Representations and characters of categorical groups

A report on joint work with Rober Usher

Projective 2-representations

following Frenkel-Zhu and Ostrik

Classical formulation

V a k-vectorspace

$\theta : G \times G \to k^\times$ 2-cocycle

homomorphism $G \longrightarrow PGL(V)$

map $G \longrightarrow GL(V)$ with $\varrho(g)\varrho(h) = \theta(g,h) \cdot \varrho(gh)$

representation of central extension $\widetilde{G}$ of G by $k^\times$

module over twisted group algebra $k^\theta[G]$

Categorification

V object of strict k-lin. 2-cat.

$\alpha : G \times G \times G \to k^\times$ 3-cocycle

homomorphism $G \longrightarrow \pi_0(GL(V))$

projective 2-representation of G with 3-cocycle α

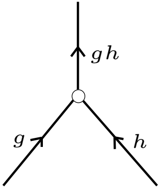
representation $\mathcal{G} \longrightarrow GL(V)$ of 2-group extension $\mathcal{G}$ of G by $pt /\!\!/ k^\times$

module over the categorified twisted group algebra $Vect_k^\alpha[G]$

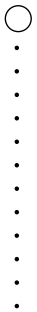
Data of a projective 2-representation with 3-cocycle α

(a) 1-automorphisms $\varrho(g) : V \longrightarrow V$, drawn as

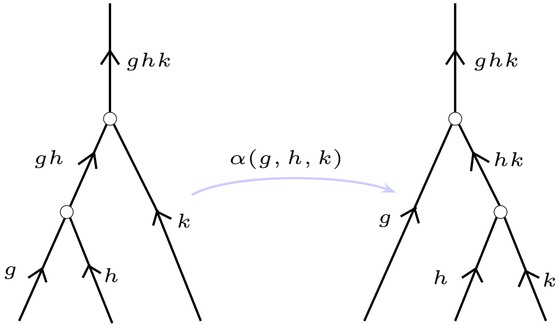
(b) 2-isomorphisms $\varrho(g)\varrho(h) \cong \varrho(gh)$, drawn as

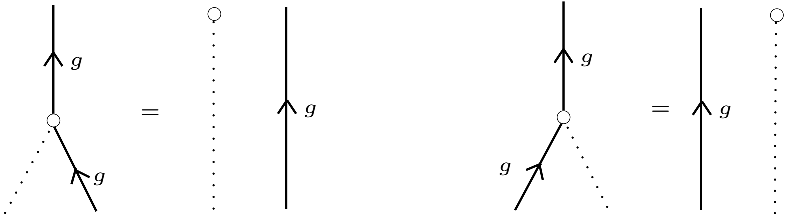


(c) a 2-isomorphism $\psi_1 : \varrho(1) \cong id_V$, drawn as



Conditions on these data





Characters

Classical picture

$$\begin{aligned} \chi : G &\longrightarrow k \\ g &\longmapsto tr(\varrho(g)) \end{aligned}$$

class function $\chi(g) = \chi(sgs^{-1})$

invariant function on inertia groupoid $\Lambda G = G /\!\!/ G^{cong}$

Categorification

$$\begin{aligned} X : G &\longrightarrow Vect_k \\ g &\longmapsto \mathbb{T}r(\varrho(g)) \end{aligned}$$

isomorphisms $X(g) \cong X(sgs^{-1})$

projective representation of ΛG,
2-cocycle the transgression $\tau(\alpha)$

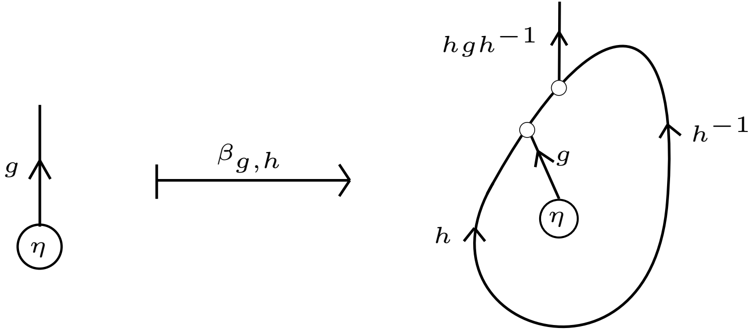
representation of $\Lambda\mathcal{G}$

Categorical Characters

If F is a 1-endomorphism of V then the categorical trace of F is defined as

$$\mathbb{T}r(F) \;:=\; \text{2-Hom}(id_V, F).$$

The isomorphism $\mathbb{T}r(\varrho(g)) \cong Tr(\varrho(sgs^{-1}))$ in terms of string diagrams:



Inertia groupoids

The inertia groupoid of a categorical group $\mathcal{G}$ is

$$\Lambda\mathcal{G} \;=\; \mathbf{Bicat}(pt/\!\!/\mathbb{Z}, pt/\!\!/\mathcal{G})\,/\,\text{2-isos}$$

objects: monoidal functors from $\mathbb{Z}$ to $\mathcal{G}$, viewed as bifunctors from $pt/\!\!/\mathbb{Z}$ to $pt/\!\!/\mathcal{G}$,

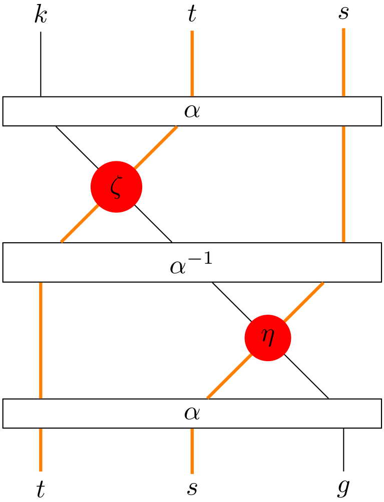
arrows: pseudonatural transformations up to modification

The inclusion of the full sub-groupoid of strict monoidal functors is an equivalence.

$$\Lambda^{strict}\mathcal{G} \longrightarrow \Lambda\mathcal{G}$$

For skeletal $\mathcal{G}$ classified by $\alpha : G \times G \times G \longrightarrow k^{\times}$

objects: G, **arrows:** 2-isomorphisms $\eta : sg \cong hs$, and
composition:



The twisted Drinfeld double

The transgression of α is the k-valued 2-cocycle on ΛG,

$$\tau(\alpha)\left(g \xrightarrow{s} h \xrightarrow{t} k\right) \;=\; \frac{\alpha(t,s,g)\cdot\alpha(k,t,s)}{\alpha(t,h,s)}.$$

Te inertia groupoid $\Lambda\mathcal{G}$ is the central extension of ΛG classified by $\tau(\alpha)$.

The twisted Drinfeld double of G is the twisted groupoid algebra

$$D^{\alpha}(G) \;=\; k^{\tau(\alpha)}[\Lambda G].$$

Theorem the categorical character of a projective 2-representation with cocycle α is a module over the twisted Drinfeld double $D^{\alpha}(G)$.

Categorical tori (construction as strict Lie 2-groups)

Start with a lattice Γ and an even symmetric bilinear form I on Γ. Choose J with $I(m,n) = J(m,n) + J(n,m)$.

objects $\qquad \mathfrak{t} = \Gamma \otimes \mathbb{R}$,

arrows $\qquad x \xrightarrow{z} x + m, \qquad x \in \mathfrak{t}, m \in \Gamma, z \in U(1)$,

composition: $\qquad$ the obvious one,

multiplication: $\qquad$ addition on objects and

$$(x \xrightarrow{z} x + m) \bullet (y \xrightarrow{w} y + n) =$$

$$x + y \xrightarrow{zw\exp(-J(m,y))} x + y + m + n$$

on arrows.

The inertia groupoid of a categorical torus

Theorem: Let $\mathfrak{T}$ be the categorical torus of the previous slide. Then the inertia groupoid of $\mathfrak{T}$ is equivalent to the Lie groupoid with objects T and arrows

$$\mathfrak{t} \times T \times U(1)/\sim$$

with

$$(x+m, s, z) \quad \sim \quad (x, s, z \cdot \exp(I(m, y)), \qquad m \in \Gamma,$$

where $y \in \mathfrak{t}$ is any element with $\exp(y) = s$, source and target of (x, s, z) equal $\exp(x)$, and composition of arrows is

$$[x, s, z] \circ [x, s', z'] \quad = \quad [x, s \cdot s', z \cdot z'].$$

The Looijenga line bundle

Fix τ in the upper half plane, and let $q = e^{2\pi i\tau}$. Over

$$T_{\mathbb{C}} \,/\, q^{\Gamma} \quad \cong \quad \mathfrak{t}_{\mathbb{C}} \,/\, (\tau\Gamma + \Gamma),$$

we have the *Looijenga line bundle for I*

$$\mathcal{L}_{Lo}(I) \quad := \quad (\, T_{\mathbb{C}} \times \mathbb{C}) \,/ \, \sim,$$

with

$$(t, z) \sim (tq^m, z\, e^{-2\pi i I^\sharp(m)}(t) \cdot q^{-\frac{1}{2} I(m,m)}), \qquad m \in \Gamma.$$

Here the weight

$$I^\sharp(m) : \Gamma \longrightarrow \mathbb{Z}$$

is the adjoint of I and $e^{2\pi i I^\sharp(m)}$ is the corresponding character.

Looijenga line bundles versus inertia groupoids

The inertia groupoid of T has objects T and arrows $T \times T$, source and target are projection onto the first factor.

Over $T \times T$, we have the principal $U(1)$-bundle

$$mor(\Lambda \mathfrak{T}) \quad \longrightarrow \quad mor(\Lambda T)$$

Theorem: The (topological) line bundle associated to this principal bundle is isomorphic to the Looijenga line bundle $\mathcal{L}_{Lo}(I)$.

Outlook: symmetries of categorical tori

Let G be a finite group of linear isometries of (Γ, I). Then the symmetries of the torus 2-group classified by (Γ, I) form a 2-group extension of G.

Examples: This yields 2-group extensions of

(1) the Conway group Co_0,

(2) the Mathieu groups M_{12} and M_{24},

(3) Weyl groups,

(4) the symmetric groups (super-duper symmetry).