{\sqrt{\frac{p}{d-p}g_\alpha^{-1}(n)}} \overset{w}{\Longrightarrow} Z_\alpha(t). \tag{2.7}$$

where the convergence is in $D([0, K], \mathbb{R}^d)$ equipped with the uniform topology.

2.1 Discussion

Our results are inspired by [1], in which the scaling limit of a class of (continuous time) trap models is considered. In that work the scaling limit is the same as in our regime when $d \geq 2$, however for $d = 1$ the scaling limit is rather different. The difference can be attributed to the following facts. In our model, each time the walk visits a new edge, the time spent on that edge is independent of all previous visits. This is not the case for the trap model studied in [1] where a random jump rate is chosen initially at each site and remains fixed thereafter. In one dimension this mutual dependence of the time spent at a particular site on successive returns remains in the scaling limit, where the time change/clock process depends on the (local time of the) Brownian motion itself.

If in the above continuous time model the jump rates at each visit to a site are instead chosen to be independent (see [13, 12]), then it is known [12] that the scaling limit is the fractional kinetics process described above. This is intuitive since in this case the process observed at times that it has just jumped to a new (nearest neighbour) site is simple random walk, and the waiting times between jumps are independent and identically distributed random variables that are independent of the position and history of the walk. For the senile reinforced random walk, the direction of the steps of the walk is dependent on the clock and we need to prove that the dependence is sufficiently weak so that it disappears in the scaling limit.

While the slowly varying functions in g_α and g_α^{-1} are not given explicitly, in many cases of interest one can use Theorem XIII.6.2 of [5] and Section 1.5.7 of [2] to explicitly construct them. For example, let $L(n) = \kappa(\log n)^\beta$ for some $\beta \geq -1$. For $\alpha = 1$ we can take

$$\ell(n) = \begin{cases} \kappa \log n, & \text{if } \beta = 0 \\ \kappa(\log\log n), & \text{if } \beta = -1 \\ |\beta^{-1}|\kappa(\log n)^{\beta+1}, & \text{otherwise,} \end{cases} \quad \text{and} \quad g_\alpha^{-1}(n) = \begin{cases} n(\kappa\log n)^{-1}, & \text{if } \beta = 0 \\ n(\kappa\log\log n)^{-1}, & \text{if } \beta = -1 \\ n|\beta|(\kappa\log n)^{-(\beta+1)}, & \text{otherwise.} \end{cases} \tag{2.8}$$

If $\alpha < 1$ we can take

$$\ell(n) = \left(\kappa\Gamma(1-\alpha)\left(\frac{\log n}{\alpha}\right)^\beta\right)^{\frac{1}{\alpha}}, \qquad \text{and} \qquad g_\alpha^{-1}(n) = n^\alpha\left(\kappa(\alpha\log n)^\beta\right)^{-\alpha}. \tag{2.9}$$

Assumption (A1) is simply to avoid the trivial cases where the walk gets stuck on a single edge (i.e. when $(1+f(n))^{-1}$ is summable [8]) or is a self-avoiding walk in one dimension. Roughly speaking, if f grows more slowly than $(2d-1)n$, then $\mathbb{E}[T] < \infty$. For linear reinforcement $f(n) = Cn$, (1.5) shows that assumption (A2) holds with $\alpha = (2d-1)/C$. It would be of interest to consider the scaling limit when $f(n)$ grows like $n\ell(n)$, where $\liminf_{n\to\infty} \ell(n) = \infty$ but such that $(1+f(n))^{-1}$ is not summable. An example is $f(n) = n\log n$, for which $\mathbb{P}(T \geq n) \sim (C\log n)^{-1}$ satisfies (2.1) with $\alpha = 0$.

The condition (2.2) when $\alpha = 1$ is so that one can apply a weak law of large numbers. The condition holds for example when $L(n) = (\log n)^k$ for any $k \geq -1$. It would be surprising if the $\alpha < 1$ case of condition (2.2) is really necessary to obtain a meaningful scaling limit. The condition holds (with $\alpha_o = \alpha_e$ and $L_o = L_e$) whenever there exists n_0 such that for all $n \geq n_0$, $f(n) \geq f(n-1) - (2d-1)$ (so in particular when f is non-decreasing). To see this, observe that for all $n \geq n_0$

$$\begin{aligned} \mathbb{P}(T \geq n, T \text{ even}) &= \sum_{m=\lfloor\frac{n+1}{2}\rfloor}^\infty \mathbb{P}(T = 2m) = \sum_{m=\lfloor\frac{n+1}{2}\rfloor}^\infty \mathbb{P}(T = 2m+1)\frac{2d + f(2m+1)}{1+f(2m)} \\ &\geq \sum_{m=\lfloor\frac{n+1}{2}\rfloor}^\infty \mathbb{P}(T = 2m+1) = \mathbb{P}(T \geq n+1, T \text{ odd}). \end{aligned} \tag{2.10}$$

Similarly, $\mathbb{P}(T \geq n, T \text{ odd}) \geq \mathbb{P}(T \geq n+1, T \text{ even})$ for all $n \geq n_0$. If $\alpha_o \neq \alpha_e$ in (2.2), then (2.1) implies that $\alpha = \alpha_o \wedge \alpha_e$ and L is the slowly varying function corresponding to $\alpha \in \{\alpha_o, \alpha_e\}$ in (2.2). If $\alpha_o = \alpha_e$ then trivially $L \sim L_o + L_e$ ($\sim L_o$ if $L_o(n)/L_e(n) \to \infty$). One can construct examples of reinforcement functions giving rise to different asymptotics for the even and odd cases in (2.2), for example by taking $f(2m) = m^2$ and $f(2m+1) = Cm$ for some well chosen constant $C > 0$ depending on the dimension.

Finally, note that in the case $d = 1$, and $f(n) = n$ we have from [8] that $p = 2(1-\log 2)$ and $\mathbb{P}(T \geq n) = 2(n+1)^{-1} \sim 2n^{-1}$. Taking $\ell(n) = 2\log n$ and $g_\alpha^{-1}(n) = n(2\log n)^{-1}$, Theorem 2.1 then implies that $(p(1-p)^{-1}g_\alpha^{-1}(n))^{-\frac{1}{2}}S_n \xrightarrow{\mathcal{D}} B(1)$, which is consistent with the result of [8] that (for $d = 1$ and $f(n) = n$) $\frac{\log n}{n}\mathbb{E}[S_n^2] \to \frac{1-\log 2}{2\log 2 - 1} > 0$.

3 Invariance principle for the time-changed walk

In this section we prove an invariance principle for any senile reinforced random walk (satisfying (A1)) observed at stopping times τ_n defined by

$$\tau_0 = 0, \quad \tau_k = \inf\{n > (\tau_{k-1} \vee 1) : S_n \neq S_{n-2}\}. \tag{3.1}$$

It is easy to see that $\tau_n = 1 + \sum_{i=1}^n T_i$ for each $n \geq 1$, where the T_i, $i \geq 1$ are independent and identically distributed random variables (with the same distribution as T), corresponding to the number of consecutive traversals of successive edges traversed by the walk.

Proposition 3.1. *For f satisfying (A1), and every $K > 0$,*

$$\frac{S_{\tau_{\lfloor nt \rfloor}}}{\sqrt{\frac{p}{d-p}n}} \overset{w}{\Longrightarrow} B_d(t), \tag{3.2}$$

where the convergence is in $D([0,K],\mathbb{R}^d)$ with the uniform topology.

The process S_{τ_n} is a simpler one than S_n and we expect that one may use many different methods to prove Proposition 3.1 (see for example the Martingale approach of [10]). We give a proof based on describing S_{τ_n} as an additive functional of a Markov chain.

Let $\mathcal{X}$ denote the collection of pairs (u,v) such that

- v is one of the unit vectors $u_i \in \mathbb{Z}^d$, for $i \in \{\pm 1, \pm 2, \cdots \pm d\}$ (labelled so that $u_{-i} = -u_i$) and
- u is either $0 \in \mathbb{Z}^d$ or one of the unit vectors $u_i \neq -v$.

The cardinality of $\mathcal{X}$ is then $|\mathcal{X}| = 2d + 2d(2d-1) = (2d)^2$.

Given a senile reinforced random walk S_n with parameter $p = \mathbb{P}(T \text{ odd}) \in (0,1]$, we define an irreducible, aperiodic Markov chain $X_n = (X_n^{[1]}, X_n^{[2]})$ with natural filtration $\mathcal{G}_n = \sigma(X_1,\dots,X_n)$, and finite-state space $\mathcal{X}$, as follows.

For $n \geq 1$, let $X_n = (S_{\tau_n - 1} - S_{\tau_{(n-1)}}, S_{\tau_n} - S_{\tau_n - 1})$, and $Y_n = X_n^{[1]} + X_n^{[2]}$. It follows immediately that $S_{\tau_n} = \sum_{m=1}^n Y_m$ and

$$\begin{aligned} X_1 =& (0, S_{\tau_1}) I_{\{T_1 \text{ even}\}} + (S_1, S_{\tau_1} - S_{\tau_1 - 1}) I_{\{T_1 \text{ odd}\}}, \quad \text{and for } n \geq 2, \\ X_n =& (0, S_{\tau_n} - S_{\tau_{(n-1)}}) I_{\{T_n \text{ odd}\}} + (-X_{n-1}^{[2]}, S_{\tau_n} - S_{\tau_n - 1}) I_{\{T_n \text{ even}\}}. \end{aligned} \tag{3.3}$$

$$\mathbb{P}(X_1 = (0,u_i)) = \frac{1-p}{2d}, \quad \text{and} \quad \mathbb{P}(X_1 = (u_i,u_j)) = \frac{p}{2d(2d-1)}, \quad \text{for each } i,j, \quad (j \neq -i). \tag{3.4}$$

Now T_n is independent of $X_1,\dots,X_{n-1}$, and conditionally on T_n being odd (resp. even), $S_{\tau_n} - S_{\tau_{(n-1)}}$ (resp. $S_{\tau_n} - S_{\tau_n - 1}$) is uniformly distributed over the $2d-1$ unit vectors in $\mathbb{Z}^d$ other than $-X_{n-1}^{[2]}$ (resp. other than $X_{n-1}^{[2]}$). Therefore for $n \geq 2$,

$$\begin{aligned} \mathbb{P}(X_n = (u,v) | X_0,\dots,X_{n-1}) =& \mathbb{P}(T_n \text{ odd}, S_{\tau_n} - S_{\tau_{(n-1)}} = v | X_0,\dots,X_{n-1}) I_{\{u=0\}} \\ &+ \mathbb{P}(T_n \text{ even}, S_{\tau_n} - S_{\tau_n - 1} = v | X_0,\dots,X_{n-1}) I_{\{u = -X_{n-1}^{[2]}\}} \\ =& \frac{p}{2d-1} I_{\{u=0\}} I_{\{v \neq -X_{n-1}^{[2]}\}} + \frac{1-p}{2d-1} I_{\{u=-X_{n-1}^{[2]}\}} I_{\{v \neq X_{n-1}^{[2]}\}}, \end{aligned} \tag{3.5}$$

which depends only on X_{n-1}. This verifies that X_n is a Markov chain with initial distribution (3.4) and transition probabilities given by

$$\mathbb{P}\left(X_n = (u,v) | X_{n-1} = (u',v')\right) = \begin{cases} \frac{p}{2d-1}, & \text{if } u = 0 \text{ and } v \neq -v', \\ \frac{1-p}{2d-1}, & \text{if } u = -v' \text{ and } v \neq v', \\ 0, & \text{otherwise.} \end{cases}. \tag{3.6}$$

That X_n is irreducible and aperiodic is obvious. By symmetry, the unique stationary distribution $\vec{\pi} = (\pi_{(0,u_{-d})},\dots,\pi_{(0,u_d)},\pi_{(u_{-d},u_{-d})},\dots,\pi_{(u_d,u_d)}) \in M_1(\mathcal{X})$ of the chain must be of the form

$$\vec{\pi} = (\pi_1,\dots,\pi_1,\pi_2,\dots,\pi_2), \tag{3.7}$$

where the first $2d$ entries are π_1 and the remaining $2d(2d-1)$ entries are π_2, and therefore

$$2d\pi_1 + 2d(2d-1)\pi_2 = 1. \tag{3.8}$$

Solving $\pi_{(0,u_1)} = \sum_{u,v} \pi_{(u,v)} P\left((u,v),(0,u_1)\right)$ with $\vec{\pi}$ as in (3.7) we get

$$\pi_1 = (2d-1)\frac{p}{2d-1}\pi_1 + (2d-1)^2 \frac{p}{2d-1}\pi_2. \tag{3.9}$$

From (3.8) and (3.9) we obtain

$$\pi_1 = \frac{p}{2d}, \qquad \pi_2 = \frac{1-p}{2d(2d-1)}. \tag{3.10}$$

It is easy to show that in general the Markov chain $\{X_n\}_{n\geq 1}$, is not stationary. However, as an irreducible, aperiodic, finite-state Markov chain, it has *exponentially fast, strong mixing.* To be precise, there exists a constant c and $t<1$ such that for every $k \geq 1$,

$$\alpha(k) \equiv \sup_n \Big\{ |\mathbb{P}(F\cap G) - \mathbb{P}(F)\mathbb{P}(G)| : F \in \sigma(X_j, j\leq n), G \in \sigma(X_j, j \geq n+k) \Big\} \leq ct^k. \tag{3.11}$$

It is obvious that if Z_j is a strongly-mixing sequence and Z'_j is measurable with respect to Z_j for each j then Z'_j is also strongly mixing, with the same (or possibly faster) mixing rate. Therefore the sequence Y_n also has exponentially fast, strong mixing. In order to verify Proposition 3.1, we will prove a multidimensional corollary of the following result of [7].

Theorem 3.2 (Corollary 1 of [7])**.** *Suppose that Z_n is a sequence of $\mathbb{R}$-valued random variables such that $\mathbb{E}[Z_n] = 0$, $\mathbb{E}[Z_n^2] < \infty$ and $\mathbb{E}[n^{-1}(\sum_{i=1}^n Z_i)^2] \to \sigma^2$ as $n\to\infty$. Further suppose that Z_n is α-strongly mixing and that there exists $\beta \in (2,\infty]$ such that*

$$\sum_{k=1}^{\infty} \alpha(k)^{1-2/\beta} < \infty, \quad \textit{and} \quad \limsup_{n\to\infty} \|Z_n\|_\beta < \infty, \tag{3.12}$$

then

$$\frac{\sum_{i=1}^{\lfloor nt \rfloor} Z_i}{\sqrt{\sigma^2 n}} \overset{w}{\Longrightarrow} B_1(t), \tag{3.13}$$

where the convergence is in $D([0,K],\mathbb{R})$ with the uniform topology.

Corollary 3.3. *Suppose that $W_n = (W_n^{(1)},\dots,W_n^{(d)})$ is a sequence of $\mathbb{R}^d$-valued random variables such that $\mathbb{E}[W_n] = 0$, $\mathbb{E}[|W_n|^2] < \infty$ and $\mathbb{E}[n^{-1}\sum_{i=1}^n\sum_{i'=1}^n W_i^{(j)}W_{i'}^{(l)}] \to \sigma^2 I_{j=l}$, as $n\to\infty$. Further suppose that W_n is α-strongly mixing and that there exists $\beta\in(2,\infty]$ such that*

$$\sum_{k=1}^{\infty} \alpha(k)^{1-2/\beta} < \infty, \quad \textit{and} \quad \limsup_{n\to\infty} \|W_n\|_\beta < \infty, \tag{3.14}$$

then

$$\mathcal{W}_n(t) \equiv \frac{\sum_{i=1}^{\lfloor nt \rfloor} W_i}{\sqrt{\sigma^2 n}} \overset{w}{\Longrightarrow} B_d(t), \tag{3.15}$$

where the convergence is in $D([0,K],\mathbb{R}^d)$ with the uniform topology.

Proof. It is sufficient to prove convergence of the finite-dimensional distributions and tightness.

Let $0 \leq t_1 < t_2 < \cdots < t_r \leq K$. For convergence of the f.d.d. we need to show that

$$(\mathcal{W}_n^{(1)}(t_1),\dots,\mathcal{W}_n^{(d)}(t_1),\dots,\mathcal{W}_n^{(1)}(t_r),\dots,\mathcal{W}_n^{(d)}(t_r)) \xrightarrow{\mathcal{D}} (B_d^{(1)}(t_1),\dots,B_d^{(d)}(t_1),\dots,B_d^{(1)}(t_r)\dots,B_d^{(d)}(t_r)). \tag{3.16}$$

Using the Cramér-Wold device (e.g. see Theorem 4.3.3. of [15]), it is enough to show that

$$\sum_{m=1}^{d}\sum_{l=1}^{r} a_{m,l}\mathcal{W}_n^{(m)}(t_l) \xrightarrow{\mathcal{D}} \sum_{m=1}^{d}\sum_{l=1}^{r} a_{m,l}B_d^{(m)}(t_l) \in \mathbb{R}, \tag{3.17}$$

for every $(a_{1,1},\dots,a_{d,1},\dots,a_{1,r},\dots,a_{d,r}) \in \mathbb{R}^{dr}$. Note that since $\{B_d^{(m)}(t)\}_{m=1,\dots,d}$ are independent and identically distributed 1-dimensional Brownian motions, we have that

$$\left(\sum_{m=1}^{d} a_{m,1}B_d^{(m)}(t_1),\dots,\sum_{m=1}^{d} a_{m,r}B_d^{(m)}(t_r)\right) \quad \text{and} \quad \left(\sum_{m=1}^{d} a_{m,1}B_1(t_1),\dots,\sum_{m=1}^{d} a_{m,r}B_1(t_r)\right) \tag{3.18}$$

have the same distribution.

Since any sequence W_n' such that W_n' is measurable with respect to W_n for each n, is also α-strongly mixing, we have that $Z_n \equiv \sum_{m=1}^d a_{m,l} W_n^{(m)} \in \mathbb{R}$ is α-strongly mixing. Finiteness of $\mathbb{E}[Z_n^2]$ and the second condition of (3.14) for Z_n follow immediately from the corresponding properties of W_n. Next,

$$\mathbb{E}[n^{-1}(\sum_{i=1}^n Z_i)^2] = \sum_{m=1}^d \sum_{m'=1}^d a_{m,l} a_{m',l} \mathbb{E}[n^{-1} \sum_{i=1}^n \sum_{i'=1}^n W_i^{(m)} W_{i'}^{(m')})] \to \sigma^2 (\sum_{m=1}^d a_{m,l})^2 \equiv \sigma^2 A_l^2, \tag{3.19}$$

so by Theorem 3.2

$$\frac{\sum_{i=1}^{\lfloor nt \rfloor} Z_i}{\sqrt{\sigma^2 A_l^2 n}} \stackrel{w}{\Longrightarrow} B(t). \tag{3.20}$$

Written in terms of $\mathcal{W}_n$, (3.20) is

$$\sum_{m=1}^d a_{m,l} \mathcal{W}_n^{(m)}(t) \stackrel{w}{\Longrightarrow} \sum_{m=1}^d a_{m,l} B(t). \tag{3.21}$$

In particular, the finite-dimensional distributions in (3.21) converge from which we get that

$$\sum_{l=1}^r \sum_{m=1}^d a_{m,l} \mathcal{W}_n^{(m)}(t_l) \stackrel{w}{\Longrightarrow} \sum_{l=1}^r \sum_{m=1}^d a_{m,l} B(t_l). \tag{3.22}$$

By (3.18), this is sufficient to prove (3.17).

To prove tightness, observe that $\{W_n^{(j)}\}_{n\geq 0}$ is also α-strongly mixing for each $j = 1, \ldots, d$. Applying Theorem 3.2 to this sequence we get that

$$\frac{\sum_{i=1}^{\lfloor nt \rfloor} W_i^{(j)}}{\sqrt{\sigma^2 n}} \stackrel{w}{\Longrightarrow} B_1(t), \tag{3.23}$$

from which tightness of $\{W_n^{(j)}\}_{n\geq 0}$ for each j follows immediately. Tightness of the joint distributions $\{(W_n^{(1)}, \ldots, W_n^{(d)})\}_{n\geq 0}$ is a trivial consequence of tightness of the marginals. $\square$

3.1 Proof of Proposition 3.1

Since $S_{\tau_n} = \sum_{m=1}^n Y_m$ where $|Y_m| \leq 2$, and the sequence $\{Y_n\}_{n\geq 0}$ has exponentially fast strong mixing, to prove Proposition 3.1 it is enough to show that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \sum_{i'=1}^n Y_i^{(j)} Y_i^{(l)}\right] \to \frac{p}{d-p} I_{j=l}. \tag{3.24}$$

By symmetry, for all $j \neq l$, and any n, m, $\mathbb{E}[(X_n^{[1]} + X_n^{[2]})^{(j)} (X_m^{[1]} + X_m^{[2]})^{(l)}] = 0$. This verifies (3.24) in the case $j \neq l$. By symmetry, it therefore remains to show that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \sum_{i'=1}^n Y_i^{(1)} Y_{i'}^{(1)}\right] \to \frac{p}{d-p}. \tag{3.25}$$

It is easy to show that $\mathbb{E}[X_n^{[2],(1)} | X_{n-1}] = \frac{2p-1}{2d-1} X_{n-1}^{[2],(1)}$ and so by induction and the Markov property, for every $n \geq m \geq 1$,

$$\mathbb{E}[X_n^{[2],(1)} | X_m] = \left(\frac{2p-1}{2d-1}\right)^{n-m} X_m^{[2],(1)}. \tag{3.26}$$

Next, observe that for $n \geq 2$,

$$\mathbb{E}[Y_n^{(1)} | X_{n-1}] = \frac{p - 2d(1-p)}{2d-1} X_{n-1}^{[2],(1)}, \tag{3.27}$$

and therefore using the fact that Y_n is X_n measurable, and the Markov property for X_n, we have that for $n > m \geq 1$,

$$\mathbb{E}[Y_n^{(1)}|X_m] = \frac{p-2d(1-p)}{2d-1}\left(\frac{2p-1}{2d-1}\right)^{n-1-m} X_m^{[2],(1)}. \tag{3.28}$$

For $n > m \geq 1$ we have

$$\begin{aligned}
\mathbb{E}[Y_n^{(1)}Y_m^{(1)}] =&\mathbb{E}[Y_m^{(1)}\mathbb{E}[Y_n^{(1)}|X_m]] = \frac{p-2d(1-p)}{2d-1}\left(\frac{2p-1}{2d-1}\right)^{n-1-m}\mathbb{E}[Y_m^{(1)}X_m^{[2],(1)}] \\
=&\frac{p-2d(1-p)}{2d-1}\left(\frac{2p-1}{2d-1}\right)^{n-1-m}\left(\mathbb{E}[X_m^{[1],(1)}X_m^{[2],(1)}]+\mathbb{E}[(X_m^{[2],(1)})^2]\right) \\
=&\frac{p-2d(1-p)}{2d-1}\left(\frac{2p-1}{2d-1}\right)^{n-1-m} \times \begin{cases} \frac{1-p}{d(2d-1)}+\frac{1}{d}, & m\geq 2 \\ \frac{p}{d(2d-1)}+\frac{1}{d}, & m=1 \end{cases}.
\end{aligned} \tag{3.29}$$

Lastly

$$\mathbb{E}[|Y_1|^2] = (1-p)+\frac{4dp}{2d-1}, \quad \text{and} \quad \mathbb{E}[|Y_m|^2] = p+\frac{4d(1-p)}{2d-1}, m\geq 2. \tag{3.30}$$

Combining all of these results, we get that

$$\begin{aligned}
\mathbb{E}\left[\sum_{l=1}^{n}\sum_{m=1}^{n}Y_l^{(1)}Y_m^{(1)}\right] =&2\sum_{l=2}^{n}\sum_{m=2}^{l-1}\mathbb{E}[Y_l^{(1)}Y_m^{(1)}]+2\sum_{l=2}^{n}\mathbb{E}[Y_l^{(1)}Y_1^{(1)}]+\sum_{l=1}^{n}\mathbb{E}[|Y_l^{(1)}|^2] \\
=&\frac{2}{d}\frac{p-2d(1-p)}{2d-1}\left(\frac{1-p}{2d-1}+1\right)\sum_{l=2}^{n}\sum_{m=1}^{l-1}\left(\frac{2p-1}{2d-1}\right)^{l-1-m}+o(n)+(n-1)\left(\frac{p}{d}+\frac{4(1-p)}{2d-1}\right) \\
=&\frac{2}{d}\frac{p-2d(1-p)}{2d-1}\left(\frac{1-p}{2d-1}+1\right)\sum_{l=2}^{n}\sum_{k=0}^{l-2}\left(\frac{2p-1}{2d-1}\right)^{k}+o(n)+(n-1)\left(\frac{p}{d}+\frac{4(1-p)}{2d-1}\right) \\
=&\frac{2}{d}\frac{p-2d(1-p)}{2d-1}\left(\frac{1-p}{2d-1}+1\right)\sum_{l=2}^{n}\frac{1-(\frac{2p-1}{2d-1})^{l-1}}{1-\frac{2p-1}{2d-1}}+o(n)+(n-1)\left(\frac{p}{d}+\frac{4(1-p)}{2d-1}\right) \\
=&\frac{2}{d}\frac{p-2d(1-p)}{2d-1}\left(\frac{1-p}{2d-1}+1\right)\frac{2d-1}{2(d-p)}(n-1)+o(n)+(n-1)\left(\frac{p}{d}+\frac{4(1-p)}{2d-1}\right) \\
=&n\left(\frac{p-2d(1-p)}{d(d-p)}\left(\frac{1-p}{2d-1}+1\right)+\frac{p}{d}+\frac{4(1-p)}{2d-1}\right)+o(n) = n\frac{p}{d-p}+o(n).
\end{aligned} \tag{3.31}$$

Taking the limit as $n \to \infty$ verifies (3.25) and thus completes the proof of Proposition 3.1. □

4 Proof of Theorem 2.1

Theorem 2.1 is a consequence of convergence of the joint distribution of the stopping time process and the random walk at those stopping times as in the following proposition.

Proposition 4.1. *Suppose that f is such that assumptions (A1) and (A2) hold for some $\alpha > 0$, then for every $K > 0$,*

$$\left(\frac{S_{\tau_{\lfloor nt\rfloor}}}{\sqrt{\frac{p}{d-p}n}}, \frac{\tau_{\lfloor nt\rfloor}}{g_\alpha(n)}\right) \overset{w}{\Longrightarrow} (B_d(t), V_\alpha(t)), \tag{4.1}$$

where the convergence is in $\left(D([0,K],\mathbb{R}^d),\mathcal{U}\right)\times(D([0,K],\mathbb{R}),J_1)$ and $\mathcal{U}, J_1$ denote the uniform and Skorokhod J_1 topologies respectively.

Proof of Theorem 2.1 assuming Proposition 4.1. Since $\lfloor g_\alpha^{-1}(n) \rfloor$ is a sequence of positive integers such that $\lfloor g_\alpha^{-1}(n)\rfloor \to \infty$ as $n \to \infty$, it follows from (4.1) that as $n \to \infty$,

$$\left(\frac{S_{\tau_{\lfloor \lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}}{\sqrt{\frac{p}{d-p}\lfloor g_\alpha^{-1}(n)\rfloor}}, \frac{\tau_{\lfloor \lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}{g_\alpha(\lfloor g_\alpha^{-1}(n)\rfloor)} \right) \overset{w}{\Longrightarrow} (B_d(t), V_\alpha(t))\,. \tag{4.2}$$

Now use the following facts,

- For all $t \le K$, $|S_{\tau_{\lfloor \lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}} - S_{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor}}| \le 2(K+1)$,
- Since g_α^{-1} is regularly varying (and w.l.o.g. monotone), $\lfloor g_\alpha^{-1}(n)\rfloor \sim g_\alpha^{-1}(n)$ as $n \to \infty$,
- Letting $\overset{\mathcal{D}}{=}$ denote equality in distribution,

$$\sup_{t\in[0,K]} \frac{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor} - \tau_{\lfloor \lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}{\lfloor g_\alpha^{-1}(n)\rfloor} \le \sup_{t\in[0,K]} \frac{\sum_{i=1+\lfloor \lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}^{\lfloor g_\alpha^{-1}(n)t\rfloor} T_i}{\lfloor g_\alpha^{-1}(n)\rfloor} \overset{\mathcal{D}}{=} \sup_{t\in[0,K]} \frac{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor - \lfloor \lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}{\lfloor g_\alpha^{-1}(n)\rfloor} \xrightarrow{P} 0,$$

to conclude that

$$\left(\frac{S_{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor}}}{\sqrt{\frac{p}{d-p} g_\alpha^{-1}(n)}}, \frac{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor}}{n} \right) \overset{w}{\Longrightarrow} (B_d(t), V_\alpha(t))\,. \tag{4.3}$$

Let

$$Y_n(t) = \frac{S_{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor}}}{\sqrt{\frac{p}{d-p} g_\alpha^{-1}(n)}}, \quad \textit{and} \quad \mathcal{T}_n(t) = \frac{\tau_{\lfloor g_\alpha^{-1}(n)t\rfloor}}{n}, \tag{4.4}$$

and let $\mathcal{T}_n^{-1}(t) \equiv \inf\{s \ge 0 : \mathcal{T}_n(s) > t\} = \inf\{s \ge 0 : \tau_{\lfloor g_\alpha^{-1}(n)s\rfloor} > nt\}$. As in (the proof of Theorem 1.3 in) [1], it follows that $Y_n(\mathcal{T}_n^{-1}(t)) \overset{w}{\Longrightarrow} B_d(V_\alpha^{-1}(t))$ in $\big(D([0,K],\mathbb{R}^d),\mathcal{U}\big)$. Thus,

$$\frac{S_{\tau_{\lfloor g_\alpha^{-1}(n)\mathcal{T}_n^{-1}(t)\rfloor}}}{\sqrt{\frac{p}{d-p} g_\alpha^{-1}(n)}} \overset{w}{\Longrightarrow} B_d(V_\alpha^{-1}(t)). \tag{4.5}$$

Since by definition of $\mathcal{T}_n^{-1}$, we have $\tau_{\lfloor g_\alpha^{-1}(n)\mathcal{T}_n^{-1}(t)\rfloor} - 1 \le nt \le \tau_{\lfloor g_\alpha^{-1}(n)\mathcal{T}_n^{-1}(t)\rfloor}$ and hence $|S_{\lfloor nt\rfloor} - S_{\tau_{\lfloor g_\alpha^{-1}(n)\mathcal{T}_n^{-1}(t)\rfloor}}| \le 3$. This fact together with (4.5) proves Theorem 2.1. $\square$

5 Proof of Proposition 4.1

The proof of Proposition 4.1 is broken into two parts. Roughly speaking, the first part is the observation that the marginal processes converge, i.e. that the time-changed walk and the time-change converge to $B_d(t)$ and $V_\alpha(t)$ respectively, while the second is to show that these two processes are asymptotically independent.

5.1 Convergence of the time-changed walk and the time-change.

Lemma 5.1. *Suppose that f is such that assumptions (A1) and (A2) hold for some $\alpha > 0$, then for every $K > 0$,*

$$\frac{S_{\tau_{\lfloor nt\rfloor}}}{\sqrt{\frac{p}{d-p}n}} \overset{w}{\Longrightarrow} B_d(t) \quad \textit{in } (D([0,K],\mathbb{R}^d),\mathcal{U}), \qquad \textit{and} \qquad \frac{\tau_{\lfloor nt\rfloor}}{g_\alpha(n)} \overset{w}{\Longrightarrow} V_\alpha(t) \quad \textit{in } (D([0,K],\mathbb{R}),J_1). \tag{5.1}$$

Proof. The first claim is the conclusion of Proposition 3.1, so we need only prove the second claim. Recall that $\tau_n = 1 + \sum_{i=1}^n T_i$ where the T_i are i.i.d. with distribution T. Since $g_\alpha(n) \to \infty$, it is enough to show convergence of $\tau^*_{\lfloor nt\rfloor} = (\tau_{\lfloor nt\rfloor} - 1)/g_\alpha(n)$.

For processes with independent and identically distributed increments, a standard result of Skorokhod essentially extends the convergence of the one-dimensional distributions to functional limit theorem. In particular when $\alpha < 1$, the result is well known (see [5] XIII.6 and [15] 4.5.3 for example).

It remains to prove convergence of the one-dimensional marginals. When $\mathbb{E}[T]$ exists, the claim is that $\tau^*_{\lfloor nt\rfloor}/n\mathbb{E}[T] \overset{w}{\Longrightarrow} t$, which is immediate from the strong law of large numbers. When $\alpha = 1$ but (2.1) is not summable, the result is immediate from the following lemma.

Lemma 5.2. *Let T_k be independent and identically distributed random variables satisfying (2.1) and (2.2) with $\alpha = 1$. Then for each $t \geq 0$,*

$$\frac{\tau^*_{\lfloor nt\rfloor}}{n\ell(n)} \xrightarrow{P} t. \tag{5.2}$$

Lemma 5.2 is a corollary of the following weak law of large numbers due to Gut [6].

Theorem 5.3 ([6] Theorem 1.3)**.** *Let X_k be i.i.d. random variables and $S_n = \sum_{k=1}^n X_k$. Let $g_n = n^{1/\alpha}\ell(n)$ for $n \geq 1$, where $\alpha \in (0, 1]$ and $\ell(n)$ is slowly varying at infinity. Then*

$$\frac{S_n - n\mathbb{E}\left[XI_{\{|X|\leq g_n\}}\right]}{g_n} \xrightarrow{P} 0, \quad \text{as } n \to \infty, \tag{5.3}$$

if and only if $n\mathbb{P}(|X| > g_n) \to 0$.

Proof of Lemma 5.2. Let $Y = TI_{\{|T|\leq n\ell(n)\}}$. Then $Y \in \{0, 1, \dots \lfloor n\ell(n)\rfloor\}$ and

$$\mathbb{E}\left[TI_{\{|T|\leq n\ell(n)\}}\right] = \sum_{j=1}^{\infty}\mathbb{P}(Y \geq j) = \sum_{j=1}^{\lfloor n\ell(n)\rfloor}\mathbb{P}(n\ell(n) \geq T \geq j) = \sum_{j=1}^{\lfloor n\ell(n)\rfloor}\mathbb{P}(T \geq j) - \lfloor n\ell(n)\rfloor\mathbb{P}(T \geq n\ell(n)). \tag{5.4}$$

Now by assumption (A2b),

$$\begin{aligned}\frac{n}{n\ell(n)}\mathbb{E}\left[TI_{\{|T|\leq n\ell(n)\}}\right] &= \frac{\sum_{j=1}^{\lfloor n\ell(n)\rfloor}\mathbb{P}(T \geq j)}{\ell(n)} - \frac{\lfloor n\ell(n)\rfloor}{\ell(n)}\mathbb{P}(T \geq n\ell(n))\\ &\sim \frac{\sum_{j=1}^{\lfloor n\ell(n)\rfloor} j^{-1}L(j)}{\ell(n)} - \frac{\lfloor n\ell(n)\rfloor}{\ell(n)}(n\ell(n))^{-1}L(n\ell(n)) \to 1.\end{aligned} \tag{5.5}$$

Theorem 5.3 then implies that $(n\ell(n))^{-1}\tau_n \xrightarrow{P} 1$, from which it follows immediately that

$$(n\ell(n))^{-1}\tau_{\lfloor nt\rfloor} = (n\ell(n))^{-1}\lfloor nt\rfloor\ell(\lfloor nt\rfloor)(\lfloor nt\rfloor\ell(\lfloor nt\rfloor))^{-1}\tau_{\lfloor nt\rfloor} \xrightarrow{P} t. \tag{5.6}$$

This completes the proof of Lemma 5.2, and hence Lemma 5.1. $\square$

5.2 Asymptotic Independence

Tightness of the joint process in Proposition 4.1 is an easy consequence of the (already established) tightness of the marginal processes, so we need only prove convergence of the finite-dimensional distributions. For $\alpha \geq 1$ this is simple and is left as an exercise. To complete the proof of Proposition 4.1, it remains to prove convergence of the finite-dimensional distributions in the case $\alpha < 1$, for which $p = \mathbb{P}(T \text{ odd}) < 1$.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be convergence determining classes of bounded, $\mathbb{C}$-valued functions on $\mathbb{R}^d$ and $\mathbb{R}_+$ respectively, each closed under conjugation and containing a non-zero constant function, then

$$\{g(x) \equiv g_1(x_1)g_2(x_2) : g_i \in \mathcal{G}_i\}$$

is a convergence determining class for $\mathbb{R}^d \times \mathbb{R}_+$. This follows as in Proposition 3.4.6 of [4] where the closure under conjugation allows us to extend the proof to complex-valued functions. Therefore, to prove convergence of the finite-dimensional distributions in (4.1) it is enough to show that for every $0 \leq t_1 < t_2 < \cdots < t_r \leq K$, $k_j \in \mathbb{R}^d$ and $\eta_j \geq 0$,

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^r k_j \cdot \frac{S_{\tau_{\lfloor nt_j\rfloor}}}{\sqrt{\frac{p}{d-p}n}}\right\}\exp\left\{-\sum_{j=1}^r \eta_j\frac{\tau_{\lfloor nt_j\rfloor}}{g_\alpha(n)}\right\}\right] \to \mathbb{E}\left[\exp\left\{i\sum_{j=1}^r \vec{k_j}\cdot B(t_j)\right\}\right]\mathbb{E}\left[\exp\left\{-\sum_{j=1}^r \eta_j V_\alpha(t_j)\right\}\right]. \tag{5.7}$$

Let $\mathcal{A}_n = \{k \in \{1,\dots,n\} : T_i \text{ is odd}\}$, $\mathcal{A}_{\lfloor n\vec{t}\rceil} = (\mathcal{A}_{\lfloor n\vec{t}_1\rfloor} \setminus \mathcal{A}_{\lfloor n\vec{t}_0\rfloor},\dots,\mathcal{A}_{\lfloor n\vec{t}_r\rfloor} \setminus \mathcal{A}_{\lfloor n\vec{t}_{r-1}\rfloor})$ and $t_0 = 0$. For fixed n and $\vec{t}$, we write $A = (A_1,\dots,A_r)$ to denote an element of the sample space of the random variable $\mathcal{A}_{\lfloor n\vec{t}\rceil}$, where $A_i \subseteq \{\lfloor nt_{i-1}\rfloor + 1,\dots \lfloor nt_i\rfloor\}$ for each $i \in 1,\dots,r$. Let $\epsilon \in (0,\frac{1}{2})$, $B_n(\vec{t}) = \{A : ||A_l| - (\lfloor nt_l p\rfloor - \lfloor nt_{l-1}p\rfloor)| \le n^{1-\epsilon} \text{ for each } l\}$ and $Q^n_{\vec{k}}(\vec{t}) = \exp\left\{i\sum_{j=1}^r k_j \cdot \frac{S_{\tau_{\lfloor nt_j\rfloor}}}{\sqrt{\frac{p}{d-p}n}}\right\}$. Note that $|Q^n_{\vec{k}}(\vec{t})| \le 1$. Then the left hand side of (5.7) is equal to

$$\begin{aligned}
&e^{-\frac{1}{g_\alpha(n)}\sum_{j=1}^r \eta_j}\sum_A \mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\exp\left\{-\sum_{j=1}^r \eta_j \frac{\mathcal{T}^*_{\lfloor nt_j\rfloor}}{g_\alpha(n)}\right\}\middle|\{\mathcal{A}_{\lfloor n\vec{t}\rceil} = A\}\right]\mathbb{P}(\mathcal{A}_{\lfloor n\vec{t}\rceil} = A)\\
&= \sum_{A\in B_n(\vec{t})} \mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\exp\left\{-\sum_{j=1}^r \eta_j \frac{\mathcal{T}^*_{\lfloor nt_j\rfloor}}{g_\alpha(n)}\right\}\middle|\{\mathcal{A}_{\lfloor n\vec{t}\rceil} = A\}\right]\mathbb{P}(\mathcal{A}_{\lfloor n\vec{t}\rceil} = A) + o(1) \qquad (5.8)\\
&= \sum_{A\in B_n(\vec{t})} \mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\middle|\{\mathcal{A}_{\lfloor n\vec{t}\rceil} = A\}\right]\mathbb{E}\left[\exp\left\{-\sum_{j=1}^r \eta_j \frac{\sum_{i=1}^{\lfloor nt_j\rfloor} T_i}{g_\alpha(n)}\right\}\middle|\{\mathcal{A}_{\lfloor n\vec{t}\rceil} = A\}\right]\mathbb{P}(\mathcal{A}_{\lfloor n\vec{t}\rceil} = A) + o(1),
\end{aligned}$$

since given $I_{\{T_i \text{ even}\}}$, $i = 1,\dots,n$, S_{τ_n} is independent of the collection $\{T_i\}_{i\ge 1}$.

Let $\eta^*_l = \sum_{j=l}^r \eta_i$. Then the last line of (5.8) is equal to

$$\sum_{A\in B_n(\vec{t})} \mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\middle|\{\mathcal{A}_{\lfloor n\vec{t}\rceil} = A\}\right]\mathbb{P}(\mathcal{A}_{\lfloor n\vec{t}\rceil} = A)\prod_{l=1}^r \mathbb{E}\left[\exp\left\{-\eta^*_l \frac{\sum_{i=\lfloor nt_{l-1}\rfloor+1}^{\lfloor nt_l\rfloor} T_i}{g_\alpha(n)}\right\}\middle|\{[\mathcal{A}_{\lfloor n\vec{t}\rceil}]_l = A_l\}\right] + o(1). \tag{5.9}$$

Let T^o_i, $i \in \mathbb{N}$ be independent, identically distributed random variables with $\mathbb{P}(T^o_i = k) = \mathbb{P}(T = k|T \text{ odd})$, and similarly define T^e_i to be i.i.d. with $\mathbb{P}(T^e_i = k) = \mathbb{P}(T = k|T \text{ even})$.

Now the l^{th} term in the product in (5.9) is equal to

$$\mathbb{E}\left[\exp\left\{-\eta^*_l \frac{\sum_{i=1}^{|A_l|} T^o_i}{g_\alpha(n)}\right\}\right]\mathbb{E}\left[\exp\left\{-\eta^*_l \frac{\sum_{i=1}^{\lfloor nt_l\rfloor - \lfloor nt_{l-1}\rfloor - |A_l|} T^e_i\}}{g_\alpha(n)}\right\}\right]. \tag{5.10}$$

Let $g^o_{\alpha_o}(n)$ and $g^e_{\alpha_e}(n)$ be defined as in (2.4) with the random variable T replaced with T_o and T_e respectively. For example, for T_o we have

$$\mathbb{P}(T_o \ge n) = \mathbb{P}(T \ge n|T \text{ odd}) = \frac{\mathbb{P}(T \ge n, T \text{ odd})}{p} \sim \frac{L_o(n)}{pn^{\alpha_o}}, \tag{5.11}$$

and there exists ℓ_o such that $(\ell_o(n))^{-\alpha_o}p^{-1}L_o(n^{\frac{1}{\alpha_o}}\ell_o(n)) \to (\Gamma(1-\alpha))^{-1}$, and define $g^o_{\alpha_o}(n) = n^{\frac{1}{\alpha_o}}\ell_o(n)$.

Observe that

$$\frac{\sum_{i=1}^{|A_l|} T^o_i}{g_\alpha(n)} = \frac{\sum_{i=1}^{n_l} T^o_i}{g^o_{\alpha_o}(n_l)}\frac{g^o_{\alpha_o}(n_l)}{g_\alpha(n_l)}\frac{g_\alpha(n_l)}{g_\alpha(n)} + \mathcal{O}\left(\frac{\sum_{i=1}^{|A_l|} T^o_i - \sum_{i=1}^{n_l} T^o_i}{g^o_{\alpha_o}(n^*_l)}\frac{g^o_{\alpha_o}(n^*_l)}{g_\alpha(n^*_l)}\frac{g_\alpha(n^*_l)}{g_\alpha(n)}\right), \tag{5.12}$$

where $n_l \equiv \lfloor nt_l p\rfloor - \lfloor nt_{l-1}p\rfloor$ and $n^*_l \equiv ||A_l| - n_l| \le n^{1-\epsilon}$ since $A \in B_n(\vec{t})$. By definition of g_α and standard results on regular variation we have that $g_\alpha(n_l)/g_\alpha(n) \to (p(t_l - t_{l-1}))^{\frac{1}{\alpha}}$ and $g_\alpha(n^*_l)/g_\alpha(n) \to 0$. Since $\alpha = \alpha_o \wedge \alpha_e \le \alpha_o$, the $\mathcal{O}$ term on the right of (5.12) converges in probability to 0. Thus, as in the second claim of Lemma 5.1, we get that

$$\frac{\sum_{i=1}^{|A_l|} T^o_i}{g_\alpha(n)} \overset{w}{\Longrightarrow} V_\alpha(1)(p(t_l - t_{l-1}))^{\frac{1}{\alpha}}\lim_{n\to\infty}\frac{g^o_{\alpha_o}(n_l)}{g_\alpha(n_l)}, \tag{5.13}$$

where for $\alpha < 1$ the limit $\rho_o \equiv \lim_{n\to\infty}\frac{g^o_{\alpha_o}(n_l)}{g_\alpha(n_l)}$ exists in $[0,\infty]$ since $\alpha \le \alpha_o$ and in the case of equality, the limit L_o/L_e exists in $[0,\infty]$. Note that we were able to replace α_o with α in various places in (5.13) due to the presence of the factor $\frac{g^o_{\alpha_o}(n_l)}{g_\alpha(n_l)}$ which is zero when $\alpha_o > \alpha$. Therefore

$$\begin{aligned}\mathbb{E}\left[\exp\left\{-\eta_l^*\frac{\sum_{i=1}^{|A_l|}T_i^o}{g_\alpha(n)}\right\}\right] &\to \mathbb{E}\left[\exp\{-\eta_l^* V_\alpha(1)(p(t_l-t_{l-1}))^{\frac{1}{\alpha}}\rho_o\}\right], \quad \text{and similarly,}\\ \mathbb{E}\left[\exp\left\{-\eta_l^*\frac{\sum_{i=1}^{\lfloor nt_l\rfloor-\lfloor nt_{l-1}\rfloor-|A_l|}T_i^e}{g_\alpha(n)}\right\}\right] &\to \mathbb{E}\left[\exp\{-\eta_l^* V_\alpha(1)((1-p)(t_l-t_{l-1}))^{\frac{1}{\alpha}}\rho_e\}\right].\end{aligned} \tag{5.14}$$

Since $\mathbb{E}[e^{-\eta V_\alpha(1)}] = \exp\{-\eta^\alpha\}$, it remains to show that

$$\left(\eta(pt)^{\frac{1}{\alpha}}\rho_o\right)^\alpha + \left(\eta((1-p)t)^{\frac{1}{\alpha}}\rho_e\right)^\alpha = t, \quad \text{i.e. that} \quad p\rho_o^\alpha + (1-p)\rho_e^\alpha = 1 \tag{5.15}$$

If $\alpha_o < \alpha_e$ (or $\alpha_o = \alpha_e$ and $L_o/L_e \to \infty$), then $\alpha = \alpha_o$, and $L \sim L_o$. It is then an easy exercise in manipulating slowly varying functions to show that $\ell_o \sim p^{-1/\alpha}\ell$ and therefore $\rho_o = p^{-1/\alpha}$ and $\rho_e = 0$, giving the desired result. Similarly if $\alpha_o > \alpha_e$ (or $\alpha_o = \alpha_e$ and $L_o/L_e \to 0$) we get the desired result. When $\alpha_o = \alpha_e < 1$ and $L_o/L_e \to \beta \in (0,\infty)$ we have that $L \sim L_o + L_e \sim (1+\beta)L_e \sim (1+\beta^{-1})L_o$. It follows that $\ell_e \sim ((1-p)(1+\beta))^{-1/\alpha}\ell$. Similarly $\ell_o \sim (p(1+\beta^{-1}))^{-1/\alpha}\ell$, and therefore $\rho_o = (p(1+\beta^{-1}))^{-1/\alpha}$ and $\rho_e = ((1-p)(1+\beta))^{-1/\alpha}$. The result follows since $(1+\beta)^{-1} + (1+\beta^{-1})^{-1} = 1$. □

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