SOLUTIONS

225 MIDTERM 2, SPRING 2007

TUESDAY, APRIL 3

(1) **(8 points)** Is the following set of three vectors linearly independent?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 5 & 3 \end{pmatrix} \text{switch} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix} \text{ echelon form}$$

$\Rightarrow$ full rank

$\boxed{\text{Yes}}$

(2) **(12 points)**

(a) Write down the matrix A that describes a counter-clockwise rotation of the plane around the origin around the angle θ.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Date: April 3, 2007.

(b) From a geometric point of view, what is A^2?

The counter-clockwise rotation around zero by the angle 2θ $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

(c) Is A an orthogonal matrix?

$\boxed{\text{Yes}}$ $A^T \circ A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$

$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(3) **(18 points)** True or false:

(a) b is in the column space of A if and only if b is a linear combination of the column vectors of A.

true (that is the definition of the column space)

(b) $Ax = b$ has a solution if and only if b is in the column space of A.

true

(c) 0 is always a solution of $Ax = b$.

false: if $b \neq 0$, then $A \cdot 0 \neq b$

(d) 0 is always in the column space of A.

true: $0 = 0 \cdot \vec{v}_1 + \ldots + 0 \cdot \vec{v}_m$, where the $\vec{v}_i$ are the column vectors of A.

(e) x is a solution of $Ax = b$ if and only if x is in the column space of A.

false

(f) Two vectors always span a plane.

false, they could span a line, e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

To save time and reduce mistakes, memorize the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(4) **(8 points)** What is the inverse of the matrix

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}?$$

$$\frac{1}{11}\begin{pmatrix} -4 & 1 \\ 3 & 2 \end{pmatrix}$$

(5) **(16 points)** Compute the projection of $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ onto the line spanned by $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

$$\frac{\langle \binom{-2}{4}, \binom{-1}{5} \rangle}{\langle \binom{-1}{5}, \binom{-1}{5} \rangle}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \frac{2+20}{1+25}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \frac{11}{13}\begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

Compute the reflection of $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ at the line spanned by $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

$$2 \cdot \frac{11}{13}\begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{22}{13} + 2 \\ \frac{110}{13} - 4 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 4 \\ 58 \end{pmatrix}$$

(6) **(8 points)** What is the matrix for the projection onto the line spanned by $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$?

$$\frac{1}{1+25}\begin{pmatrix} 1 & -5 \\ -5 & 25 \end{pmatrix} = \frac{1}{26}\begin{pmatrix} 1 & -5 \\ -5 & 25 \end{pmatrix}$$

(7) **(16 points)** What are the determinants of

(a) a rotation around the origin

1 (preserves angles, distances & orientation)

(b) a reflection of the plane at a line through the origin

−1 (preserves angles & distances, reverses orientation)

(c) the projection of the plane onto a line through the origin

0 (singular)

(d) the linear transformation that stretches the plane by a factor of 7 along the x-axis and by a factor of 2 along the y-axis

$$7 \cdot 2 = 14$$

(8) **(10 points)** Write

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(9) **(4 points)** Give an example of three linearly dependent vectors in $\mathbb{R}^3$ none of which is a multiple of another. How many dimensions do these three vectors span?

example 1 $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

example 2 $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$

2 dimensions (they are in the same plane, but not in the same line).