Product Structure of Bott-Samelson Classes of Type G_2

Katie Bowles

Supervisor: Dr. Nora Ganter
Co-Supervisor: Prof. Arun Ram

Department of Mathematics and Statistics
The University of Melbourne

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Abstract

This work attempts understand the product structure the Bott-Samelson classes of type G_2 in T-equivariant complex cobordism by expressing each product as a linear combination of the Bott-Samelson classes. First we devise an algorithm which determines the linear expansion of the product. By then using another algorithm we show that the coefficients of the expansion are well-defined in the case of K-Theory, and compare to the results in [GR2]. Finally, we attempt to show that the coefficients are well-defined in complex cobordism.

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Contents

Chapter 1

Schubert Calculus and Moment Graphs

Let G be a connected complex reductive group, T be a maximal torus in G and B be a Borel subgroup which contains T and let h_T^* be a T-equivariant multiplicative cohomology theory with Chern classes. Schubert Calculus is interested in the cohomology of the flag variety G/B, (ie. the graded ring $h_T^*(G/B)$). The universal example of such a cohomology theory is T-equivariant complex cobordism which is the case of interest in this thesis. From [HK, Prop. 3.1], a basis for the graded ring $h_T^*(G/B)$ is the Bott-Samelson classes.

The Schubert Problem:
The Schubert Problem is to understand the product structure of the cohomology ring of flag varieties. In other words, the problem is to express each product of two Bott-Samelson classes as a linear combination of Bott-Samelson classes. This problem is interesting because the process produces coefficients which must be well-defined but do not appear to be. This work solves the Schubert Problem for Type G_2 in T-equivariant complex cobordism.

The elements of $h_T^*(G/B)$ can be determined by the map induced by the inclusion of the T-fixed points.

The T-fixed points of G/B are (T acts G be left multiplication):

$$(G/B)^T \cong (K/T)^T \cong N(T)/T \cong W$$

where K is a compact subgroup of G, $N(T)$ the normalizer of T in K and W is the Weyl group, [Sti, Ch. 8]. A presentation of the Weyl group is given in Section 1.2.
Let $\iota : (G/B)^T \to G/B$ be the inclusion of the T-fixed points, then the induced map

$$\iota^* : h_T^*(G/B) \to \bigoplus_{w \in W} h_T(pt)$$

is injective.

There is a well known result for a variety of cohomology theories (including [HHH] for T-equivariant complex cobordism) that the image of ι^* is

$$\operatorname{im}(\iota^*) = \left\{ (g_w)_{w\in W} \,\middle|\, g_w \in h_T(pt) \text{ and } (g_w - g_{ws_\alpha}) \in y_{-\alpha}S \text{ for all } \alpha \in R^+, w \in W \right\}$$

where y_α is a 1-dimensional representation of T with weight α,
R^+ is the positive roots of the character lattice, $\mathfrak{h}_{\mathbb{Z}}^* = \operatorname{Hom}(T, \mathbb{C}^\times)$,
$S = \mathbb{L}[[y_\lambda | \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]]$ and $\mathbb{L}$ generated by the symbols a_{ij} for $i.j \in \mathbb{Z}_{>0}$.

A useful tool for the calculations that follow is the *Moment Graph* which contains all the necessary information to determine which elements of $\bigoplus_{w\in W} h_T(pt)$ are in the image ι^*. The moment graph of the flag variety G/B is constructed as follows: the vertices are the elements of the Weyl group, W and there is a directed edge from a reduced word w to another reduced word u if there a root, $\alpha \in R^+$ such that $u = ws_\alpha$, this edge is labelled $y_{-\alpha}$. Reduced words are defined in Section 1.2.

An element $(g_w)_{w\in W} \in \bigoplus_{w\in W} h_T(pt)$ can be written in moment graph form by replacing the vertex w with g_w and for simplicity the edges are removed. An element $(g_w)_{w\in W} \in \bigoplus_{w\in W} h_T(pt)$ is in the image of ι^* if when it is written in moment graph form whenever two vertices are adjacent then the difference between the vertices is divisible by the label of the edge connecting them. For a good introduction to moment graphs (for equivariant ordinary cohomology) see [Tym].

1.1 Example: Type A_2

Type A_2 occurs when $G = GL_3$ and has two simple roots α_1, α_2 and the root system as shown in [Fig.1.1]. The Weyl group of A_2 is generated by s_1 and s_2 with the relations $s_1^2 = s_2^2 = 1$ and $s_1s_2s_1 = s_2s_1s_1$.

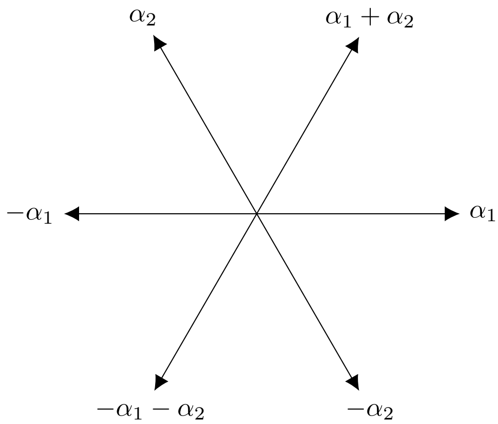


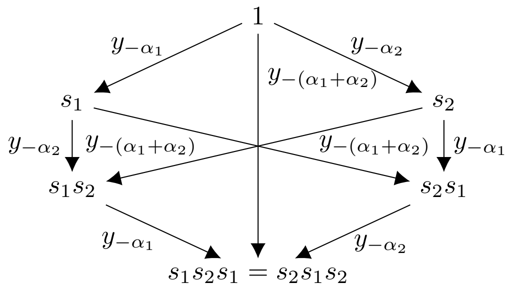
Figure 1.1: The root system of type A_2

Root	Reflection
α_1	s_1
α_2	s_2
$\alpha_1 + \alpha_2$	$s_1s_2s_1$

and

$$s_1\alpha_1 = -\alpha_1 \qquad s_1\alpha_2 = \alpha_1 + \alpha_2$$
$$s_2\alpha_1 = \alpha_1 + \alpha_2 \qquad s_2\alpha_2 = -\alpha_2$$

Then the moment graph can be easily drawn:



The simplest possible element one can conceive would have only one non-zero entry, in order for the divisibility requirement to be met the non-zero entry must be $y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}$. So if this entry is in the position of the vertex 1, (the top vertex), then element in moment graph form is:

$$y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}$$

$$0 \qquad\qquad 0$$

$$0 \qquad\qquad 0$$

$$0$$

This is a Bott-Samelson class and is called $[Z_{pt}]$. Some other examples of elements in the image of ι^* are:

$$y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}$$

$$y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)} \qquad\qquad 0$$

$$0 \qquad\qquad 0$$

$$0$$

$[Z_1]$

$$y_{-\alpha_1} y_{-(\alpha_1+\alpha_2)}$$

$$0 \qquad\qquad y_{-\alpha_1} y_{-(\alpha_1+\alpha_2)}$$

$$0 \qquad\qquad 0$$

$$0$$

$[Z_2]$

$$y_{-(\alpha_1+\alpha_2)}$$

$$y_{-(\alpha_1+\alpha_2)} \qquad\qquad y_{-\alpha_1}$$

$$y_{-\alpha_1} \qquad\qquad 0$$

$$0$$

$[Z_{12}]$

$$y_{-(\alpha_1+\alpha_2)}$$

$$y_{-\alpha_2} \qquad\qquad y_{-(\alpha_1+\alpha_2)}$$

$$0 \qquad\qquad y_{-\alpha_2}$$

$$0$$

$[Z_{21}]$

$$\begin{array}{ccc} & \Delta & \\ \Delta & & 1 \\ 1 & & 1 \\ & 1 & \end{array} \qquad \text{where } \Delta = y_{-(\alpha_1+\alpha_2)}/y_{-\alpha_1} + y_{-\alpha_2}/y_{\alpha_1}$$

$[Z_{121}]$

These elements are in fact the Bott-Samelson classes of Type A_2 and form a basis of the image of ι^*. However this is not obvious that they form a basis or that the last three are even in the image, we do not prove either.

We want to write products of Bott-Samelson classes as linear combinations of the Bott-Samelson classes. For example consider the product $[Z_{12}]\,[Z_{21}]$ which is (since multiplication is pointwise):

$$\begin{array}{ccc} & y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} & \\ y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)} & & y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} \\ 0 & & 0 \\ & 0 & \end{array}$$

We note that the entry at vertex s_1 are the same for our product and $[Z_1]$. Then we can annihilate this entry but subtracting $1 \cdot [Z_1]$ which results in:

$$\left(\begin{array}{ccc} & y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} & \\ y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)} & & y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} \\ 0 & & 0 \\ & 0 & \end{array}\right) - \left(\begin{array}{ccc} & y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)} & \\ y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)} & & 0 \\ 0 & & 0 \\ & 0 & \end{array}\right)$$

$$= \left(\begin{array}{ccc} & y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)} & \\ 0 & & y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} \\ 0 & & 0 \\ & 0 & \end{array}\right)$$

Similarly, the vertices in position s_2 in $[Z_{12}]\,[Z_{21}] - [Z_1]$ and $[Z_2]$ are the same. So subtracting $[Z_2]$ from $[Z_{12}]\,[Z_{21}] - [Z_1]$ annihilates the entry s_2, leaving us with:

$$[Z_{12}]\,[Z_{21}] - [Z_1] - [Z_2] = \left(\begin{array}{ccc} & y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} & \\ 0 & & 0 \\ 0 & & 0 \\ & 0 & \end{array}\right)$$

Finally, $[Z_{12}][Z_{21}] - [Z_1] - [Z_2]$ is scalar multiple of $[Z_{pt}]$:

$$[Z_{12}][Z_{21}] - [Z_1] - [Z_2] = \frac{y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_1}y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)}}[Z_{pt}]$$

Therefore $[Z_{12}][Z_{21}]$ can be written as a linear combination

$$[Z_{12}][Z_{21}] = [Z_1] + [Z_2] + \left(\frac{y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_1}y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)}}\right)[Z_{pt}]$$

This coefficient of $[Z_{pt}]$ does not seem to be well-defined even in this baby example. However we know it must be well-defined, to understand why we use the formal group law. Every cohomolgy theory comes eqipped with a formal group law and for T-equivariant complex cobordism it is:

$$y_{\alpha+\beta} = y_\alpha + y_\beta + y_\alpha y_\beta p[y_\alpha, y_\beta]$$

where $p[y_\alpha, y_\beta] = \sum_{i,j\in\mathbb{Z}_{>0}} a_{ij}y_\alpha^{i-1}y_\beta^{j-1}$

The coefficient of $[Z_{pt}]$ is:

$$\frac{y_{-(\alpha_1+\alpha_2)}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_1}y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_1}y_{-\alpha_2}y_{-(\alpha_1+\alpha_2)}} = \frac{y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_1} - y_{-\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}}$$

Then the formal group law tells us $y_{-(\alpha_1+\alpha_2)} = y_{-\alpha_1} + y_{-\alpha_2} - y_{-\alpha_1}y_{-\alpha_2}p[y_{-\alpha_1}, y_{-\alpha_2}]$, which gives us:

$$\frac{y_{-(\alpha_1+\alpha_2)} - y_{-\alpha_1} - y_{-\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}} = \frac{y_{-\alpha_1} + y_{-\alpha_2} - y_{-\alpha_1}y_{-\alpha_2}p[y_{-\alpha_1}, y_{-\alpha_2}] - y_{-\alpha_1} - y_{-\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}} = -p[y_{-\alpha_1}, y_{-\alpha_2}]$$

Then we can re-express the linear expression of the product $[Z_{12}][Z_{21}]$ without fractions:

$$[Z_{12}][Z_{21}] = [Z_1] + [Z_2] - p[y_{-\alpha_1}, y_{-\alpha_2}][Z_{pt}]$$

If we wanted to convert this result to the setting of K-Theory, we would set $p[y_{-\alpha_1}, y_{-\alpha_2}] = 1$, see Section 1.3 for an explanation. The rank two results for K-Theory have been computed previously in [GR2], therefore we can compare the result above to their result. For this example our result is the same as theirs.

1.2 The Bott-Samelson Classes

Here we construct the Bott-Samelson classes explicitly.
First we provide a presentation of the Weyl group, W of G, which has the presentation:

$$\langle s_1, \ldots, s_n \quad | \quad s_i^2 = 1, \quad (s_i s_j)^{m_{i,j}} = 1\rangle$$

where the s_i are called the simple reflections and m_{ij} is 2,3,4 or 6 and is determined by whether the roots corresponding to s_i and s_j make an angle of 90° , 120°, 135° and 150°. A reduced word for an element $w \in W$ is $s_{i_1}\ldots s_{i_n}$ such that n is minimum and $w = s_{i_1}\ldots s_{i_n}$. Length of the reduced word is n and is denoted by $l(w)$. A reduced word may not be unique.

For general type, the Bott-Samelson class $[Z_{pt}]$ has y_{R^-} where $y_{R^-} = \prod_{\alpha\in R^+} y_{-\alpha}$ for vertex 1 and it is the only non-zero entry.

Before we can define the remaining classes we need to make some definitions:
Let b_w be the standard basis for $\bigoplus_{w\in W} h_T(pt)$, (ie. b_w has a 1 in position w and zeroes elsewhere), in general b_w is not in the image of ι^* and therefore is not a basis for the image.
Let $x_\lambda = (y_{w^{-1}\lambda})_{w\in W}$ where $s_i\alpha$ is determined by the action of the reflection s_i on the root α as defined in the root system.

For s_i are simple reflection let t_{s_i} be an operator on $\bigoplus_{w \in W} h_T(pt)$ which takes the element in position w to $s_i w$, (ie. $t_{s_i} b_w = b_{s_i w}$).

Then the Bott-Samelson classes are defined as:

$$[Z_{i_1 \dots i_n}] = A_{i_1} \dots A_{i_n} [Z_{pt}]$$

where $s_{i_1} \dots s_{i_n}$ is a reduced word, the multplication of the moment graphs is pointwise and $A_i = (1 + t_{s_i}) \frac{1}{x - \alpha_i}$ is the BGG-Demazure operator from [BE1].

If, for every element $w_k \in W \setminus \{1\}$ we choose one reduced word, $\vec{w_k}$, then the Bott-Samelson classes

$$\left\{ [Z_{i_1, \dots, i_n}] \,\middle|\, s_{i_1} \dots s_{i_n} \text{is one of the chosen reduced words } \vec{w_k} \right\}$$

along with $[Z_{pt}]$ form a basis regardless of the choice of fixed reduced words, [HK, Prop. 3.1].

We note that the 'lowest' non-zero entry of the Bott-Samelson class $[Z_{i_1 \dots i_n}] \neq [Z_{pt}]$ is at vertex $s_{i_1} \dots s_{i_n}$. This can be easily proved by induction, but we do not prove it here. Then for every element $w \in W$ there is a Bott-Samelson class with a non-zero entry in position w and a zero in all posititions w' such that $l(w') \geq l(w)$.

1.3 Formal Group Law

Every oriented cohomology theory comes equipped with a formal group law which provides a relationship between $y_{\alpha+\beta}$ and y_α and y_β. Whilst our case of interest is complex cobordism, by simplifying the formal group law we can transfer the results to other T-equivariant cohomology theories.

In the case of complex cobordism the formal group law is:

$$y_{\alpha+\beta} = y_\alpha + y_\beta + \sum_{i,j \in \mathbb{Z}_{>0}} a_{ij} y_\alpha^i y_\beta^j = y_\alpha + y_\beta + y_\alpha y_\beta \sum_{i,j \in \mathbb{Z}_{>0}} a_{ij} y_\alpha^{i-1} y_\beta^{j-1}$$

We denote $\sum_{i,j \in \mathbb{Z}_{>0}} a_{ij} y_\alpha^{i-1} y_\beta^{j-1}$ by $p[y_\alpha, y_\beta]$

To transfer to the case of K-Theory we note that y_α is $1 - e^\alpha$ so the formal group law for complex cobordism in this setting becomes:

$$y_{\alpha+\beta} = y_\alpha + y_\beta - p[y_\alpha, y_\beta] y_\alpha y_\beta = (1 - e^\alpha) + \left(1 - e^\beta\right) - p\left[(1 - e^\alpha) + \left(1 - e^\beta\right)\right] (1 - e^\alpha) \left(1 - e^\beta\right)$$

The formal group law for K-Theory is:

$$e^{\lambda+\mu} = e^\lambda e^\mu$$

Using this on relation on $\left(1 - e^{(\alpha+\beta)}\right)$ we get:

$$\left(1 - e^{(\alpha+\beta)}\right) = (1 - e^\alpha) + \left(1 - e^\beta\right) - (1 - e^\alpha) \left(1 - e^\beta\right)$$

Therefore we can turn results for complex cobordism into results for K-Theory by setting $p[y_{-\alpha}, y_{-\beta}] = 1$ for all α and β.

1.4 Type G_2

In this thesis the case interest is type G_2, which has the root system as shown in [Fig. 1.2]. There are two simple reflections, s_1 and s_2, and $m_{12} = 6$.

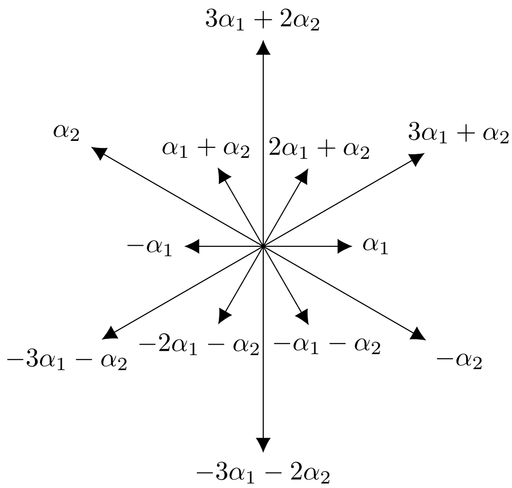


Figure 1.2: The root system of type G_2

Root	Reflection
α_1	s_1
α_2	s_2
$\alpha_1 + \alpha_2$	$s_2 s_1 s_2$
$3\alpha_1 + \alpha_2$	$s_1 s_2 s_1$
$2\alpha_1 + \alpha_2$	$s_1 s_2 s_1 s_2 s_1$
$3\alpha_1 + 2\alpha_2$	$s_2 s_1 s_2 s_1 s_2$

and

$$s_1\alpha_1 = -\alpha_1 \qquad s_1\alpha_2 = 3\alpha_1 + \alpha_2$$
$$s_2\alpha_1 = \alpha_1 + \alpha_2 \qquad s_2\alpha_2 = -\alpha_2$$

As previously, this is all the information required to construct the moment graph and it is shown in [Fig. 1.3].

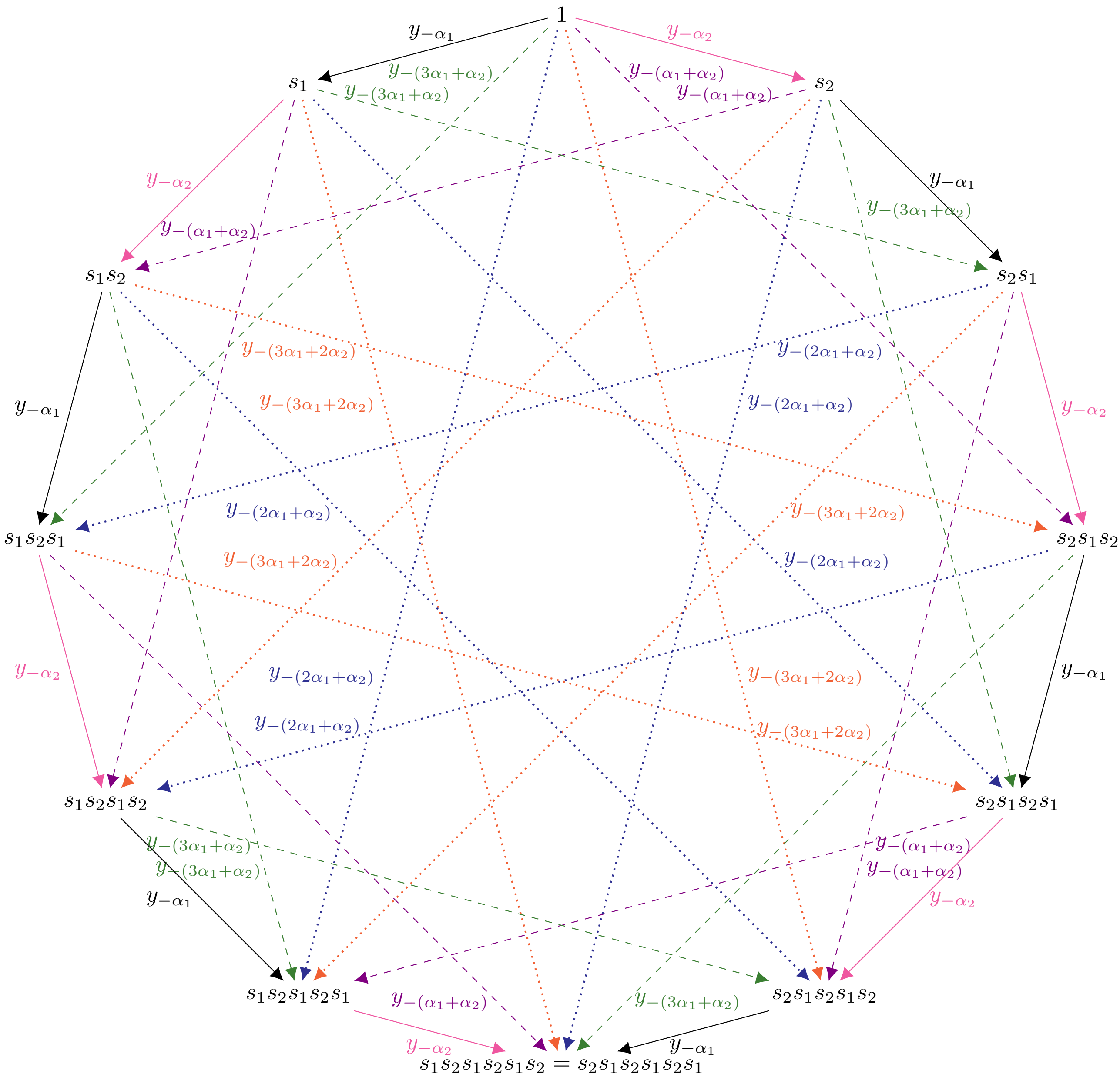


Figure 1.3: The moment graph for the root system of type G_2
In grayscale the labels are as follows:
darker solid edge: $y_{-\alpha_1}$
lighter solid edge: $y_{-\alpha_2}$
darker dashed edge: $y_{-(\alpha_1+\alpha_2)}$
lighter dashed edge: $y_{-(3\alpha_1+\alpha_2)}$
darker dotted edge: $y_{-(2\alpha_1+\alpha_2)}$
lighter dotted edge: $y_{-(3\alpha_1+2\alpha_2)}$

1.4.1 The Bott-Samelson Classes of Type G_2

The Bott-Samelson classes of type G_2 as defined previously in Section 1.2 are given below:
where $y_{R^-} = y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)} y_{-(2\alpha_1+\alpha_2)} y_{-(3\alpha_1+\alpha_2)} y_{-(3\alpha_1+2\alpha_2)}$, the remaining notation is defined at the end of the list.

y_{R^-}

0 0

0 0

0 0

0 0

0 0

0

$[Z_{pt}]$

Δ_1

Δ_1 0

0 0

0 0

0 0

0 0

0

$[Z_1]$

Δ_2

0 Δ_2

0 0

0 0

0 0

0 0

0

$[Z_2]$

Δ_{12}

Δ_{12} Γ_{12}

Γ_{12} 0

0 0

0 0

0 0

0

$[Z_{12}]$

Δ_{21}

Γ_{21} Δ_{21}

0 Γ_{21}

0 0

0 0

0 0

0

$[Z_{21}]$

Δ_{121}

Δ_{121} Γ_{121}

Γ_{121} K_{121}

K_{121} 0

0 0

0 0

0

$[Z_{121}]$

Δ_{212}

Γ_{212} Δ_{212}

K_{212} Γ_{212}

0 K_{212}

0 0

0 0

0

$[Z_{212}]$

Δ_{1212}		Δ_{2121}	
Δ_{1212}	Γ_{1212}	Γ_{2121}	Δ_{2121}
Γ_{1212}	K_{1212}	K_{2121}	Γ_{2121}
K_{1212}	L_{1212}	L_{2121}	K_{2121}
L_{1212}	0	0	L_{2121}
0	0	0	0
0		0	
$[Z_{1212}]$		$[Z_{2121}]$	

Δ_{12121}		Δ_{21212}	
Δ_{12121}	Γ_{12121}	Γ_{21212}	Δ_{21212}
Γ_{12121}	K_{12121}	K_{21212}	Γ_{21212}
K_{12121}	L_{12121}	L_{21212}	K_{21212}
L_{12121}	M_{12121}	M_{21212}	L_{21212}
M_{12121}	0	0	M_{21212}
0		0	
$[Z_{12121}]$		$[Z_{21212}]$	

Δ_{121212}	
Δ_{121212}	Γ_{121212}
Γ_{121212}	K_{121212}
K_{121212}	L_{121212}
L_{121212}	1
1	1
1	
$[Z_{121212}]$	

The remaining coefficients are listed below or obtained by interchanging 1 and 2 in those listed.

$$\Delta_1 = \frac{y_{R^-}}{y_{-\alpha_1}}$$

$$\Delta_{12} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2}}$$

$$\Gamma_{12} = \frac{\Delta_2}{y_{-s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_2} y_{-s_2\alpha_1}}$$

$$\Delta_{121} = \frac{\Delta_{21}}{y_{-\alpha_1}} + \frac{\Gamma_{21}}{y_{-s_1\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1}} \left(\frac{1}{y_{-\alpha_1} y_{-\alpha_2}} + \frac{1}{y_{-s_1\alpha_1} y_{-s_1\alpha_2}} \right)$$

$$\Gamma_{121} = \frac{\Delta_{21}}{y_{-s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-s_2\alpha_1}}$$

$$K_{121} = \frac{\Gamma_{21}}{y_{-s_1s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-s_1\alpha_2} y_{-s_1s_2\alpha_1}}$$

$$\Delta_{1212} = \frac{\Delta_{212}}{y_{-\alpha_1}} + \frac{\Gamma_{212}}{y_{-s_1\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2}} \left(\frac{1}{y_{-\alpha_1} y_{-\alpha_2}} + \frac{1}{y_{-s_1\alpha_1} y_{-s_1\alpha_2}} + \frac{1}{y_{-s_2\alpha_2} y_{-s_2\alpha_1}} \right)$$

$$\Gamma_{1212} = \frac{\Delta_{212}}{y_{-s_2\alpha_1}} + \frac{K_{212}}{y_{-s_2s_1\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_2} y_{-s_2\alpha_1}} \left(\frac{1}{y_{-\alpha_1} y_{-\alpha_2}} + \frac{1}{y_{-s_2\alpha_2} y_{-s_2\alpha_1}} + \frac{1}{y_{-s_2s_1\alpha_1} y_{-s_2s_1\alpha_2}} \right)$$

$$K_{1212} = \frac{\Gamma_{212}}{y_{-s_1s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-s_1\alpha_2} y_{-s_1s_2\alpha_1}}$$

$$L_{1212} = \frac{K_{212}}{y_{-s_2s_1s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_2} y_{-s_2\alpha_1} y_{-s_2s_1\alpha_2} y_{-s_2s_1s_2\alpha_1}}$$

$$\begin{aligned}\Delta_{12121} &= \frac{\Delta_{2121}}{y_{-\alpha_1}} + \frac{\Gamma_{2121}}{y_{-s_1\alpha_1}} \\ &= \frac{y_{R^-}}{y_{-\alpha_1}} \left(\frac{2}{y_{-\alpha_1} y_{-\alpha_2} y_{-s_1\alpha_2} y_{-s_1\alpha_1}} + \frac{1}{(y_{-s_1\alpha_1} y_{-s_1\alpha_2})^2} + \frac{1}{y_{-s_1s_2\alpha_2} y_{-s_1s_2\alpha_1} y_{-s_1\alpha_2} y_{-s_1\alpha_1}} \right. \\ &\left. + \frac{1}{(y_{-\alpha_1} y_{-\alpha_2})^2} + \frac{1}{y_{-\alpha_1} y_{-\alpha_2} y_{-s_2\alpha_2} y_{-s_2\alpha_1}} \right)\end{aligned}$$

$$\Gamma_{12121} = \frac{\Delta_{2121}}{y_{-s_2\alpha_1}} + \frac{K_{2121}}{y_{-s_2s_1\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-s_2\alpha_1}} \left(\frac{1}{y_{-\alpha_1} y_{-\alpha_2}} + \frac{1}{y_{-s_1\alpha_1} y_{-s_1\alpha_2}} + \frac{1}{y_{-s_2\alpha_2} y_{-s_2\alpha_1}} + \frac{1}{y_{-s_2s_1\alpha_1} y_{-s_2s_1\alpha_2}} \right)$$

$$\begin{aligned}K_{12121} &= \frac{\Gamma_{2121}}{y_{-s_1s_2\alpha_1}} + \frac{L_{2121}}{y_{-s_1s_2s_1\alpha_1}} \\ &= \frac{y_{R^-}}{y_{-\alpha_1} y_{-s_1\alpha_2} y_{-s_1s_2\alpha_1}} \left(\frac{1}{y_{-\alpha_1} y_{-\alpha_2}} + \frac{1}{y_{-s_1\alpha_1} y_{-s_1\alpha_2}} + \frac{1}{y_{-s_1s_2\alpha_2} y_{-s_1s_2\alpha_1}} + \frac{1}{y_{-s_1s_2s_1\alpha_1} y_{-s_1s_2s_1\alpha_2}} \right)\end{aligned}$$

$$L_{12121} = \frac{K_{2121}}{y_{-s_2s_1s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-s_2\alpha_1} y_{-s_2s_1\alpha_2} y_{-s_2s_1s_2\alpha_1}}$$

$$M_{12121} = \frac{L_{2121}}{y_{-s_1s_2s_1s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_1} y_{-s_1\alpha_2} y_{-s_1s_2\alpha_1} y_{-s_1s_2s_1\alpha_2} y_{-s_1s_2s_1s_2\alpha_1}}$$

$$\Delta_{121212} = \frac{\Delta_{21212}}{y_{-\alpha_1}} + \frac{\Gamma_{21212}}{y_{-s_1\alpha_1}}$$

$$\Gamma_{121212} = \frac{\Delta_{21212}}{y_{-s_2\alpha_1}} + \frac{K_{21212}}{y_{-s_2s_1\alpha_1}}$$

$$K_{121212} = \frac{\Gamma_{21212}}{y_{-s_1s_2\alpha_1}} + \frac{L_{21212}}{y_{-s_1s_2s_1\alpha_1}}$$

$$L_{121212} = \frac{K_{21212}}{y_{-s_2s_1s_2\alpha_1}} + \frac{M_{21212}}{y_{-s_2s_1s_2s_1\alpha_1}}$$

We make two observations about the Bott-Samelson classes:

First every Bott-Samelson class has a diagonal symmetry, all the elements in a moment graph can be paired up such that both elements in a pair have the same value and they are from consecutive rows. We call this phenomenon 'diagonal symmetry'.

Second for every element w, of the Weyl group there is a Bott-Samelson class with a non-zero entry in position w and all other entries in the same or lower rows are zero.

Chapter 2

Expansion of products

As explained in Chapter 1 we are interested in the product structure of the Bott-Samelson classes. In this chapter we express each product of the Bott-Samelson classes as a linear combination of the Bott-Samelson classes. This is always possible since the Bott-Samelson classes are a basis. The techniques and algorithm that follow are applicable to products of all types however we focus on type G_2 unless stated otherwise.

2.1 Example computation of $[Z_2][Z_{12}]$

As an illustrative example of how these expansion were computed the full details for $[Z_2][Z_{12}]$ are shown here. The multiplication of the Bott-Samelson classes is simply pointwise multiplication, so the product $[Z_2][Z_{12}]$ is

$$
\begin{array}{ccc}
 & \Delta_2 \cdot \Delta_{12} & \\
0 \cdot \Delta_{12} & & \Delta_2 \cdot \Gamma_{12} \\
0 \cdot \Gamma_{12} & & 0 \\
0 & & 0 \\
0 & & 0 \\
0 & & 0 \\
 & 0 &
\end{array}
\quad = \quad
\begin{array}{ccc}
 & \Delta_2 \Delta_{12} & \\
0 & & \Delta_2 \Gamma_{12} \\
0 & & 0 \\
0 & & 0 \\
0 & & 0 \\
0 & & 0 \\
 & 0 &
\end{array}
$$

We want to expanded the product as a linear combination of Bott-Samelson classes. To do this a row elimination type algorithm is used to repeatedly eliminate the lowest element. For the product $[Z_2][Z_{12}]$, the lowest non-zero element is $\Delta_2\Gamma_{12}$. This element is eliminated by the Bott-Samelson class $[Z_2]$ multiplied by Γ_{12}.

$$
[Z_2][Z_{12}] - \Gamma_{12}[Z_2] \quad = \quad
\begin{array}{ccc}
 & \Delta_{12}\Delta_2 - \Gamma_{12}\Delta_2 & \\
0 & & 0 \\
0 & & 0 \\
0 & & 0 \\
0 & & 0 \\
0 & & 0 \\
 & 0 &
\end{array}
$$

This leaves only the topmost element which is annihilated by $[Z_{pt}]$ with the coefficient $(\Gamma_{12}\Delta_2 - \Delta_{12}\Delta_2)/y_{R^-}$. Therefore the expansion of the product as a linear combination is

$$[Z_2]\,[Z_{12}] = \Gamma_{12}\,[Z_2] + \frac{\Delta_{12}\Delta_2 - \Gamma_{12}\Delta_2}{y_{R^-}}\,[Z_{pt}]$$

2.2 General Products and the Algorithm

In the preceding example we were able to expand the product because we could cancel the entries with scalar multiples of the Bott-Samelson classes. To be able to do this for any product, we need that for every element w of the Weyl group there is a Bott-Samelson class which has a non-zero entry in position w and all other entries on the same or lower rows are zero. We observed that this is true of the Bott-Samelson classes of type G_2 in Subsection 1.4.1 and we noted it is always true in Section 1.2.

2.2.1 The Algorithm

The algorithm is detailed below.

Let P be a general moment graph and $E = 0$, and let $D = P$.

Let d_L denote the left most non-zero element in the lowest row with non-zero elements in D.

Let $[Z]$ be the Bott-Samelson class with its lowest non-zero entry in the position of d_L and let this entry be Z_L.

Add $(d_L/Z_L)\,[Z]$ to E and create a new D' which is $D - (d_l/Z_l)\,[Z]$

If every entry in D' is zero then E is the linear expansion of P, otherwise set $D = D'$ and repeat the process above starting with assigning new d_L.

This algorithm works because each run through removes the entry d_L without changing any of the entries which are in the same or lower rows. Therefore the next run through removes an element to the right of d_L or in a higher row. Hence by repeated application we remove all entries and therefore find a linear combination as desired.

2.2.2 Products Involving $[Z_{121212}]$

To understand the product structure of the Bott-Samelson classes we should consider products involving all of the classes. However since the lowest entry $[Z_{121212}]$ is a one it can be written as a sum of the multiplicative identity, I (which has a 1 at every vertex) and a linear combination of the other Bott-Samelson classes. By applying the algorithm above to $[Z_{121212}] - I$ we obtain:

$$\begin{aligned}
[Z_{121212}] - I = {} & \left(\frac{L_{121212}-1}{L_{1212}}\right)[Z_{1212}] + \frac{1}{K_{121}}\left\{K_{121212} - 1 + K_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right)\right\}[Z_{121}] \\
& + \frac{1}{\Gamma_{12}}\left\{\Gamma_{121212} - 1 + \Gamma_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right) - \frac{\Gamma_{121}}{K_{121}}\left(K_{121212} - 1 + K_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right)\right)\right\}[Z_{12}] \\
& \frac{1}{\Delta_1}\left\{\Delta_{121212} - 1 - \Delta_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right) - \frac{\Delta_{121}}{K_{121}}\left(K_{121212} - 1 + K_{1212}\frac{L_{121212}-1}{L_{1212}}\right)\right. \\
& \left. - \frac{\Delta_{12}}{\Gamma_{12}}\left(\Gamma_{121212} - 1 + \Gamma_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right) - \frac{\Gamma_{121}}{K_{121}}\left[K_{121212} - 1 + K_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right)\right]\right)\right\}[Z_1]
\end{aligned}$$

which is well defined since both I and $[Z_{121212}]$ are in the image and therefore $[Z_{121212}] - I$ is also in the image and the coefficients above are well-defined. Hence any product $[Z_w]\,[Z_{121212}]$ can be written as a sum of $[Z_w]$

and other products:

$$
\begin{aligned}
[Z_w]\,[Z_{121212}] &= [Z_w]\,(I + [Z_{121212}] - I)\\
&= [Z_w] + [Z_w]\,([Z_{121212}] - I)\\
&= [Z_w] + \left(\frac{L_{121212}-1}{L_{1212}}\right)[Z_w]\,[Z_{1212}] + \frac{1}{K_{121}}\left\{K_{121212} - 1 + K_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right)\right\}[Z_w]\,[Z_{121}]\\
&+ \frac{1}{\Gamma_{12}}\left\{\Gamma_{121212} - 1 + \Gamma_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right) - \frac{\Gamma_{121}}{K_{121}}\left(K_{121212} - 1 + K_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right)\right)\right\}[Z_w]\,[Z_{12}]\\
&\frac{1}{\Delta_1}\left\{\Delta_{121212} - 1 - \Delta_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right) - \frac{\Delta_{121}}{K_{121}}\left(K_{121212} - 1 + K_{1212}\frac{L_{121212}-1}{L_{1212}}\right)\right.\\
&\left. - \frac{\Delta_{12}}{\Gamma_{12}}\left(\Gamma_{121212} - 1 + \Gamma_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right) - \frac{\Gamma_{121}}{K_{121}}\left[K_{121212} - 1 + K_{1212}\left(\frac{L_{121212}-1}{L_{1212}}\right)\right]\right)\right\}[Z_w]\,[Z_1]
\end{aligned}
$$

Products involving $[Z_{121212}]$ are not considered in the rest of this of this work apart from a brief discussion in Subsection 3.4.1.

2.2.3 Formula for General Expansions

Whilst this algorithm could be used to calculate every product individually, doing so would be repetitive and time consuming. Instead consider a general product, M of type G_2 defined below:

$$
M = \begin{array}{ccc}
 & a & \\
b & & c\\
d & & e\\
f & & g\\
h & & i\\
j & & k\\
 & 0 &
\end{array}
$$

Therefore we only need to apply the algorithm above to M which yields the following formula for the linear expansion of M. This formula can then be used to calculate the linear expansion of any product of the Bott-Samelson classes By the choice of the values of a to i.

$$
\begin{aligned}
&\frac{1}{y_{R^-}}\left\{\boldsymbol{a}-\boldsymbol{b}-\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}-\boldsymbol{f}-\boldsymbol{g}+\boldsymbol{h}+\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\right\}[Z_{pt}] \\
&+\frac{1}{\Delta_1}\left\{\boldsymbol{b}-\boldsymbol{j}\frac{\Delta_{12121}}{M_{12121}}-\frac{\Delta_{1212}}{L_{1212}}\left(\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right)-\frac{\Delta_{121}}{K_{121}}\left[\boldsymbol{f}-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{j}\frac{K_{12121}}{M_{12121}}-\frac{K_{1212}}{L_{1212}}\left(\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right)\right]\right. \\
&\qquad -\boldsymbol{e}+\boldsymbol{f}-\boldsymbol{i}+\boldsymbol{j}-\frac{\Delta_{12}}{\Gamma_{12}}\left(\boldsymbol{d}-\frac{\Gamma_{121}}{K_{121}}\left[\boldsymbol{f}-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{j}\frac{K_{12121}}{M_{12121}}-\frac{K_{1212}}{L_{1212}}\left(\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right)\right]\right. \\
&\qquad \left.\left.-\boldsymbol{g}+\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{\Gamma_{12121}}{M_{12121}}-\frac{\Gamma_{1212}}{L_{1212}}\left[\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right]\right)\right\}[Z_1] \\
&+\frac{1}{\Delta_2}\left\{\boldsymbol{c}-\boldsymbol{k}\frac{\Delta_{21212}}{M_{21212}}-\frac{\Delta_{2121}}{L_{2121}}\left(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right)-\frac{\Delta_{212}}{K_{212}}\left[\boldsymbol{g}-\boldsymbol{h}+\boldsymbol{k}-\boldsymbol{k}\frac{K_{21212}}{M_{21212}}-\frac{K_{2121}}{L_{2121}}\left(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right)\right]\right. \\
&\qquad -\boldsymbol{d}+\boldsymbol{g}-\boldsymbol{h}+\boldsymbol{k}-\frac{\Delta_{21}}{\Gamma_{21}}\left(\boldsymbol{e}-\frac{\Gamma_{212}}{K_{212}}\left[\boldsymbol{g}-\boldsymbol{h}+\boldsymbol{k}-\boldsymbol{k}\frac{K_{21212}}{M_{21212}}-\frac{K_{2121}}{L_{2121}}\left(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right)\right]\right. \\
&\qquad \left.\left.-\boldsymbol{f}+\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{\Gamma_{21212}}{M_{21212}}-\frac{\Gamma_{2121}}{L_{2121}}\left[\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right]\right)\right\}[Z_2] \\
&+\frac{1}{\Gamma_{12}}\left\{\boldsymbol{d}-\boldsymbol{g}+\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{\Gamma_{12121}}{M_{12121}}-\frac{\Gamma_{1212}}{L_{1212}}\left(\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right)\right. \\
&\qquad \left.-\frac{\Gamma_{121}}{K_{121}}\left[\boldsymbol{f}-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{j}\frac{K_{12121}}{M_{12121}}-\frac{K_{1212}}{L_{1212}}\left(\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right)\right]\right\}[Z_{12}] \\
&+\frac{1}{\Gamma_{21}}\left\{\boldsymbol{e}-\boldsymbol{f}+\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{\Gamma_{21212}}{M_{21212}}-\frac{\Gamma_{2121}}{L_{2121}}\left(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right)\right. \\
&\qquad \left.-\frac{\Gamma_{212}}{K_{212}}\left[\boldsymbol{g}-\boldsymbol{h}+\boldsymbol{k}-\boldsymbol{k}\frac{K_{21212}}{M_{21212}}-\frac{K_{2121}}{L_{2121}}\left(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right)\right]\right\}[Z_{21}] \\
&+\frac{1}{K_{121}}\left\{\boldsymbol{f}-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{j}\frac{K_{12121}}{M_{12121}}-\frac{K_{1212}}{L_{1212}}\left(\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right)\right\}[Z_{121}] \\
&+\frac{1}{K_{212}}\left\{\boldsymbol{g}-\boldsymbol{h}+\boldsymbol{k}-\boldsymbol{k}\frac{K_{21212}}{M_{21212}}-\frac{K_{2121}}{L_{2121}}\left(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right)\right\}[Z_{212}]+\frac{1}{L_{1212}}\left\{\boldsymbol{h}-\boldsymbol{k}-\boldsymbol{j}\frac{L_{12121}}{M_{12121}}\right\}[Z_{1212}] \\
&+\frac{1}{L_{2121}}\left\{\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\frac{L_{21212}}{M_{21212}}\right\}[Z_{2121}]+\frac{1}{M_{12121}}\left\{\boldsymbol{j}\right\}[Z_{12121}]+\frac{1}{M_{21212}}\left\{\boldsymbol{k}\right\}[Z_{21212}]
\end{aligned}
$$

The expansions as obtained by direct application of this formula are not explored further in this work but for completeness are listed in Appendix A.

As complicated as this looks, the coefficient of $[Z_{pt}]$ is quite simple and is the next point of interest before continuing with the formula above.

2.2.4 The $[Z_{pt}]$ coefficient

The simplicity of the $[Z_{pt}]$ coefficient in the formula above arises from the diagonal symmetry of the Bott-Samelson classes which was observed in Subsection 1.4.1. This symmetry is present in all Bott-Samelson classes and not just in those of Type G_2. More precisely if $\zeta_{pt}(G)$ is the coefficient of $[Z_{pt}]$ in a linear expansion of G then we have the following lemma:

Lemma 2.2.1 *Let $G = (g_w)_{w \in W}$ be a moment graph of general type (not necessarily of type G_2), then*

$$\zeta_{pt}\left((g_w)_{w\in W}\right) = \sum_{w \in W} \frac{(-1)^{l(w)} g_w}{y_{R^-}}$$

where $l(w)$ is the length of a reduced word for w

Proof ζ_{pt} is uniquely determined by being linear and that $\zeta_{pt}([Z_u])$ is $\delta_{u,pt}$ for a Bott-Samelson class, $[Z_u]$. Define

$$h\left((g_w)_{w\in W}\right) = \sum_{w\in W} \frac{(-1)^{l(w)} g_w}{y_{R^-}}$$

Then we will show that h is linear and $h([Z_u]) = \delta_{u,pt}$ for a Bott-Samelson clas $[Z_u]$.
Let $N = \{n_w\}_{w\in W}$ and $N' = \{n'_w\}_{w\in W}$ be two moment graphs of the same type and let $r, s \in h_T(pt)$.
Then $rN + sN' = \{rn_w\}_{w\in W} + \{sn'_w\}_{w\in W} = \{rn_w + sn'_w\}_{w\in W}$
So:

$$\begin{aligned}
h(rN + sN') &= h\left(\{rn_w + sn'_w\}_{w\in W}\right) \\
&= \sum_{w\in W} \frac{(-1)^{l(w)}(rn_w + sn'_w)}{y_{R^-}} \\
&= \sum_{w\in W} \frac{(-1)^{l(w)}(rn_w)}{y_{R^-}} + \sum_{w\in W} \frac{(-1)^{l(w)}(sn'_w)}{y_{R^-}} \\
&= r\sum_{w\in W} \frac{(-1)^{l(w)}(n_w)}{y_{R^-}} + s\sum_{w\in W} \frac{(-1)^{l(w)}(n'_w)}{y_{R^-}} \\
&= r(h(N)) + s(h(N'))
\end{aligned}$$

Therefore h is linear.

Remains to show that $h([Z_u]) = \delta_{u,pt}$ for a Bott-Samelson class, $[Z_u]$.
If $[Z_u] = [Z_{pt}]$ then $[Z_u] = \{g_w\}_{w\in W}$ with $g_w = \begin{cases} y_{R^-} & w = 1 \\ 0 & w \neq 1 \end{cases}$ then $h([Z_u]) = y_{R^-}/y_{R^-} = 1$
If $[Z_u]$ is a Bott-Samelson class other than $[Z_{pt}]$ then $[Z_u]$ is defined as $(1 - t_{s_i})M$ for some moment graph M and simple reflection s_i.
For any moment graph $f = (f_w)_{w\in W}$ the action of t_{s_i} gives $(f_{s_i w})_{w\in W}$, then $(1 + t_{s_i})f = (f_w + f_{s_i w})_{w\in W}$.
Then

$$\begin{aligned}
h\left((f_w + f_{s_i w})_{w\in W}\right) &= \sum_{w\in W} \frac{(-1)^{l(w)}(f_w + f_{s_i w})}{y_{R^-}} \\
&= \sum_{w\in W} \frac{(-1)^{l(w)}(f_w)}{y_{R^-}} + \sum_{w\in W} \frac{(-1)^{l(w)}(f_{s_i w})}{y_{R^-}} \\
&= \sum_{w\in W} \frac{(-1)^{l(w)}(f_w)}{y_{R^-}} + \sum_{w\in W} \frac{-(-1)^{l(s_i w)}(f_{s_i w})}{y_{R^-}} && \text{since } l(s_i w) = l(w) \pm 1 \\
&= \sum_{w\in W} \frac{(-1)^{l(w)}(f_w)}{y_{R^-}} + \sum_{s_i w\in W} \frac{-(-1)^{l(s_i w)}(f_{s_i w})}{y_{R^-}} && \text{since the map } w \mapsto s_i w \text{ is bijective} \\
&= \sum_{w\in W} \frac{(-1)^{l(w)}(f_w)}{y_{R^-}} + \sum_{w\in W} \frac{-(-1)^{l(w)}(f_w)}{y_{R^-}} \\
&= 0
\end{aligned}$$

Therefore $h([Z_u]) = \delta_{u,pt}$ for a Bott-Samelson class $[Z_u]$.
Hence we conclude $h = \zeta_{u,pt}$. ∎

Chapter 3

Removing Division in the Linear Expansions in K-Theory

This section shows how the linear expansions of products of the Bott-Samelson classes can be expressed in such a way that it is obvious that they are well defined in the case of K-Theory. More precisely we will re-express the coefficients of the linear expansion such that they have no division or multiplication (excluding possible common factors) when the simplification to K-Theory is made. Expressing the coefficients in this way, allows for much easier comparison to the results in [GR2], which are for the K-Theory case. Another algorithm was used to do this, which unlike the previous algorithm was implemented on a computer due to the large number of calculations and their length. However many of the calculations that follow were also done by hand.
Recall that in Section 1.1 we used the formal group law (for T-equivariant complex cobordism),

$$y_{\alpha+\beta} = y_\alpha + y_\beta - p\left[y_\alpha, y_\beta\right] y_\alpha y_\beta \tag{3.1}$$

to show that the coefficient was well-defined. We will do the same here.
Additionally the formal group law also has the relation:

$$y_0 = 0$$

Together these yield:

$$\begin{aligned} y_0 &= y_{-\beta} + y_\beta - p\left[y_{-\beta}, y_\beta\right] y_{-\beta} y_\beta \\ 0 &= y_{-\beta} + y_\beta - p\left[y_{-\beta}, y_\beta\right] y_{-\beta} y_\beta \\ y_\beta &= \frac{y_{-\beta}}{p\left[y_{-\beta}, y_\beta\right] y_{-\beta} - 1} \\ \frac{1}{y_\beta} &= p\left[y_\beta, y_{-\beta}\right] - \frac{1}{y_{-\beta}} \end{aligned} \tag{3.2}$$

Recall from page 7 that the simplification to K-Theory is $p\left[y_{-\alpha}, y_{-\beta}\right] = 1$ for all α and all β.
For comparison to [GR2] we note that their notation is:

$$\begin{aligned} \alpha_{rs} &= e^{-(r\alpha_1 + s\alpha_2)} - 1 = -y_{-(r\alpha_1+s\alpha_2)} \\ y_{rs} &= e^{-(r\alpha_1 + s\alpha_2)} = \alpha_{rs} + 1 = 1 - y_{-(r\alpha_1+s\alpha_2)} \\ \left[s_{i_1} \ldots s_{i_n}\right] &= \left[Z_{i_n \ldots i_1}\right] \\ [1] &= \left[Z_{pt}\right] \end{aligned}$$

3.1 Example computation of $[Z_2]\,[Z_{12}]$

From our previous example in Section 2.1 we know $[Z_2]\,[Z_{12}] = \Gamma_{12}\,[Z_2] + \left(\Delta_{12}\Delta_2 - \Gamma_{12}\Delta_2\right)/y_{R^-}\,[Z_{pt}]$. We want to express these coefficients in a way that makes it clear they are well-defined. In this example it is clear that the coefficient of $[Z_2]$ is well-defined, since it is only Γ_{12} and from its definition we obtain:

$$\Gamma_{12} = \frac{y_{R^-}}{y_{-\alpha_2} y_{-s_2\alpha_1}} = \frac{y_{R^-}}{y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}} = y_{-\alpha_1} y_{-(2\alpha_1+\alpha_2)} y_{-(3\alpha_1+\alpha_2)} y_{-(3\alpha_1+2\alpha_2)}$$

which is nicely defined.
However the coefficient of $[Z_{pt}]$ is less trivial, here it is far from obvious that $(\Delta_{12}\Delta_2 - \Gamma_{12}\Delta_2)/y_{R^-}$ is well-defined, however using the relations above we can rewrite this in a more conducive form:

$$\begin{aligned}
\frac{\Gamma_{12}\Delta_2 - \Delta_{12}\Delta_2}{y_{R^-}} &= \frac{\Gamma_{12}\Delta_2}{y_{R^-}} - \frac{\Delta_{12}\Delta_2}{y_{R^-}} \\
&= \frac{y_{R^-}}{y_{-\alpha_2} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}} - \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-\alpha_2}} \\
&= \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}} \left\{ \frac{y_{-\alpha_1}}{y_{-\alpha_2}} - \frac{y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_2}} \right\} \\
&= \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}} \left\{ \frac{y_{-\alpha_1}}{y_{-\alpha_2}} - \frac{y_{-\alpha_1} + y_{-\alpha_2} - y_{-\alpha_1} y_{-\alpha_2} p\left[y_{-\alpha_1}, y_{-\alpha_2}\right]}{y_{-\alpha_2}} \right\} \\
&= \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}} \left\{ \frac{-y_{-\alpha_2} + y_{-\alpha_1} y_{-\alpha_2} p\left[y_{-\alpha_1}, y_{-\alpha_2}\right]}{y_{-\alpha_2}} \right\} \\
&= \frac{y_{R^-}}{y_{-\alpha_1} y_{-\alpha_2} y_{-(\alpha_1+\alpha_2)}} \left\{ y_{-\alpha_1} p\left[y_{-\alpha_1}, y_{-\alpha_2}\right] - 1 \right\} \\
&= y_{-(2\alpha_1+\alpha_2)} y_{-(3\alpha_1+\alpha_2)} y_{-(3\alpha_1+2\alpha_2)} \left\{ y_{-\alpha_1} p\left[y_{-\alpha_1}, y_{-\alpha_2}\right] - 1 \right\}
\end{aligned}$$

Now it is clear that this coefficient is also well-defined, also after setting $p = 1$ it only takes a moment to check this is the same as the corresponding product, $[s_2][s_2 s_1] = -\alpha_{21}\alpha_{31}\alpha_{32} y_{10}[1] + \alpha_{10}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]$, in [GR2].

3.2 General Coefficients

In the example above we explored the divisibility directly from the linear expansion. In general we could directly apply the formula in Subsection 2.2.3 to the products, however by instead applying the formula to specific types of products, $M\,[Z_u]$ where $[Z_u]$ is a Bott-Samelson class and M is defined as below. Then using the definitions of Δ_1, Δ_2, Δ_{12}, Δ_{21}, Γ_{12}, Γ_{21} etc, which were defined on page 11, we obtain the following formulae. These formulae were used instead of those obtained by direct application of the formula Subsection 2.2.3 because they expose repeated factors greatly simplifying the calculations.

$$M = \begin{array}{ccc} & a & \\ b & & c \\ d & & e \\ f & & g \\ h & & i \\ j & & k \\ & 0 & \end{array}$$

$$
\begin{aligned}
M[Z_{pt}] &= a[Z_{pt}] \\
M[Z_1] &= b[Z_1] + E_{a,b}[Z_{pt}] \\
M[Z_2] &= c[Z_2] + E_{a,c}[Z_{pt}] \\
M[Z_{12}] &= d[Z_{12}] + E_{c,d}[Z_2] + E_{b,d}[Z_1] + F_{ab,cd}[Z_{pt}] \\
M[Z_{21}] &= e[Z_{21}] + E_{c,e}[Z_2] + E_{b,e}[Z_1] + F_{ac,be}[Z_{pt}] \\
M[Z_{121}] &= f[Z_{121}] + E_{e,f}[Z_{21}] + E_{d,f}[Z_{12}] + F_{cd,ef}[Z_2] + G_{bd,bf,ef}[Z_1] + G_{abcd,ab,ef}[Z_{pt}] \\
M[Z_{212}] &= g[Z_{212}] + E_{e,g}[Z_{21}] + E_{d,g}[Z_{12}] + G_{ce,cg,dg}[Z_2] + F_{be,dg}[Z_1] + G_{acbe,ac,dg}[Z_{pt}] \\
M[Z_{1212}] &= h[Z_{1212}] + E_{g,h}[Z_{212}] + E_{f,h}[Z_{121}] + F_{ef,gh}[Z_{21}] + G_{df,dh,gh}[Z_{12}] + G_{cdef,cd,gh}[Z_2] \\
&\quad + H[Z_1] + J[Z_{pt}] \\
M[Z_{2121}] &= i[Z_{2121}] + E_{g,i}[Z_{212}] + E_{f,i}[Z_{121}] + G_{eg,ei,fi}[Z_{21}] + F_{dg,fi}[Z_{12}] + K[Z_2] \\
&\quad + G_{bedg,be,fi}[Z_1] + L[Z_{pt}] \\
M[Z_{12121}] &= j[Z_{12121}] + E_{i,j}[Z_{2121}] + E_{h,j}[Z_{1212}] + F_{gh,ij}[Z_{212}] + G_{fh,fj,ij}[Z_{121}] + G_{efgh,ef,ij}[Z_{21}] \\
&\quad + N[Z_{12}] + O[Z_2] + P[Z_1] + Q[Z_{pt}] \\
M[Z_{21212}] &= k[Z_{21212}] + E_{i,k}[Z_{2121}] + E_{h,k}[Z_{1212}] + G_{gi,gk,hk}[Z_{212}] + F_{fi,hk}[Z_{121}] + R[Z_{21}] \\
&\quad + G_{dgfi,dg,hk}[Z_{12}] + S[Z_2] + U[Z_1] + V[Z_{pt}]
\end{aligned}
$$

By defining the following:

$$
\begin{aligned}
E_{a,b} &= \frac{a-b}{y_{-\alpha_1}} & E_{a,c} &= \frac{a-c}{y_{-\alpha_2}} & & \\
E_{b,d} &= \frac{b-d}{y_{-\alpha_2}} & E_{c,d} &= \frac{c-d}{y_{-s_2\alpha_1}} & F_{ab,cd} &= \frac{E_{a,b}-E_{c,d}}{y_{-\alpha_2}} \\
E_{c,e} &= \frac{c-e}{y_{-\alpha_1}} & E_{b,e} &= \frac{b-e}{y_{-s_1\alpha_2}} & F_{ac,be} &= \frac{E_{a,c}-E_{b,e}}{y_{-\alpha_1}} \\
E_{d,f} &= \frac{d-f}{y_{-\alpha_1}} & E_{e,f} &= \frac{e-f}{y_{-s_1s_2\alpha_1}} & F_{cd,ef} &= \frac{E_{c,d}-E_{e,f}}{y_{-\alpha_1}} \\
E_{e,g} &= \frac{e-g}{y_{-\alpha_2}} & E_{d,g} &= \frac{d-g}{y_{-s_2s_1\alpha_2}} & F_{be,dg} &= \frac{E_{b,e}-E_{d,g}}{y_{-\alpha_2}} \\
E_{f,h} &= \frac{f-h}{y_{-\alpha_2}} & E_{g,h} &= \frac{g-h}{y_{-s_2s_1s_2\alpha_1}} & F_{ef,gh} &= \frac{E_{e,f}-E_{g,h}}{y_{-\alpha_2}} \\
E_{g,i} &= \frac{g-i}{y_{-\alpha_1}} & E_{f,i} &= \frac{f-i}{y_{-s_1s_2s_1\alpha_2}} & F_{dg,fi} &= \frac{E_{d,g}-E_{f,i}}{y_{-\alpha_1}} \\
E_{h,j} &= \frac{h-j}{y_{-\alpha_1}} & E_{i,j} &= \frac{i-j}{y_{-s_1s_2s_1s_2\alpha_1}} & F_{gh,ij} &= \frac{E_{g,h}-E_{i,j}}{y_{-\alpha_1}} \\
E_{i,k} &= \frac{i-k}{y_{-\alpha_2}} & E_{h,k} &= \frac{h-k}{y_{-s_2s_1s_2s_1\alpha_2}} & F_{fi,hk} &= \frac{E_{f,i}-E_{h,k}}{y_{-\alpha_2}}
\end{aligned}
$$

$$G_{bd,bf,ef} = \frac{E_{b,d}}{y_{-\alpha_1}} + \frac{b-f}{y_{-s_1\alpha_2}y_{-s_1\alpha_1}} - \frac{E_{e,f}}{y_{-s_1\alpha_2}}$$
$$G_{ce,cg,dg} = \frac{E_{c,e}}{y_{-\alpha_2}} + \frac{c-g}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} - \frac{E_{d,g}}{y_{-s_2\alpha_1}}$$
$$G_{df,dh,gh} = \frac{E_{d,f}}{y_{-\alpha_2}} + \frac{d-h}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} + \frac{d-h}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} - \frac{E_{g,h}}{y_{-s_2s_1\alpha_2}}$$
$$G_{eg,ei,fi} = \frac{E_{e,g}}{y_{-\alpha_1}} + \frac{e-i}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{e-i}{y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{E_{f,i}}{y_{-s_1s_2\alpha_1}}$$
$$G_{fh,fj,ij} = \frac{E_{f,h}}{y_{-\alpha_1}} + \frac{f-j}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{f-j}{y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} + \frac{f-j}{y_{-s_1s_2s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}} - \frac{E_{i,j}}{y_{-s_1s_2s_1\alpha_2}}$$
$$G_{gi,gk,hk} = \frac{E_{g,i}}{y_{-\alpha_2}} + \frac{g-k}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{g-k}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} + \frac{g-k}{y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}} - \frac{E_{h,k}}{y_{-s_2s_1s_2\alpha_1}}$$

$$G_{abcd,ab,ef} = \frac{F_{ab,cd}}{y_{-\alpha_1}} + \frac{E_{a,b}}{y_{-s_1\alpha_2}y_{-s_1\alpha_1}} + \frac{E_{e,f}}{y_{-\alpha_1}y_{-s_1\alpha_2}}$$
$$G_{acbe,ac,dg} = \frac{F_{ac,be}}{y_{-\alpha_2}} + \frac{E_{a,c}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} + \frac{E_{d,g}}{y_{-\alpha_2}y_{-s_2\alpha_1}}$$
$$G_{cdef,cd,gh} = \frac{F_{cd,ef}}{y_{-\alpha_2}} + \frac{E_{c,d}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} + \frac{E_{c,d}}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} + \frac{E_{g,h}}{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} - \frac{E_{g,h}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}}$$
$$G_{bedg,be,fi} = \frac{F_{be,dg}}{y_{-\alpha_1}} + \frac{E_{b,e}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{E_{b,e}}{y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} + \frac{E_{f,i}}{y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} - \frac{E_{f,i}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}}$$
$$G_{efgh,ef,ij} = \frac{F_{ef,gh}}{y_{-\alpha_1}} + \frac{E_{e,f}-E_{i,j}}{y_{-s_1\alpha_2}y_{-s_1\alpha_1}} + \frac{E_{e,f}-E_{i,j}}{y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} + \frac{E_{e,f}}{y_{-s_1s_2s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}} + \frac{E_{i,j}}{y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}}$$
$$G_{dgfi,dg,hk} = \frac{F_{dg,fi}}{y_{-\alpha_2}} + \frac{E_{d,g}-E_{h,k}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} + \frac{E_{d,g}-E_{h,k}}{y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}} + \frac{E_{d,g}}{y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}} + \frac{E_{h,k}}{y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}$$

$$H = \frac{G_{bd,bf,ef}}{y_{-\alpha_2}} + \frac{E_{b,d}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} + \frac{E_{g,h}}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}} - \frac{d-h}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}}$$
$$K = \frac{G_{ce,cg,dg}}{y_{-\alpha_1}} + \frac{E_{c,e}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{E_{f,i}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}} - \frac{e-i}{y_{-\alpha_1}y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}}$$
$$N = \frac{G_{df,dh,gh}}{y_{-\alpha_1}} + \frac{E_{d,f}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1s_2s_1\alpha_2}} - \frac{f-j}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{f-j}{y_{-\alpha_1}y_{-s_1s_2s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}}$$
$$R = \frac{G_{eg,ei,fi}}{y_{-\alpha_2}} + \frac{E_{e,g}}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2s_1s_2\alpha_1}} - \frac{g-k}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} - \frac{g-k}{y_{-\alpha_2}y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}}$$

$$J = \frac{G_{abcd,ab,ef}}{y_{-\alpha_2}} + \frac{F_{ab,cd}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} - \frac{E_{c,d}}{y_{-\alpha_2}y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}} - \frac{E_{g,h}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}$$
$$L = \frac{G_{acbe,ac,dg}}{y_{-\alpha_1}} + \frac{F_{ac,be}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} - \frac{E_{b,e}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{E_{f,i}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}$$
$$O = \frac{G_{cdef,cd,gh}}{y_{-\alpha_1}} + \frac{F_{cd,ef}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{E_{i,j}-E_{e,f}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{E_{e,f}}{y_{-\alpha_1}y_{-s_1s_2s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}} - \frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}}$$
$$U = \frac{G_{bedg,be,fi}}{y_{-\alpha_2}} + \frac{F_{be,dg}}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{E_{h,k}-E_{d,g}}{y_{-\alpha_2}y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}} - \frac{E_{d,g}}{y_{-\alpha_2}y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}} - \frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}$$

$$
\begin{aligned}
P = \frac{H}{y_{-\alpha_1}} + \frac{G_{bd,bf,ef}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{G_{bd,bf,ef}}{y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{E_{b,d}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{f-j}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} \\
- \frac{E_{e,f}}{y_{-s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} + \frac{E_{i,j}}{y_{-s_1\alpha_2}y_{-s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} - \frac{E_{i,j}}{y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}} \\
S = \frac{K}{y_{-\alpha_2}} + \frac{G_{ce,cg,dg}}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{G_{ce,cg,dg}}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} - \frac{E_{c,e}}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} - \frac{g-k}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}} \\
- \frac{E_{d,g}}{y_{-s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}} + \frac{E_{h,k}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}} - \frac{E_{h,k}}{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}
\end{aligned}
$$

$$
\begin{aligned}
Q = \frac{J}{y_{-\alpha_1}} + \frac{G_{abcd,ab,ef}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{E_{a,b}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} + \frac{E_{e,f}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} \\
+ \frac{E_{e,f}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} + \frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}} \\
V = \frac{L}{y_{-\alpha_2}} + \frac{G_{acbe,ac,dg}}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{E_{a,c}}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} + \frac{E_{d,g}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} \\
+ \frac{E_{d,g}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}} + \frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}
\end{aligned}
$$

There is actually more symmetry and patterns in these formulae than this list shows, however these are easier to compute. For completeness the more symmetric version are included below.

$$
\begin{aligned}
M[Z_{pt}] &= a[Z_{pt}] \\
M[Z_1] &= b[Z_1] + E_{a,b}[Z_{pt}] \\
M[Z_2] &= c[Z_2] + E_{a,c}[Z_{pt}] \\
M[Z_{12}] &= d[Z_{12}] + E_{c,d}[Z_2] - \frac{d}{y_{-\alpha_2}}[Z_1] - \frac{E_{c,d}}{y_{-\alpha_2}}[Z_{pt}] + \frac{M[Z_1]}{y_{-\alpha_2}} \\
M[Z_{21}] &= e[Z_{21}] + E_{b,e}[Z_1] - \frac{e}{y_{-\alpha_1}}[Z_2] - \frac{E_{b,e}}{y_{-\alpha_1}}[Z_{pt}] + \frac{M[Z_2]}{y_{-\alpha_1}} \\
M[Z_{121}] &= f[Z_{121}] + E_{e,f}[Z_{21}] - \frac{f}{y_{-\alpha_1}}[Z_{12}] - \frac{E_{e,f}}{y_{-\alpha_1}}[Z_2] - \left\{\frac{f}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{E_{e,f}}{y_{-s_1\alpha_2}}\right\} Z_1 + \frac{E_{e,f}}{y_{-\alpha_1}y_{-s_1\alpha_2}}[Z_{pt}] \\
&\quad + \frac{M[Z_{12}]}{y_{-\alpha_1}} + \frac{M[Z_1]}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} \\
M[Z_{212}] &= g[Z_{121}] + E_{d,g}[Z_{12}] - \frac{g}{y_{-\alpha_2}}[Z_{21}] - \frac{E_{d,g}}{y_{-\alpha_2}}[Z_1] - \left\{\frac{g}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{E_{d,g}}{y_{-s_2\alpha_1}}\right\}[Z_2] + \frac{E_{d,g}}{y_{-\alpha_2}y_{-s_2\alpha_1}}[Z_{pt}] \\
&\quad + \frac{M[Z_{21}]}{y_{-\alpha_2}} + \frac{M[Z_2]}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} \\
M[Z_{1212}] &= h[Z_{1212}] + E_{g,h}[Z_{212}] - \frac{h}{y_{-\alpha_2}}[Z_{121}] - \frac{E_{g,h}}{y_{-\alpha_2}}[Z_{21}] \\
&\quad - \left\{\frac{h}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{h}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} + \frac{E_{g,h}}{y_{-s_2s_1\alpha_2}}\right\}[Z_{12}] \\
&\quad + \left\{\frac{E_{g,h}}{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} - \frac{E_{g,h}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}}\right\}[Z_2] + \left\{\frac{E_{g,h}}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}} + \frac{h}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}}\right\}[Z_1] \\
&\quad - \frac{E_{d,g}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}[Z_{pt}] + \frac{M[Z_{121}]}{y_{-\alpha_2}} + \frac{M[Z_{12}]}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{M[Z_{12}]}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} - \frac{M[Z_1]}{y_{-\alpha_2}y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}}
\end{aligned}
$$

$$
\begin{aligned}
M\left[Z_{2121}\right] = {} & i\left[Z_{2121}\right] + E_{f,i}\left[Z_{121}\right] - \frac{i}{y_{-\alpha_1}}\left[Z_{212}\right] - \frac{E_{f,i}}{y_{-\alpha_1}}\left[Z_{21}\right] \\
& - \left\{\frac{i}{y_{-s_1\alpha_2}y_{-s_1\alpha_1}} + \frac{i}{y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} + \frac{E_{f,i}}{y_{-s_1s_2\alpha_1}}\right\}\left[Z_{21}\right] \\
& + \left\{\frac{E_{f,i}}{y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} - \frac{E_{f,i}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}}\right\}\left[Z_1\right] + \left\{\frac{E_{f,i}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}} + \frac{i}{y_{-\alpha_1}y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}}\right\}\left[Z_2\right] \\
& - \frac{E_{f,i}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}\left[Z_{pt}\right] + \frac{M\left[Z_{212}\right]}{y_{-\alpha_1}} + \frac{M\left[Z_{21}\right]}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{M\left[Z_{21}\right]}{y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} - \frac{M\left[Z_2\right]}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}}
\end{aligned}
$$

$$
\begin{aligned}
M\left[Z_{12121}\right] = {} & j\left[Z_{12121}\right] + E_{i,j}\left[Z_{2121}\right] - \frac{j}{y_{-\alpha_1}}\left[Z_{1212}\right] - \frac{E_{i,j}}{y_{-\alpha_1}}\left[Z_{212}\right] - \Bigg\{\frac{j}{y_{-s_1\alpha_2}y_{-s_1\alpha_1}} + \frac{j}{y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} \\
& + \frac{j}{y_{-s_1s_2s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}} + \frac{E_{i,j}}{y_{-s_1s_2s_1\alpha_2}}\Bigg\}\left[Z_{121}\right] + \Bigg\{\frac{E_{i,j}}{y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}} - \frac{E_{i,j}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} \\
& - \frac{E_{i,j}}{y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}}\Bigg\}\left[Z_{21}\right] + \Bigg\{\frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1s_2s_1\alpha_2}} + \frac{j}{y_{-\alpha_1}y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} \\
& + \frac{j}{y_{-\alpha_1}y_{-s_1s_2s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}}\Bigg\}\left[Z_{12}\right] + \left\{\frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1s_2\alpha_2}y_{-s_1s_2\alpha_1}} - \frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}}\right\}\left[Z_2\right] \\
& + \left\{\frac{j}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} + \frac{E_{i,j}}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2s_1\alpha_2}} - \frac{E_{i,j}}{y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}}\right\}\left[Z_1\right] \\
& + \left\{\frac{E_{i,j}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_2s_1\alpha_2}}\right\}\left[Z_{pt}\right] + \frac{M\left[Z_{1212}\right]}{y_{-\alpha_1}} + \frac{M\left[Z_{121}\right]}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}} + \frac{M\left[Z_{121}\right]}{y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} \\
& + \frac{M\left[Z_{121}\right]}{y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} - \frac{M\left[Z_{12}\right]}{y_{-\alpha_1}y_{-s_1s_2\alpha_1}y_{-s_1s_2\alpha_2}} - \frac{M\left[Z_{12}\right]}{y_{-\alpha_1}y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}} \\
& - \frac{M\left[Z_1\right]}{y_{-s_1\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2s_1\alpha_1}y_{-s_1s_2s_1\alpha_2}}
\end{aligned}
$$

$$
\begin{aligned}
M\left[Z_{21212}\right] = {} & k\left[Z_{21212}\right] - \frac{k}{y_{-\alpha_2}}\left[Z_{2121}\right] + E_{h,k}\left[Z_{1212}\right] - \frac{E_{h,k}}{y_{-\alpha_2}}\left[Z_{121}\right] - \Bigg\{\frac{k}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}} + \frac{k}{y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} \\
& + \frac{k}{y_{-s_2s_1s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}} + \frac{E_{h,k}}{y_{-s_2s_1s_2\alpha_1}}\Bigg\}\left[Z_{212}\right] + \Bigg\{\frac{E_{h,k}}{y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}} - \frac{E_{h,k}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} \\
& - \frac{E_{h,k}}{y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}}\Bigg\}\left[Z_{12}\right] + \Bigg\{\frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2s_1s_2\alpha_1}} + \frac{k}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} \\
& + \frac{k}{y_{-\alpha_2}y_{-s_2s_1s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}}\Bigg\}\left[Z_{21}\right] + \left\{\frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1\alpha_1}} - \frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}\right\}\left[Z_1\right] \\
& + \left\{\frac{k}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}} + \frac{E_{h,k}}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}} - \frac{E_{h,k}}{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}\right\}\left[Z_2\right] \\
& + \left\{\frac{E_{h,k}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}y_{-s_2s_1s_2\alpha_1}}\right\}\left[Z_{pt}\right] + \frac{M\left[Z_{2121}\right]}{y_{-\alpha_2}} + \frac{M\left[Z_{212}\right]}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}} + \frac{M\left[Z_{212}\right]}{y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}} \\
& + \frac{M\left[Z_{212}\right]}{y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}} - \frac{M\left[Z_{21}\right]}{y_{-\alpha_2}y_{-s_2s_1\alpha_1}y_{-s_2s_1\alpha_2}} - \frac{M\left[Z_{21}\right]}{y_{-\alpha_2}y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}} \\
& - \frac{M\left[Z_2\right]}{y_{-s_2\alpha_1}y_{-s_2\alpha_2}y_{-s_2s_1s_2\alpha_1}y_{-s_2s_1s_2\alpha_2}}
\end{aligned}
$$

3.3 The Algorithm

In the example above it was easy to see the steps to take to get the desired expression however even with any simplifications from above, in general it is far from obvious what we should do to re-express the coefficients of an expansion. In general to rewrite each coefficient in the product expansion into the desired form, as detailed

in above we use the following algorithm.
Since the scenario of interest here is when $p[y_{-\alpha}, y_{-\beta}] = 1$ then $p[y_{-\alpha}, y_{-\beta}]$ would be thought to 'cancel' with -1 or $-p[y_{-\alpha'}, y_{-\beta'}]$. Any such cancelling terms for the coefficient $[Z_\delta]$ in the product $[Z_\gamma][Z_\zeta]$ are collected in the expression $D[Z_\gamma Z_\zeta]_\delta$. In the algorithm that follows collecting any 'cancelling' terms means moving any terms that 'cancel' when $p[y_{-\alpha}, y_{-\beta}] = 1$ to the corresponding $D[Z_\gamma Z_\zeta]_\delta$.

Let C be a general coefficient and D be its collection of 'cancelling' terms. Expand any brackets in C and rewrite C without s_1 and s_2, (i.e. $y_{-s_2 s_1 \alpha_2} \to y_{-(3\alpha_1+2\alpha_2)}$)

Remove any $y_{\gamma\alpha_1+\delta\alpha_2}$ where $\gamma, \delta \leq 0$, by using (Eq. 3.2).

After doing any division cancellations and collecting any 'cancelling' terms, if any of $y_{-\alpha_1}$, $y_{-\alpha_2}$, $y_{-(\alpha_1+\alpha_2)}$, $y_{-(2\alpha_1+\alpha_2)}$, $y_{-(3\alpha_1+\alpha_2)}$, $y_{-(3\alpha_1+2\alpha_2)}$ is a common factor then factorise as such, however each one may only contribute to one common factor. Call these common factors F. From now on consider $C' = C/F$.

Remove Products: For every product (in a numerator) $zy_{-\beta}y_{-\gamma}$ in C' remove it by adding $zy_\gamma + zy_\beta - zy_\gamma y_\beta p[y_\gamma, y_\beta] - zy_{\gamma+\beta} = 0$. Repeat this until there are no products.

Do any division cancellations and collect any 'cancelling' terms and add them to D making sure to multiply the 'canceling' terms by F.

Divides:

For each term identify the 'largest' denominator (where the size of $y_{-(\gamma\alpha_1+\beta\alpha_2)}$ is $\gamma+\beta$), and calculate the difference between the 'largest' denominator and the numerator (where the difference between $y_{-(\gamma\alpha_1+\beta\alpha_2)}$ and $y_{-(\nu\alpha_1+\delta\alpha_2)}$ is $\gamma - \nu + \delta - \beta$ and is considered good if $\gamma - \nu$ and $\delta - \beta$ are non-negative).

Then choose the terms with the largest 'largest denominator' and out of these choose the one with the largest good difference. (If there is more than one pick one, if there is none skip to Common Denominator). Let the chosen term be $(zy_{-(\gamma\alpha_1+\beta\alpha_2)})/(z'y_{-(\nu\alpha_1+\delta\alpha_2)})$.

Add the following terms to C':

$$\frac{z}{z'} - \frac{zy_{-((\gamma-\nu)\alpha_1+(\beta-\delta)\alpha_2)}}{z'} + \frac{zy_{-((\gamma-\nu)\alpha_1+(\beta-\delta)\alpha_2)}}{z'y_{-(\nu\alpha_1+\delta\alpha_2)}} - \frac{zy_{-(\gamma\alpha_1+\beta\alpha_2)}}{z'y_{-(\nu\alpha_1+\delta\alpha_2)}} = 0$$

Do any division cancellations and collect any 'cancelling' terms and add them to D making sure to multiply the 'canceling' terms by F.

Return to divides.

Common Denominator: Put all the terms which have non-trivial denominator on a common denominator.

Remove Products: For every product (in a numerator) $zy_{-\beta}y_{-\gamma}$ in C' remove it be adding $zy_\gamma + zy_\beta - zy_\gamma y_\beta p[y_\gamma, y_\beta] - zy_{\gamma+\beta} = 0$. Repeat this until there are no products.

Do any division cancellations and collect any 'cancelling' terms and add them to D making sure to multiply the 'canceling' terms by F.

We do not offer a proof of this algorithm since we only have a finite number of coefficients and for all of the coefficients it produces results in the desired form (no division when simplified to K-Thoery).

3.4 The Coefficients without Division when Simplified to K-Theory

The results of the algorithm above quickly become quite messy, with many terms which cancel when $p = 1$. In this section we give the results of the algorithm however for simplicity, the terms which cancel when $p = 1$ are not included. Instead these terms are considered in Chapter 4, where $D[Z_\chi Z_\epsilon]_w$ is the cancelling terms in the coefficient of $[Z_w]$ in the expansion of the product $[Z_\chi][Z_\epsilon]$. For compactness the following notation is introduced, $y_{-(\gamma\alpha_1+\delta\alpha_2)} = y_{(\gamma,\delta)}$ and $y_{\gamma\alpha_1+\delta\alpha_2} = y_{-(\gamma,\delta)}$.

All of these results agree with those in [GR2] except for $[Z_{12}][Z_{121}]$, $[Z_{12}][Z_{212}]$, $[Z_{12}][Z_{2121}]$, $[Z_{121}][Z_{212}]$, $[Z_{121}][Z_{1212}]$, $[Z_{212}][Z_{2121}]$, $[Z_{212}][Z_{1212}]$, $[Z_{1212}][Z_{2121}]$ and $[Z_{1212}][Z_{12121}]$.

$$
\begin{aligned}
[Z_{pt}]\,[Z_{pt}] &= y_{R^-}\,[Z_{pt}] \\
[Z_{pt}]\,[Z_1] &= \frac{y_{R^-}}{y_{(1,0)}}\,[Z_{pt}] \\
[Z_{pt}]\,[Z_2] &= \frac{y_{R^-}}{y_{(0,1)}}\,[Z_{pt}] \\
[Z_{pt}]\,[Z_{12}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}}\,[Z_{pt}] \\
[Z_{pt}]\,[Z_{21}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}}\,[Z_{pt}] \\
[Z_{pt}]\,[Z_{121}] &= \left\{\frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\Big(3 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) + D\left[Z_{pt}Z_{121}\right]_{pt}\right\}[Z_{pt}] \\
[Z_{pt}]\,[Z_{212}] &= \left\{\frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}} + D\left[Z_{pt}Z_{212}\right]_{pt}\right\}[Z_{pt}] \\
[Z_{pt}]\,[Z_{1212}] &= \left\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right]\Big) + D\left[Z_{pt}Z_{1212}\right]_{pt}\right\}[Z_{pt}] \\
[Z_{pt}]\,[Z_{2121}] &= \left\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{pt}Z_{2121}\right]_{pt}\right\}[Z_{pt}] \\
[Z_{pt}]\,[Z_{12121}] &= \left\{y_{(3,2)}\Big(2 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right]\Big) + D\left[Z_{pt}Z_{12121}\right]_{pt}\right\}[Z_{pt}] \\
[Z_{pt}]\,[Z_{21212}] &= \left\{y_{(2,1)}\Big(2 - y_{(2,1)}\Big) + D\left[Z_{pt}Z_{21212}\right]_{pt}\right\}[Z_{pt}]
\end{aligned}
$$

$$
[Z_1]\,[Z_1] = \frac{y_{R^-}}{y_{(1,0)}}\,[Z_1] \qquad [Z_1]\,[Z_2] = \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}}\,[Z_{pt}] \qquad [Z_1]\,[Z_{12}] = \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}}\,[Z_1]
$$

$$
\begin{aligned}
[Z_1]\,[Z_{21}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(3,1)}}\,[Z_1] + \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\Big(3 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \\
&\quad - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right]\Big)\,[Z_{pt}] \\
[Z_1]\,[Z_{121}] &= \left\{\frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\Big(3 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) + D\left[Z_1Z_{121}\right]_1\right\}[Z_1] \\
[Z_1]\,[Z_{212}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\,[Z_1] + \Big\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) \\
&\quad + D\left[Z_1Z_{212}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_1]\,[Z_{1212}] &= \left\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right]\Big) + D\left[Z_1Z_{1212}\right]_1\right\}[Z_1] \\
[Z_1]\,[Z_{2121}] &= \left\{y_{(1,1)}y_{(3,2)}\Big(2 - y_{(1,1)}p\left[y_{(1,1)}, y_{(3,1)}\right]\Big) + D\left[Z_1Z_{2121}\right]_1\right\}[Z_1] + \Big\{y_{(3,2)}\Big(2 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] \\
&\quad - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right]\Big) + D\left[Z_1Z_{2121}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_1]\,[Z_{12121}] &= \left\{y_{(3,2)}\Big(2 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right]\Big) + D\left[Z_1Z_{12121}\right]_1\right\}[Z_1] \\
[Z_1]\,[Z_{21212}] &= \left\{y_{(3,2)} + D\left[Z_1Z_{21212}\right]_1\right\}[Z_1] + \left\{\Big(1 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right]\Big) + D\left[Z_1Z_{21212}\right]_{pt}\right\}[Z_{pt}]
\end{aligned}
$$

$$
\begin{aligned}
[Z_2]\,[Z_2] &= \frac{y_{R^-}}{y_{(0,1)}}\,[Z_2] \\
[Z_2]\,[Z_{12}] &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}}\,[Z_2] + \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}}\Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big)\,[Z_{pt}] \\
[Z_2]\,[Z_{21}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}}\,[Z_2] \\
[Z_2]\,[Z_{121}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}}\,[Z_2] + \Big\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) \\
&\quad + D\left[Z_2Z_{121}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_2]\,[Z_{212}] &= \Big\{\frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}} + D\left[Z_2Z_{212}\right]_2\Big\}\,[Z_2] \\
[Z_2]\,[Z_{1212}] &= \Big\{y_{(2,1)}y_{(3,1)}\Big(2 - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + D\left[Z_2Z_{1212}\right]_2\Big\}\,[Z_2] \\
&\quad + \Big\{y_{(2,1)}\Big(2 - y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_2Z_{1212}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_2]\,[Z_{2121}] &= \Big\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) + D\left[Z_2Z_{2121}\right]_2\Big\}\,[Z_2] \\
[Z_2]\,[Z_{12121}] &= \Big\{y_{(2,1)}\Big(3 - y_{(4,2)}p\left[y_{(1,1)}, y_{(4,2)}\right] - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + D\left[Z_2Z_{12121}\right]_2\Big\}\,[Z_2] \\
&\quad + \Big\{\Big(1 - y_{(6,3)}p\left[y_{(0,1)}, y_{(6,3)}\right]\Big) + D\left[Z_2Z_{12121}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_2]\,[Z_{21212}] &= \Big\{y_{(2,1)}\Big(2 - y_{(2,1)}\Big) + D\left[Z_2Z_{21212}\right]_2\Big\}\,[Z_2]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{12}]\,[Z_{12}] &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}}Z_{12} + \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}}\Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big)\,[Z_1] \\
[Z_{12}]\,[Z_{21}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}}\,[Z_2] + \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\,[Z_1] + \Big\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(1,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \\
&\quad - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{12}Z_{21}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_{12}]\,[Z_{121}] &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}}\,[Z_{12}] + \Big\{y_{(2,1)}y_{(3,2)}\Big(2 - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) \\
&\quad + D\left[Z_{12}Z_{121}\right]_1\Big\}\,[Z_1] \\
[Z_{12}]\,[Z_{212}] &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}y_{(3,2)}}\,[Z_{12}] + \Big\{y_{(2,1)}y_{(3,1)}\Big(2 - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right] - y_{(1,0)}p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) \\
&\quad + D\left[Z_{12}Z_{212}\right]_2\Big\}\,[Z_2] + \Big\{y_{(2,1)}\Big(y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
&\quad - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{12}Z_{212}\right]_1\Big\}\,[Z_1] + \Big\{y_{(2,1)}\Big(2 - y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] \\
&\quad - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] - y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] + y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{12}Z_{212}\right]_{pt}\Big\}\,[Z_{pt}]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{12}]\,[Z_{1212}] = &\Big\{ y_{(2,1)}y_{(3,1)}\Big(2 - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{12}Z_{1212}\right]_{12} \Big\}[Z_{12}] + \Big\{ y_{(2,1)}\Big(2 - y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] \\
&- y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{12}Z_{1212}\right]_{1} \Big\}[Z_{1}] \\
[Z_{12}]\,[Z_{2121}] = &\; y_{(2,1)}y_{(3,1)}\,[Z_{12}] + \Big\{ y_{(2,1)}\Big(3 - y_{(4,2)}p\left[y_{(1,1)}, y_{(4,2)}\right] - y_{(3,1)}p\left[y_{(1,1)}, y_{(3,1)}\right] - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) \\
&+ D\left[Z_{12}Z_{2121}\right]_{2} \Big\}[Z_{2}] + \Big\{ \Big(y_{(5,3)}p\left[y_{(0,1)}, y_{(5,3)}\right] + y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
&- y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{12}Z_{2121}\right]_{1} \Big\}[Z_{1}] + \Big\{ \Big(1 - y_{(6,3)}p\left[y_{(0,1)}, y_{(6,3)}\right] - y_{(5,3)}p\left[y_{(0,1)}, y_{(5,3)}\right] \\
&- y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] + y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] + y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{12}Z_{2121}\right]_{pt} \Big\}[Z_{pt}] \\
[Z_{12}]\,[Z_{12121}] = &\Big\{ y_{(2,1)}\Big(3 - y_{(4,2)}p\left[y_{(1,1)}, y_{(4,2)}\right] - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{12}Z_{12121}\right]_{12} \Big\}[Z_{12}] \\
&+ \Big\{ \Big(1 - y_{(6,3)}p\left[y_{(0,1)}, y_{(6,3)}\right]\Big) + D\left[Z_{12}Z_{12121}\right]_{1} \Big\}[Z_{1}] \\
[Z_{12}]\,[Z_{21212}] = &\Big\{ y_{(3,1)} + D\left[Z_{12}Z_{21212}\right]_{12} \Big\}[Z_{12}] + \Big\{ \Big(1 - y_{(3,1)}p\left[y_{(1,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{12}Z_{21212}\right]_{2} \Big\}[Z_{2}] \\
&+ \Big\{ \Big(1 - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{12}Z_{21212}\right]_{1} \Big\}[Z_{1}] \\
&+ \Big\{ \Big(-1 + y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{12}Z_{21212}\right]_{pt} \Big\}[Z_{pt}]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{21}]\,[Z_{21}] = &\;\frac{y_{R^-}}{y_{(1,0)}y_{(3,1)}}\,[Z_{21}] + \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\Big(3 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \\
&- y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right]\Big)[Z_{2}] \\
[Z_{21}]\,[Z_{121}] = &\;\frac{y_{R^-}}{y_{(1,0)}y_{(2,1)}y_{(3,1)}}\,[Z_{21}] + \Big\{ y_{(3,2)}\Big(y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] \\
&- y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] - 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right]\Big) \\
&+ D\left[Z_{21}Z_{121}\right]_{2} \Big\}[Z_{2}] + \Big\{ y_{(1,1)}y_{(3,2)}\Big(2 - y_{(1,1)}p\left[y_{(1,1)}, y_{(3,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) \\
&+ D\left[Z_{21}Z_{121}\right]_{1} \Big\}[Z_{1}] + \Big\{ y_{(3,2)}\Big(2 - y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] - 2y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] \\
&- 2y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] + 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \\
&+ y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right]\Big) + D\left[Z_{21}Z_{121}\right]_{pt} \Big\}[Z_{pt}] \\
[Z_{21}]\,[Z_{212}] = &\;\frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\,[Z_{21}] + \Big\{ y_{(2,1)}y_{(3,2)}\Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) \\
&+ D\left[Z_{21}Z_{212}\right]_{2} \Big\}[Z_{2}]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{21}]\,[Z_{1212}] = {} & y_{(1,1)}y_{(3,2)}\,[Z_{21}] + \Big\{ \Big(y_{(6,3)}p\left[y_{(1,0)}, y_{(6,3)}\right] + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] \\
& - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{21}Z_{1212}\right]_2 \Big\}\,[Z_2] \\
& + \Big\{ y_{(3,2)}\Big(1 - y_{(1,1)}p\left[y_{(1,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{21}Z_{1212}\right]_1 \Big\}\,[Z_1] \\
& + \Big\{ \Big(1 - y_{(6,3)}p\left[y_{(1,0)}, y_{(6,3)}\right] - y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] \\
& + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{21}Z_{1212}\right]_{pt} \Big\}\,[Z_{pt}] \\
[Z_{21}]\,[Z_{2121}] = {} & \Big\{ y_{(1,1)}y_{(3,2)}\Big(2 - y_{(1,1)}p\left[y_{(1,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{21}Z_{2121}\right]_{21} \Big\}\,[Z_{21}] \\
& + \Big\{ y_{(3,2)}\Big(2 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right]\Big) + D\left[Z_{21}Z_{2121}\right]_2 \Big\}\,[Z_2] \\
[Z_{21}]\,[Z_{12121}] = {} & \Big\{ y_{(1,1)}\Big(3 - y_{(2,2)}p\left[y_{(2,1)}, y_{(2,2)}\right] - y_{(1,1)}p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{21}Z_{12121}\right]_{21} \Big\}\,[Z_{21}] \\
& + \Big\{ \Big(3 - y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(3,3)}p\left[y_{(1,0)}, y_{(3,3)}\right]\Big) + D\left[Z_{21}Z_{12121}\right]_2 \Big\}\,[Z_2] \\
& + \Big\{ \Big(1 - y_{(3,3)}p\left[y_{(3,1)}, y_{(3,3)}\right]\Big) + D\left[Z_{21}Z_{12121}\right]_1 \Big\}\,[Z_1] + \Big\{ \Big(-3 + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] \\
& + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] + y_{(3,3)}p\left[y_{(1,0)}, y_{(3,3)}\right] \Big) + D\left[Z_{21}Z_{12121}\right]_{pt} \Big\}\,[Z_{pt}] \\
[Z_{21}]\,[Z_{21212}] = {} & \Big\{ y_{(3,2)} + D\left[Z_{21}Z_{21212}\right]_{21} \Big\}\,[Z_{21}] + \Big\{ \Big(1 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right]\Big) + D\left[Z_{21}Z_{21212}\right]_2 \Big\}\,[Z_2]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{121}]\,[Z_{121}] = {} & \frac{y_{R^-}}{y_{(1,0)}y_{(2,1)}y_{(3,1)}}\,[Z_{121}] + \Big\{ y_{(3,2)}\Big(y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] \\
& - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] - 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) \\
& + D\left[Z_{121}Z_{121}\right]_{12} \Big\}\,[Z_{12}] + \Big\{ y_{(3,2)}\Big(2 - y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] - 2y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] \\
& - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{121}\right]_1 \Big\}\,[Z_1] \\
[Z_{121}]\,[Z_{212}] = {} & y_{(1,1)}y_{(3,2)}\,[Z_{21}] + y_{(2,1)}y_{(3,1)}\,[Z_{12}] + \Big\{ \Big(y_{(5,2)} - y_{(2,1)} - y_{(3,1)} + y_{(6,3)}p\left[y_{(1,0)}, y_{(6,3)}\right] \\
& + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] \\
& - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{212}\right]_2 \Big\}\,[Z_2] + \Big\{ \Big(- y_{(2,1)}p\left[y_{(1,0)}, y_{(0,1)}\right] + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] \\
& + y_{(5,2)}p\left[y_{(1,0)}, y_{(5,2)}\right] + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(3,1)}p\left[y_{(1,0)}, y_{(3,1)}\right] \\
& - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{212}\right]_1 \Big\}\,[Z_1] + \Big\{ \Big(1 + y_{(2,1)}p\left[y_{(1,0)}, y_{(0,1)}\right] - y_{(6,3)}p\left[y_{(1,0)}, y_{(6,3)}\right] \\
& - 2y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] - y_{(5,2)}p\left[y_{(1,0)}, y_{(5,2)}\right] - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] \\
& + y_{(3,1)}p\left[y_{(1,0)}, y_{(3,1)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{212}\right]_{pt} \Big\}\,[Z_{pt}]
\end{aligned}
$$

$$
\begin{aligned}
\left[Z_{121}\right]\left[Z_{1212}\right] = {} & y_{(1,1)}y_{(3,2)}\left[Z_{121}\right] + \Big\{ \Big(y_{(6,3)}p\left[y_{(1,0)}, y_{(6,3)}\right] + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] \\
& - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{1212}\right]_{12} \Big\} \left[Z_{12}\right] \\
& + \Big\{ \Big(1 - y_{(6,3)}p\left[y_{(1,0)}, y_{(6,3)}\right] - y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] \Big) + D\left[Z_{121}Z_{1212}\right]_{1} \Big\} \left[Z_{1}\right] \\
\left[Z_{121}\right]\left[Z_{2121}\right] = {} & y_{(0,1)}y_{(1,1)}\left[Z_{121}\right] + \Big\{ y_{(1,1)}\Big(3 - y_{(2,2)}p\left[y_{(2,1)}, y_{(2,2)}\right] - y_{(1,1)}p\left[y_{(1,1)}, y_{(2,1)}\right] \\
& - y_{(0,1)}p\left[y_{(0,1)}, y_{(2,1)}\right] \Big) + D\left[Z_{121}Z_{2121}\right]_{21} \Big\} \left[Z_{21}\right] + \Big\{ \Big(y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] \\
& + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] \\
& - 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\left[Z_{121}Z_{2121}\right]_{12} \Big\} \left[Z_{12}\right] \\
& + \Big\{ \Big(3 - y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] \\
& - y_{(3,3)}p\left[y_{(1,0)}, y_{(3,3)}\right] - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] \\
& - y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] + 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] + y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\left[Z_{121}Z_{2121}\right]_{2} \Big\} \left[Z_{2}\right] \\
& + \Big\{ \Big(2 - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] - y_{(3,3)}p\left[y_{(1,0)}, y_{(3,3)}\right] - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] \\
& - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] + 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \\
& + y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\left[Z_{121}Z_{2121}\right]_{1} \Big\} \left[Z_{1}\right] + \Big\{ \Big(-4 + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] \\
& + 2y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] + y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] + y_{(3,3)}p\left[y_{(1,0)}, y_{(3,3)}\right] + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] \\
& + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] - 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \\
& - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\left[Z_{121}Z_{2121}\right]_{pt} \Big\} \left[Z_{pt}\right] \\
\left[Z_{121}\right]\left[Z_{12121}\right] = {} & \Big\{ y_{(1,1)}\Big(3 - y_{(2,2)}p\left[y_{(2,1)}, y_{(2,2)}\right] - y_{(1,1)}p\left[y_{(1,1)}, y_{(2,1)}\right] \Big) + D\left[Z_{121}Z_{12121}\right]_{121} \Big\} \left[Z_{121}\right] \\
& + \Big\{ \Big(3 - y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(3,3)}p\left[y_{(1,0)}, y_{(3,3)}\right] \Big) \\
& + D\left[Z_{121}Z_{12121}\right]_{12} \Big\} \left[Z_{12}\right] + \Big\{ \Big(-2 + y_{(5,3)}p\left[y_{(1,0)}, y_{(5,3)}\right] + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] \Big) \\
& + D\left[Z_{121}Z_{12121}\right]_{1} \Big\} \left[Z_{1}\right] \\
\left[Z_{121}\right]\left[Z_{21212}\right] = {} & y_{(1,1)}\left[Z_{121}\right] + \Big\{ \Big(1 - y_{(1,1)}p\left[y_{(1,1)}, y_{(2,1)}\right] \Big) + D\left[Z_{121}Z_{21212}\right]_{21} \Big\} \left[Z_{21}\right] \\
& + \Big\{ \Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{21212}\right]_{12} \Big\} \left[Z_{12}\right] \\
& + \Big\{ \Big(-2 + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{21212}\right]_{2} \Big\} \left[Z_{2}\right] \\
& + \Big\{ \Big(-2 + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{21212}\right]_{1} \Big\} \left[Z_{1}\right] \\
& + \Big\{ \Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\left[Z_{121}Z_{21212}\right]_{pt} \Big\} \left[Z_{pt}\right]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{212}]\,[Z_{212}] &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}y_{(3,2)}}\,[Z_{212}] + \Big\{ y_{(2,1)}\Big(y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
&\quad - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{212}Z_{212}\right]_{21}\Big\}\,[Z_{21}] \\
&\quad + \Big\{ y_{(2,1)}\Big(2 - y_{(2,1)}p\left[y_{(1,1)}, y_{(2,1)}\right] - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right]\Big) + D\left[Z_{212}Z_{212}\right]_{2}\Big\}\,[Z_2] \\
[Z_{212}]\,[Z_{1212}] &= y_{(1,0)}y_{(3,1)}\,[Z_{212}] + \Big\{\Big(y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
&\quad - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{212}Z_{1212}\right]_{21}\Big\}\,[Z_{21}] + \Big\{ y_{(3,1)}\Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(3,2)}\right]\Big) \\
&\quad + D\left[Z_{212}Z_{1212}\right]_{12}\Big\}\,[Z_{12}] + \Big\{\Big(2 - y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] - y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] \\
&\quad + y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{212}Z_{1212}\right]_{2}\Big\}\,[Z_2] + \Big\{\Big(1 - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] - y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] \\
&\quad + y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{212}Z_{1212}\right]_{1}\Big\}\,[Z_1] + \Big\{\Big(-2 + y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] \\
&\quad + y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) + D\left[Z_{212}Z_{1212}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_{212}]\,[Z_{2121}] &= y_{(2,1)}y_{(3,1)}\,[Z_{212}] + \Big\{\Big(y_{(5,3)}p\left[y_{(0,1)}, y_{(5,3)}\right] + y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
&\quad - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) + D\left[Z_{212}Z_{2121}\right]_{21}\Big\}\,[Z_{21}] \\
&\quad + \Big\{\Big(1 - y_{(5,3)}p\left[y_{(0,1)}, y_{(5,3)}\right]\Big) + D\left[Z_{212}Z_{2121}\right]_{2}\Big\}\,[Z_2] \\
[Z_{212}]\,[Z_{12121}] &= y_{(3,1)}\,[Z_{212}] + \Big\{\Big(2 - y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{212}Z_{12121}\right]_{21}\Big\}\,[Z_{21}] \\
&\quad + \Big\{\Big(1 - y_{(3,1)}p\left[y_{(3,1)}, y_{(3,2)}\right]\Big) + D\left[Z_{212}Z_{12121}\right]_{12}\Big\}\,[Z_{12}] + \Big\{\Big(-3 + y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] \\
&\quad + y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] + y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{212}Z_{12121}\right]_{2}\Big\}\,[Z_2] \\
&\quad + \Big\{\Big(-2 + y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] + y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{212}Z_{12121}\right]_{1}\Big\}\,[Z_1] \\
&\quad + \Big\{\Big(3 - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] - y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) \\
&\quad + D\left[Z_{212}Z_{12121}\right]_{pt}\Big\}\,[Z_{pt}] \\
[Z_{212}]\,[Z_{21212}] &= \Big\{ y_{(3,1)} + D\left[Z_{212}Z_{21212}\right]_{212}\Big\}\,[Z_{212}] + \Big\{\Big(1 - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{212}Z_{21212}\right]_{21}\Big\}\,[Z_{21}] \\
&\quad + \Big\{ D\left[Z_{212}Z_{21212}\right]_{2}\Big\}\,[Z_2]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{1212}]\,[Z_{1212}] = {}& y_{(1,0)}y_{(3,1)}\,[Z_{1212}] + \Big\{ \Big(y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(4,1)}p\left[y_{(0,1)}, y_{(4,1)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
& - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) + D\left[Z_{1212}Z_{1212}\right]_{121} \Big\} [Z_{121}] + \Big\{ \Big(2 - y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] \\
& - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] \Big) + D\left[Z_{1212}Z_{1212}\right]_{12} \Big\} [Z_{12}] + \Big\{ \Big(-1 + y_{(5,2)}p\left[y_{(0,1)}, y_{(5,2)}\right] \Big) \\
& + D\left[Z_{1212}Z_{1212}\right]_{1} \Big\} [Z_{1}] \\
[Z_{1212}]\,[Z_{2121}] = {}& y_{(3,1)}\,[Z_{212}] + y_{(1,1)}\,[Z_{121}] + \Big\{ \Big(2 - y_{(1,1)}p\left[y_{(0,1)}, y_{(2,1)}\right] - y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] \\
& - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \Big) + D\left[Z_{1212}Z_{2121}\right]_{21} \Big\} [Z_{21}] + \Big\{ \Big(2 - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
& - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(0,1)}, y_{(1,1)}\right] \Big) + D\left[Z_{1212}Z_{2121}\right]_{12} \Big\} [Z_{12}] \\
& + \Big\{ \Big(-4 + y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] + y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \\
& + y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(0,1)}, y_{(1,1)}\right] \Big) + D\left[Z_{1212}Z_{2121}\right]_{2} \Big\} [Z_{2}] \\
& + \Big\{ \Big(-3 + y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] + y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] + y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right] \\
& + y_{(1,1)}p\left[y_{(0,1)}, y_{(1,1)}\right] \Big) + D\left[Z_{1212}Z_{2121}\right]_{1} \Big\} [Z_{1}] + \Big\{ \Big(4 - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] \\
& - y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] - y_{(2,1)}p\left[y_{(0,1)}, y_{(2,1)}\right] \\
& - y_{(1,1)}p\left[y_{(0,1)}, y_{(1,1)}\right] \Big) + D\left[Z_{1212}Z_{2121}\right]_{pt} \Big\} [Z_{pt}] \\
[Z_{1212}]\,[Z_{12121}] = {}& y_{(3,1)}\,[Z_{1212}] + \Big\{ \Big(2 - y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] - y_{(3,1)}p\left[y_{(0,1)}, y_{(3,1)}\right] \Big) \\
& + D\left[Z_{1212}Z_{12121}\right]_{121} \Big\} [Z_{121}] + \Big\{ \Big(-2 + y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] + y_{(3,2)}p\left[y_{(0,1)}, y_{(3,2)}\right] \Big) \\
& + D\left[Z_{1212}Z_{12121}\right]_{12} \Big\} [Z_{12}] + \Big\{ \Big(1 - y_{(4,2)}p\left[y_{(0,1)}, y_{(4,2)}\right] \Big) + D\left[Z_{1212}Z_{12121}\right]_{1} \Big\} [Z_{1}] \\
[Z_{1212}]\,[Z_{21212}] = {}& y_{(1,0)}\,[Z_{1212}] + \Big\{ \Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(2,1)}\right] \Big) + D\left[Z_{1212}Z_{21212}\right]_{212} \Big\} [Z_{212}] \\
& + \Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) [Z_{121}] + \Big\{ \Big(-1 + y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) \\
& + D\left[Z_{1212}Z_{21212}\right]_{21} \Big\} [Z_{21}] + \Big\{ \Big(-1 + y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) + D\left[Z_{1212}Z_{21212}\right]_{12} \Big\} [Z_{12}] \\
& + \Big\{ \Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) + D\left[Z_{1212}Z_{21212}\right]_{2} \Big\} [Z_{2}] + \Big\{ \Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) \\
& + D\left[Z_{1212}Z_{21212}\right]_{1} \Big\} [Z_{1}] + \Big\{ \Big(-1 + y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right] \Big) + D\left[Z_{1212}Z_{21212}\right]_{pt} \Big\} [Z_{pt}]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{2121}]\,[Z_{2121}] = {} & y_{(0,1)}y_{(1,1)}\,[Z_{2121}] + \Big\{ \Big(y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] \\
& - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,2)}p\left[y_{(1,0)}, y_{(1,2)}\right] - 2y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) \\
& + D\,[Z_{2121}Z_{2121}]_{212} \Big\}\,[Z_{212}] + \Big\{ \Big(2 - y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] - y_{(4,2)}p\left[y_{(1,0)}, y_{(4,2)}\right] \\
& - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) \\
& + D\,[Z_{2121}Z_{2121}]_{21} \Big\}\,[Z_{21}] + \Big\{ \Big(-1 + y_{(4,3)}p\left[y_{(1,0)}, y_{(4,3)}\right] \Big) + D\,[Z_{2121}Z_{2121}]_{2} \Big\}\,[Z_2] \\
[Z_{2121}]\,[Z_{12121}] = {} & y_{(0,1)}\,[Z_{2121}] + \Big(3 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big)\,[Z_{212}] \\
& + \Big\{ \Big(1 - y_{(0,1)}p\left[y_{(0,1)}, y_{(3,2)}\right] \Big) + D\,[Z_{2121}Z_{12121}]_{121} \Big\}\,[Z_{121}] + \Big\{ \Big(-4 + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] \\
& + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] + y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\,[Z_{2121}Z_{12121}]_{21} \Big\}\,[Z_{21}] \\
& + \Big\{ \Big(-3 + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] + y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) \\
& + D\,[Z_{2121}Z_{12121}]_{12} \Big\}\,[Z_{12}] + \Big\{ \Big(5 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] \\
& - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\,[Z_{2121}Z_{12121}]_{2} \Big\}\,[Z_2] \\
& + \Big\{ \Big(4 - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) \\
& + D\,[Z_{2121}Z_{12121}]_{1} \Big\}\,[Z_1] + \Big\{ \Big(-5 + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] \\
& + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] + y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big) + D\,[Z_{2121}Z_{12121}]_{pt} \Big\}\,[Z_{pt}] \\
[Z_{2121}]\,[Z_{21212}] = {} & y_{(1,1)}\,[Z_{2121}] + \Big\{ \Big(2 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \Big) + D\,[Z_{2121}Z_{21212}]_{212} \Big\}\,[Z_{212}] \\
& + \Big\{ \Big(-1 + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] \Big) + D\,[Z_{2121}Z_{21212}]_{21} \Big\}\,[Z_{21}] + \Big\{ D\,[Z_{2121}Z_{21212}]_{2} \Big\}\,[Z_2]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{12121}]\,[Z_{12121}] = {} & y_{(0,1)}\,[Z_{12121}] + \Big(3 - y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] - y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] \\
& - y_{(0,1)}p\left[y_{(0,1)}, y_{(1,0)}\right] \Big)\,[Z_{1212}] + \Big\{ -3 + y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + y_{(2,1)}p\left[y_{(1,0)}, y_{(2,1)}\right] \\
& + y_{(1,1)}p\left[y_{(1,0)}, y_{(1,1)}\right] + D\,[Z_{12121}Z_{12121}]_{121} \Big\}\,[Z_{121}] \\
& + \Big\{ 2 - y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] - y_{(2,2)}p\left[y_{(1,0)}, y_{(2,2)}\right] + D\,[Z_{12121}Z_{12121}]_{12} \Big\}\,[Z_{12}] \\
& + \Big\{ -1 + y_{(3,2)}p\left[y_{(1,0)}, y_{(3,2)}\right] + D\,[Z_{12121}Z_{12121}]_{1} \Big\}\,[Z_1]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{12121}]\,[Z_{21212}] = [Z_{2121}] + [Z_{1212}] &+ \Big\{ -p\left[y_{(1,0)}, y_{(2,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{212} \Big\} [Z_{212}] \\
&+ \Big\{ -p\left[y_{(1,0)}, y_{(0,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{121} \Big\} [Z_{121}] \\
&+ \Big\{ p\left[y_{(1,0)}, y_{(0,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{21} \Big\} [Z_{21}] \\
&+ \Big\{ p\left[y_{(1,0)}, y_{(0,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{12} \Big\} [Z_{12}] \\
&+ \Big\{ -p\left[y_{(1,0)}, y_{(0,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{2} \Big\} [Z_{2}] \\
&+ \Big\{ -p\left[y_{(1,0)}, y_{(2,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{1} \Big\} [Z_{1}] \\
&+ \Big\{ p\left[y_{(1,0)}, y_{(0,1)}\right] + D\left[Z_{12121}Z_{21212}\right]_{pt} \Big\} [Z_{pt}]
\end{aligned}
$$

$$
\begin{aligned}
[Z_{21212}]\,[Z_{21212}] = y_{(1,0)}\,[Z_{21212}] &+ \Big(1 - y_{(1,0)}p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) [Z_{2121}] + \Big\{ D\left[Z_{21212}Z_{21212}\right]_{212} \Big\} [Z_{212}] \\
&+ \Big\{ D\left[Z_{21212}Z_{21212}\right]_{21} \Big\} [Z_{21}] + \Big\{ D\left[Z_{21212}Z_{21212}\right]_{2} \Big\} [Z_{2}]
\end{aligned}
$$

3.4.1 The Products Involving $[Z_{121212}]$ when Simplified to K-Theory

In this section we give the results of directly applying the algorithm in Section 3.3 to the vertices of $[Z_{121212}]$. We continue to use the notation, $y_{-(\gamma\alpha_1+\delta\alpha_2)} = y_{(\gamma,\delta)}$ and $y_{\gamma\alpha_1+\delta\alpha_2} = y_{-(\gamma,\delta)}$.

$$
\begin{aligned}
L_{121212} &= \frac{K_{21212}}{y_{(2,1)}} + \frac{M_{21212}}{y_{-(2,1)}} \\
&= 1 + \frac{y_{(1,0)}y_{(3,1)}}{y_{(1,1)}}\Big(p\left[y_{-(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\Big) + \frac{D\left[Z_{212}Z_{21212}\right]_{212}}{y_{(2,1)}} \\
K_{121212} &= \frac{\Gamma_{21212}}{y_{(2,1)}} + \frac{L_{21212}}{y_{-(2,1)}} \\
&= 1 + \frac{y_{(1,0)}y_{(3,2)}}{y_{(2,1)}}\Big(p\left[y_{-(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + \frac{D\left[Z_{21}Z_{21212}\right]_{21}}{y_{(2,1)}} \\
\Gamma_{121212} &= \frac{\Delta_{21212}}{y_{(1,1)}} + \frac{K_{21212}}{y_{-(1,1)}} \\
&= 1 + \frac{y_{(2,1)}y_{(2,1)}}{y_{(1,1)}}\Big(p\left[y_{(2,1)}, y_{(2,1)}\right] - 1\Big) + y_{(3,1)}\Big(p\left[y_{-(1,1)}, y_{(1,1)}\right] - p\left[y_{(1,1)}, y_{(3,1)}\right]\Big) \\
&+ \frac{D\left[Z_{2}Z_{21212}\right]_{2}}{y_{(1,1)}} + \frac{D\left[Z_{212}Z_{21212}\right]_{212}}{y_{-(1,1)}} \\
\Delta_{121212} &= \frac{\Delta_{21212}}{y_{(1,0)}} + \frac{\Gamma_{21212}}{y_{-(1,0)}} \\
&= 1 + \frac{y_{(2,1)}y_{(2,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)}, y_{(2,1)}\right] - 1\Big) + y_{(3,1)}\Big(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(3,2)}\right]\Big) \\
&+ \frac{D\left[Z_{2}Z_{21212}\right]_{2}}{y_{(1,0)}} + \frac{D\left[Z_{21}Z_{21212}\right]_{21}}{y_{-(1,0)}}
\end{aligned}
$$

Therefore in the case of K-Theory $[Z_{121212}]$ is exactly the multiplicative identity.

Chapter 4

The Cancelling Terms, $D\left[Z_\gamma Z_\mu\right]_w$

In manipulating the terms which cancel in K-Theory so that they are clearly well-defined we obtain sums of fractions that must be well-defined. This provides a source of expressions which despite appearances are divisible. We give a couple of examples below, including why they are well defined.

The re-expressed cancelling expressions are listed in Section 4.1 and those which have not been re-expressed are listed in Appendix B.

Since these manipulations were done from the computer outputs from Chapter 3 so we continue to use the notation introduced in Section 3.4, $y_{-(\gamma\alpha_1+\delta\alpha_2)} = y_{(\gamma,\delta)}$ and $y_{\gamma\alpha_1+\delta\alpha_2} = y_{-(\gamma,\delta)}$.

For example the coefficient of $[Z_{pt}]$ in $[Z_{pt}]\,[Z_{2121}]$ has the terms, (which are the only terms with fractions):

$$y_{(2,1)}y_{(3,2)}y_{(1,1)}\left\{\frac{y_{(1,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(1,1)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\Big)+\frac{y_{(2,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(2,1)}\right]\Big)\right\}$$

This suggests that $p\left[y_\gamma,y_\beta\right]-p\left[y_\gamma,y_{\beta+\alpha}\right]$ is divisible by y_α, which is true since:

$$\begin{aligned}
p\left[y_\gamma,y_\beta\right]-p\left[y_\gamma,y_{\beta+\alpha}\right] &= \sum_{j,k\geq 1} a_{jk}y_\gamma^{j-1}y_\beta^{k-1} - \sum_{j,k\geq 1} a_{jk}y_\gamma^{j-1}y_{\beta+\alpha}^{k-1}\\
&= \sum_{j,k\geq 1} a_{jk}y_\gamma^{j-1}y_\beta^{k-1} - \sum_{j,k\geq 1} a_{jk}y_\gamma^{j-1}\left(y_\alpha+y_\beta-y_\alpha y_\beta p\left[y_\alpha,y_\beta\right]\right)^{k-1}\\
&= \sum_{j,k\geq 1} a_{jk}y_\gamma^{j-1}\left(y_\beta^{k-1}-\left(y_\alpha+y_\beta-y_\alpha y_\beta p\left[y_\alpha,y_\beta\right]\right)^{k-1}\right)\\
&= \sum_{j\geq 1,k>1} a_{jk}y_\gamma^{j-1}\left(y_\beta^{k-1}-\sum_{i=0}^{k-1}\binom{k-1}{i}y_\beta^{k-1-i}\left(y_\alpha-y_\alpha y_\beta p\left[y_\alpha,y_\beta\right]\right)^{i}\right)\\
&= \sum_{j\geq 1,k>1} a_{jk}y_\gamma^{j-1}\left(y_\beta^{k-1}-y_\beta^{k-1}-\sum_{i=1}^{k-1}\binom{k-1}{i}y_\beta^{k-1-i}\left(y_\alpha-y_\alpha y_\beta p\left[y_\alpha,y_\beta\right]\right)^{i}\right)\\
&= -\sum_{j\geq 1,k>1} a_{jk}y_\gamma^{j-1}\left(\sum_{i=1}^{k-1}\binom{k-1}{i}y_\beta^{k-1-i}\left(y_\alpha-y_\alpha y_\beta p\left[y_\alpha,y_\beta\right]\right)^{i}\right)\\
&= -y_\alpha\sum_{j\geq 1,k>1} a_{jk}y_\gamma^{j-1}\left(\sum_{i=1}^{k-1}\binom{k-1}{i}y_\beta^{k-1-i}y_\alpha^{i-1}\left(1-y_\beta p\left[y_\alpha,y_\beta\right]\right)^{i}\right)
\end{aligned}$$

From this we obtain $p\left[y_\gamma, y_\beta\right] - p\left[y_{\gamma+m\alpha}, y_{\beta+n\alpha}\right]$ is also divisible by y_α since:

$$\begin{aligned} p\left[y_\gamma, y_\beta\right] - p\left[y_{\gamma+m\alpha}, y_{\beta+n\alpha}\right] &= \sum_{i=1}^{m}\left(p\left[y_{\gamma+(i-1)\alpha}, y_\beta\right] - p\left[y_{\gamma+i\alpha}, y_\beta\right]\right) + p\left[y_{\gamma+m\alpha}, y_\beta\right] - p\left[y_{\gamma+m\alpha}, y_{\beta+n\alpha}\right] \\ &= \sum_{i=1}^{m}\left(p\left[y_{\gamma+(i-1)\alpha}, y_\beta\right] - p\left[y_{\gamma+i\alpha}, y_\beta\right]\right) \\ &\quad + \sum_{i=1}^{n}\left(p\left[y_{\gamma+m\alpha}, y_{\beta+(i-1)\alpha}\right] - p\left[y_{\gamma+m\alpha}, y_{\beta+i\alpha}\right]\right) \end{aligned}$$

Another example comes from the coefficient of $[Z_1]$ in $[Z_{21}]\,[Z_{1212}]$ which has the fractions

$$y_{(3,2)}y_{(2,1)}\left\{\frac{y_{(2,1)}}{y_{(3,1)}}\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right) + \frac{y_{(1,1)}}{y_{(3,1)}}\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right)\right\}$$

which using the divisible expressions above and the formal group law we obtain:

$$\begin{aligned} & y_{(3,2)}y_{(2,1)}\left\{\frac{\left(y_{(3,1)} + y_{-(1,0)} - y_{(3,1)}y_{-(1,0)}p\left[y_{(3,1)}, y_{-(1,0)}\right]\right)}{y_{(3,1)}}\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right)\right. \\ & \left. + \frac{\left(y_{(3,1)} + y_{-(2,0)} - y_{(3,1)}y_{-(2,0)}p\left[y_{(3,1)}, y_{-(2,0)}\right]\right)}{y_{(3,1)}}\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right)\right\} \\ & = y_{(3,2)}y_{(2,1)}\left\{\left(1 - y_{-(1,0)}p\left[y_{(3,1)}, y_{-(1,0)}\right]\right)\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right)\right. \\ & + \frac{y_{-(1,0)}}{y_{(3,1)}}\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right) + \frac{y_{-(2,0)}}{y_{(3,1)}}\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right) \\ & \left. + \left(1 - y_{-(2,0)}p\left[y_{(3,1)}, y_{-(2,0)}\right]\right)\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right)\right\} \\ & = y_{(3,2)}y_{(2,1)}\left\{\left(1 - y_{-(1,0)}p\left[y_{(3,1)}, y_{-(1,0)}\right]\right)\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right)\right. \\ & + \left(1 - y_{-(2,0)}p\left[y_{(3,1)}, y_{-(2,0)}\right]\right)\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right) \\ & + \frac{y_{-(1,0)}\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{-(1,0)}, y_{-(1,0)}\right]\right)}{y_{(3,1)}} + \frac{y_{-(1,0)}\left(p\left[y_{-(1,0)}, y_{-(1,0)}\right] - p\left[y_{(1,0)}, y_{-(1,0)}\right]\right)}{y_{(3,1)}} \\ & + \frac{y_{-(1,0)}\left(p\left[y_{(1,0)}, y_{-(1,0)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right)}{y_{(3,1)}} + \frac{y_{-(2,0)}\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{-(2,0)}\right]\right)}{y_{(3,1)}} \\ & \left. + \frac{y_{-(2,0)}\left(p\left[y_{(1,0)}, y_{-(2,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right)}{y_{(3,1)}}\right\} \end{aligned}$$

Then the only terms which in which the divisibility isn't understood are:

$$y_{(3,2)}y_{(2,1)}\left\{\frac{y_{-(1,0)}\left(p\left[y_{-(1,0)}, y_{-(1,0)}\right] - p\left[y_{(1,0)}, y_{-(1,0)}\right]\right)}{y_{(3,1)}} + \frac{y_{-(2,0)}\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{-(2,0)}\right]\right)}{y_{(3,1)}}\right\}$$

Terms with numerators of the form $y_{2\alpha}\left(p\left[y_\alpha, y_{-\alpha}\right] - p\left[y_{2\alpha}, y_{-\alpha}\right]\right) + y_\alpha\left(p\left[y_\alpha, y_\alpha\right] - p\left[y_\alpha, y_{-\alpha}\right]\right)$ also occur in other coefficients with a different denominators, leading to the idea that the numerator is zero, which is the case by:

$$
\begin{aligned}
\begin{matrix} y_{2\alpha}\left(p\left[y_\alpha, y_{-\alpha}\right]-p\left[y_{2\alpha}, y_{-\alpha}\right]\right) \\ +y_\alpha\left(p\left[y_\alpha, y_\alpha\right]-p\left[y_\alpha, y_{-\alpha}\right]\right) \end{matrix} &= y_\alpha\left(p\left[y_\alpha, y_\alpha\right]-p\left[y_\alpha, y_{-\alpha}\right]\right)-y_{2\alpha} p\left[y_{2\alpha}, y_{-\alpha}\right]+p\left[y_\alpha, y_{-\alpha}\right]\left(2y_\alpha-y_\alpha^2 p\left[y_\alpha, y_\alpha\right]\right) \\
&= y_\alpha p\left[y_\alpha, y_\alpha\right]-y_{2\alpha} p\left[y_{2\alpha}, y_{-\alpha}\right]+y_\alpha p\left[y_\alpha, y_{-\alpha}\right]\left(1-y_\alpha p\left[y_\alpha, y_\alpha\right]\right) \\
&= -y_{2\alpha} p\left[y_{2\alpha}, y_{-\alpha}\right]+y_\alpha p\left[y_\alpha, y_\alpha\right]+y_\alpha p\left[y_\alpha, y_{-\alpha}\right]-y_\alpha^2 p\left[y_\alpha, y_\alpha\right] p\left[y_\alpha, y_{-\alpha}\right] \\
&= -y_{2\alpha} p\left[y_{2\alpha}, y_{-\alpha}\right]+y_\alpha p\left[y_\alpha, y_{-\alpha}\right]+y_\alpha p\left[y_\alpha, y_\alpha\right]\left(1-y_\alpha p\left[y_\alpha, y_{-\alpha}\right]\right) \\
&= \frac{1}{y_{-\alpha}}\Big(-y_{-\alpha} y_{2\alpha} p\left[y_{2\alpha}, y_{-\alpha}\right]+y_{-\alpha} y_\alpha p\left[y_\alpha, y_{-\alpha}\right] \\
&\qquad +y_\alpha p\left[y_\alpha, y_\alpha\right]\left(y_{-\alpha}-y_{-\alpha} y_\alpha p\left[y_\alpha, y_{-\alpha}\right]\right)\Big) \\
&= \frac{1}{y_{-\alpha}}\left(y_\alpha-y_{-\alpha}-y_{2\alpha}+y_{-\alpha}+y_\alpha+y_\alpha p\left[y_\alpha, y_\alpha\right]\left(-y_\alpha\right)\right) \\
&= \frac{1}{y_{-\alpha}}\left(y_\alpha-y_{2\alpha}+y_\alpha-y_\alpha^2 p\left[y_\alpha, y_\alpha\right]\right) \\
&= 0
\end{aligned}
$$

4.1 The Expressions $D\left[Z_\gamma Z_\mu\right]_w$ with only understood fractions

The expressions which are zero in the case of K-Theory and have been written so they are obviously well-defined are listed here. The only fractions included are well-defined from relations on page 36. All of these were manipulated by hand from the computer outputs from the algorithm in Section 3.3. Those that have not been re-written so they are clearly well-defined are listed in Appendix B.
The $D\left[Z_\gamma, Z_\mu\right]_w$ not included here are listed in page 35 and their divisions have not been studied in this work.

$$
\begin{aligned}
D\left[Z_{pt} Z_{121}\right]_{pt} &= \frac{y_{R^-}}{y_{(1,0)} y_{(3,1)}}\Big(p\left[y_{-(1,0)}, y_{(1,0)}\right]-p\left[y_{(0,1)}, y_{(1,0)}\right]\Big) \\
D\left[Z_{pt} Z_{212}\right]_{pt} &= \frac{y_{R^-}}{y_{(0,1)} y_{(1,1)}}\Big(p\left[y_{-(0,1)}, y_{(0,1)}\right]-p\left[y_{(1,0)}, y_{(0,1)}\right]\Big) \\
D\left[Z_{pt} Z_{1212}\right]_{pt} &= y_{(2,1)} y_{(3,2)}\Big\{y_{(1,1)}\Big(p\left[y_{-(1,0)}, y_{(1,0)}\right]-p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) \\
&\qquad + y_{(3,1)}\Big(p\left[y_{-(0,1)}, y_{(0,1)}\right]-p\left[y_{(1,0)}, y_{(0,1)}\right]\Big)\Big\} \\
D\left[Z_{pt} Z_{2121}\right]_{pt} &= y_{(2,1)} y_{(3,2)}\Bigg\{y_{(1,1)} y_{(2,1)}\left(\frac{p\left[y_{(1,1)}, y_{(2,1)}\right]-p\left[y_{(1,0)}, y_{(2,1)}\right]}{y_{(0,1)}}\right) \\
&\qquad +\left(y_{(1,1)}\right)^2\left(\frac{p\left[y_{(1,1)}, y_{(1,1)}\right]-p\left[y_{(1,0)}, y_{(1,1)}\right]}{y_{(0,1)}}\right)+y_{(3,1)}\Big(p\left[y_{-(0,1)}, y_{(0,1)}\right]-p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) \\
&\qquad + y_{(1,1)}\Big(p\left[y_{-(1,0)}, y_{(1,0)}\right]-p\left[y_{(0,1)}, y_{(1,0)}\right]\Big)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{pt}Z_{21212}\right]_{pt} = y_{(2,1)}\Big\{ & y_{(2,1)}y_{(3,1)}\Big(p\left[y_{(3,1)},y_{(2,1)}\right] - p\left[y_{-(0,1)},y_{(0,1)}\right]\Big) + y_{(5,2)}\Big(1 - p\left[y_{(0,1)},y_{(5,2)}\right]\Big) \\
& - y_{(3,1)}y_{(3,2)}\Big(y_{(1,0)}p\left[y_{(1,0)},y_{(0,1)}\right] + 1\Big)\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right] - p\left[y_{(1,0)},y_{(0,1)}\right]}{y_{(1,1)}}\right) \\
& + y_{(3,1)}y_{(3,2)}y_{(1,0)}p\left[y_{(0,1)},y_{-(0,1)}\right]\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right] - p\left[y_{(1,0)},y_{(0,1)}\right]}{y_{(1,1)}}\right) \\
& + y_{(1,0)}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right] - p\left[y_{(1,0)},y_{(1,1)}\right]\Big)\Big(1 + y_{(3,1)}p\left[y_{(0,1)},y_{-(0,1)}\right] - y_{(3,1)}p\left[y_{(3,1)},y_{(0,1)}\right] \\
& - y_{(3,2)}p\left[y_{(1,0)},y_{(0,1)}\right]\Big) + y_{(1,0)}y_{(3,1)}\left(\frac{p\left[y_{-(1,1)},y_{(1,1)}\right] - p\left[y_{-(1,0)},y_{(1,0)}\right]}{y_{-(0,1)}}\right) \\
& + y_{(2,1)}y_{(3,1)}\left(\frac{p\left[y_{(1,1)},y_{(2,1)}\right] - p\left[y_{(1,0)},y_{(2,1)}\right]}{y_{(0,1)}}\right) \\
& + y_{(2,1)}y_{(3,1)}\left(\frac{p\left[y_{(2,1)},y_{(3,2)}\right] - p\left[y_{(2,1)},y_{(3,1)}\right]}{y_{(0,1)}}\right) \\
& + y_{(2,1)}y_{(3,1)}\left(p\left[y_{(3,2)},y_{(2,1)}\right] - p\left[y_{(1,0)},y_{(2,1)}\right]\right)p\left[y_{(3,1)},y_{(0,1)}\right] \\
& + y_{(2,1)}\left(p\left[y_{(3,2)},y_{(2,1)}\right] - p\left[y_{(1,0)},y_{(2,1)}\right]\right) + y_{(2,1)}y_{(3,1)}\left(1 - p\left[y_{(1,1)},y_{(2,1)}\right]\right)p\left[y_{-(0,1)},y_{(0,1)}\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_1Z_{121}\right]_1 &= D\left[Z_{pt}Z_{121}\right]_{pt} \\
D\left[Z_1Z_{212}\right]_{pt} &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(1,1)}}\Big(p\left[y_{-(0,1)},y_{(0,1)}\right] - p\left[y_{(0,1)},y_{(1,0)}\right]\Big) \\
D\left[Z_1Z_{1212}\right]_1 &= D\left[Z_{pt}Z_{1212}\right]_{pt} \\
D\left[Z_1Z_{21212}\right]_1 &= D\left[Z_{21}Z_{1212}\right]_2 + y_{(1,1)}y_{(3,2)}\Big(p\left[y_{-(3,1)},y_{(3,1)}\right] - p\left[y_{(1,1)},y_{(3,1)}\right]\Big)
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_2Z_{121}\right]_{pt} &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\Bigg\{y_{(1,1)}\left(\frac{p\left[y_{(1,1)},y_{(1,1)}\right] - p\left[y_{(1,0)},y_{(1,1)}\right]}{y_{(0,1)}}\right) \\
&\quad + y_{(2,1)}\left(\frac{p\left[y_{(1,1)},y_{(2,1)}\right] - p\left[y_{(1,0)},y_{(2,1)}\right]}{y_{(0,1)}}\right) + \left(p\left[y_{-(1,0)},y_{(1,0)}\right] - p\left[y_{(1,0)},y_{(0,1)}\right]\right)\Bigg\} \\
D\left[Z_2Z_{212}\right]_2 &= D\left[Z_{pt}Z_{212}\right]_{pt} \\
D\left[Z_2Z_{1212}\right]_2 &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}y_{(3,2)}}\Bigg\{\Big(p\left[y_{-(1,1)},y_{(1,1)}\right] - p\left[y_{(1,0)},y_{(1,1)}\right]\Big) \\
&\quad + y_{(3,2)}\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right] - p\left[y_{(1,0)},y_{(0,1)}\right]}{y_{(1,1)}}\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_2Z_{1212}\right]_{pt} = y_{(2,1)}\Big\{ & y_{(2,1)}\left(1-y_{(3,1)}p\left[y_{(3,1)},y_{(0,1)}\right]\right)\left(p\left[y_{(3,2)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(2,1)}\right]\right)\\
&+y_{(2,1)}y_{(3,1)}\left(\frac{p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(2,1)}\right]}{y_{(0,1)}}\right)+y_{(2,1)}y_{(3,1)}\left(\frac{p\left[y_{(3,2)},y_{(2,1)}\right]-p\left[y_{(3,1)},y_{(2,1)}\right]}{y_{(0,1)}}\right)\\
&+y_{(3,2)}\left(1-y_{(1,0)}p\left[y_{(1,0)},y_{(0,1)}\right]\right)\left(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\right)\\
&+y_{(1,0)}\left(1-y_{(3,1)}p\left[y_{(3,1)},y_{(0,1)}\right]\right)\left(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\right)\\
&+y_{(3,1)}y_{(3,2)}\left(1-y_{(1,0)}p\left[y_{(1,0)},y_{(0,1)}\right]\right)\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right]-p\left[y_{(0,1)},y_{(1,0)}\right]}{y_{(1,1)}}\right)\\
&+y_{(1,0)}y_{(3,1)}\left(\frac{p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{-(1,1)},y_{(1,1)}\right]}{y_{(0,1)}}\right)\Big\}\\
D\left[Z_2Z_{2121}\right]_2 = {} & D\left[Z_{pt}Z_{2121}\right]_{pt}\\
D\left[Z_2Z_{12121}\right]_2 = y_{(2,1)}\Big\{ & y_{(2,1)}y_{(3,2)}\left(1-y_{(1,0)}p\left[y_{(1,0)},y_{(1,1)}\right]\right)\left(p\left[y_{(3,2)},y_{(2,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\right)\\
&+y_{(2,1)}y_{(3,2)}\left(1-y_{(2,0)}p\left[y_{(2,0)},y_{(1,1)}\right]\right)\left(p\left[y_{-(0,1)},y_{(0,1)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\right)\\
&+y_{(3,1)}\left(p\left[y_{-(1,1)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\right)+y_{(1,0)}y_{(2,1)}y_{(3,2)}\left(\frac{p\left[y_{(3,2)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(1,0)}\right]}{y_{(1,1)}}\right)\\
&+y_{(1,0)}y_{(2,1)}y_{(3,2)}\left(\frac{p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]}{y_{(1,1)}}\right)\\
&+y_{(2,0)}y_{(2,1)}y_{(3,2)}\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]}{y_{(1,1)}}\right)\\
&+y_{(2,0)}y_{(2,1)}y_{(3,2)}\left(\frac{p\left[y_{-(1,0)},y_{(2,0)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]}{y_{(1,1)}}\right)\Big\}\\
D\left[Z_2Z_{21212}\right]_2 = {} & D\left[Z_{pt}Z_{21212}\right]_{pt}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{12}Z_{21}\right]_{pt} &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\Big\{y_{(1,1)}\left(\frac{p\left[y_{(1,1)},y_{(1,1)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]}{y_{(0,1)}}\right)\\
&\quad+y_{(2,1)}\left(\frac{p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(2,1)}\right]}{y_{(0,1)}}\right)\Big\}\\
D\left[Z_{12}Z_{121}\right]_1 &= D\left[Z_2Z_{121}\right]_{pt}\\
D\left[Z_{12}Z_{212}\right]_2 &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}}\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right]-p\left[y_{(0,1)},y_{(1,0)}\right]}{y_{(1,1)}}\right)\\
D\left[Z_{12}Z_{212}\right]_1 &= y_{(2,1)}\Big\{y_{(1,0)}\left(1-p\left[y_{(1,1)},y_{(3,2)}\right]\right)\left(1-y_{(3,2)}p\left[y_{(1,0)},y_{(0,1)}\right]-y_{(3,1)}p\left[y_{(3,1)},y_{(0,1)}\right]\right)\\
&\quad+y_{(3,2)}\left(1-p\left[y_{(1,1)},y_{(3,2)}\right]\right)+y_{(1,0)}y_{(3,1)}\left(\frac{p\left[y_{(1,0)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(3,2)}\right]}{y_{(0,1)}}\right)\Big\}\\
D\left[Z_{12}Z_{1212}\right]_{12} &= \frac{y_{R^-}}{y_{(0,1)}y_{(1,1)}y_{(3,2)}}\Big\{\left(p\left[y_{-(1,1)},y_{(1,1)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\right)\\
&\quad+\left(\frac{p\left[y_{-(0,1)},y_{(0,1)}\right]-p\left[y_{(1,0)},y_{(0,1)}\right]}{y_{(1,1)}}\right)\Big\}
\end{aligned}
$$

$$D\left[Z_{12}Z_{1212}\right]_1 = D\left[Z_2 Z_{1212}\right]_{pt}$$
$$D\left[Z_{12}Z_{2121}\right]_2 = D\left[Z_2 Z_{12121}\right]_2 - y_{(3,1)}\left(p\left[y_{-(1,1)}, y_{(1,1)}\right] - p\left[y_{(1,1)}, y_{(3,1)}\right]\right)$$

$$\begin{aligned}
D\left[Z_{21}Z_{212}\right]_2 &= D\left[Z_1 Z_{212}\right]_{pt} \\
D\left[Z_{21}Z_{1212}\right]_1 &= y_{(3,2)}y_{(2,1)}\Big\{ \left(1 - y_{-(1,0)}p\left[y_{(3,1)}, y_{-(1,0)}\right]\right)\left(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\right) \\
&+ \left(1 - y_{-(2,0)}p\left[y_{(3,1)}, y_{-(2,0)}\right]\right)\left(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\right) \\
&+ y_{-(1,0)}\left(\frac{p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{-(1,0)}, y_{-(1,0)}\right]}{y_{(3,1)}}\right) + \left(p\left[y_{-(0,1)}, y_{(0,1)}\right] - p\left[y_{(1,0)}, y_{(0,1)}\right]\right) \\
&+ y_{-(1,0)}\left(\frac{p\left[y_{(1,0)}, y_{-(1,0)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]}{y_{(3,1)}}\right) + y_{-(2,0)}\left(\frac{p\left[y_{(1,0)}, y_{-(2,0)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]}{y_{(3,1)}}\right)\Big\} \\
D\left[Z_{21}Z_{21212}\right]_{21} &= y_{(3,2)}y_{(1,1)}\Big(p\left[y_{-(3,1)}, y_{(3,1)}\right] - p\left[y_{(1,1)}, y_{(3,1)}\right]\Big) + D\left[Z_{21}Z_{1212}\right]_1
\end{aligned}$$

Appendix A

The Linear Expansions from Direct Application of Formula in Subsection 2.2.3

The expansions of products as computed from Subsection 2.2.3 are listed here, where the expansions not listed can be obtained by the interchanging 1 and 2.

$$
\begin{aligned}
Z_{pt}Z_{pt} &= y_{R^-}Z_{pt}\\
Z_{pt}Z_1 &= \Delta_1 Z_{pt}\\
Z_{pt}Z_2 &= \Delta_2 Z_{pt}\\
Z_{pt}Z_{12} &= \Delta_{12} Z_{pt}\\
Z_{pt}Z_{21} &= \Delta_{21} Z_{pt}\\
Z_{pt}Z_{121} &= \Delta_{121} Z_{pt}\\
Z_{pt}Z_{212} &= \Delta_{212} Z_{pt}\\
Z_{pt}Z_{1212} &= \Delta_{1212} Z_{pt}\\
Z_{pt}Z_{2121} &= \Delta_{2121} Z_{pt}\\
Z_{pt}Z_{12121} &= \Delta_{12121} Z_{pt}\\
Z_{pt}Z_{21212} &= \Delta_{21212} Z_{pt}
\end{aligned}
$$

$$
\begin{aligned}
Z_1Z_1 &= \Delta_1 Z_1\\
Z_1Z_2 &= \frac{1}{y_{R^-}}\left\{\Delta_1\Delta_2\right\} Z_{pt}\\
Z_1Z_{12} &= \Delta_{12} Z_1\\
Z_1Z_{21} &= \Gamma_{21}Z_1 + \frac{1}{y_{R^-}}\left\{\Delta_{21}\Delta_1 - \Delta_1\Gamma_{21}\right\} Z_{pt}\\
Z_1Z_{121} &= \Delta_{121} Z_1\\
Z_1Z_{212} &= \Gamma_{212}Z_1 + \frac{1}{y_{R^-}}\left\{\Delta_{212}\Delta_1 - \Delta_1\Gamma_{212}\right\} Z_{pt}\\
Z_1Z_{1212} &= \Delta_{1212} Z_1\\
Z_1Z_{2121} &= \Gamma_{2121}Z_1 + \frac{1}{y_{R^-}}\left\{\Delta_{2121}\Delta_1 - \Delta_1\Gamma_{2121}\right\} Z_{pt}
\end{aligned}
$$

$$Z_1 Z_{12121} = \Delta_{12121} Z_1$$
$$Z_1 Z_{21212} = \Gamma_{21212} Z_1 + \frac{1}{y_{R^-}} \left\{\Delta_{21212}\Delta_1 - \Delta_1\Gamma_{21212}\right\} Z_{pt}$$

$$\begin{aligned}
Z_{12}Z_{12} &= \Gamma_{12}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Delta_{12} - \Delta_{12}\Gamma_{12}\right\} Z_1 \\
Z_{12}Z_{21} &= \frac{1}{\Delta_1}\left\{\Delta_{12}\Gamma_{21}\right\} Z_1 + \frac{1}{\Delta_2}\left\{\Delta_{21}\Gamma_{12}\right\} Z_2 + \frac{1}{y_{R^-}}\left\{\Delta_{21}\Delta_{12} - \Delta_{12}\Gamma_{21} - \Delta_{21}\Gamma_{12}\right\} Z_{pt} \\
Z_{12}Z_{121} &= \Gamma_{121}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Delta_{121} - \Delta_{12}\Gamma_{121}\right\} Z_1 \\
Z_{12}Z_{212} &= K_{212}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Gamma_{212} - \Delta_{12}K_{212}\right\} Z_1 + \frac{1}{\Delta_2}\left\{\Gamma_{12}\Delta_{212} - \Gamma_{12}K_{212}\right\} Z_2 \\
&\quad + \frac{1}{y_{R^-}}\left\{\Delta_{12}\Delta_{212} - \Delta_{12}\Gamma_{212} - \Gamma_{12}\Delta_{212} + \Gamma_{12}K_{212}\right\} Z_{pt} \\
Z_{12}Z_{1212} &= \Gamma_{1212}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Delta_{1212} - \Delta_{12}\Gamma_{1212}\right\} Z_1 \\
Z_{12}Z_{2121} &= K_{2121}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Gamma_{2121} - \Delta_{12}K_{2121}\right\} Z_1 + \frac{1}{\Delta_2}\left\{\Gamma_{12}\Delta_{2121} - \Gamma_{12}K_{2121}\right\} Z_2 \\
&\quad + \frac{1}{y_{R^-}}\left\{\Delta_{12}\Delta_{2121} - \Delta_{12}\Gamma_{2121} - \Gamma_{12}\Delta_{2121} + \Gamma_{12}K_{2121}\right\} Z_{pt} \\
Z_{12}Z_{12121} &= \Gamma_{12121}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Delta_{12121} - \Delta_{12}\Gamma_{12121}\right\} Z_1 \\
Z_{12}Z_{21212} &= K_{21212}Z_{12} + \frac{1}{\Delta_1}\left\{\Delta_{12}\Gamma_{21212} - \Delta_{12}K_{21212}\right\} Z_1 + \frac{1}{\Delta_2}\left\{\Gamma_{12}\Delta_{21212} - \Gamma_{12}K_{21212}\right\} Z_2 \\
&\quad + \frac{1}{y_{R^-}}\left\{\Delta_{12}\Delta_{21212} - \Delta_{12}\Gamma_{21212} - \Gamma_{12}\Delta_{21212} + \Gamma_{12}K_{21212}\right\} Z_{pt}
\end{aligned}$$

$$\begin{aligned}
Z_{121}Z_{121} &= K_{121}Z_{121} + \frac{1}{\Gamma_{12}}\left\{\Gamma_{121}\Gamma_{121} - \Gamma_{121}K_{121}\right\}Z_{12} \\
&\quad + \frac{1}{\Delta_1}\left\{\Delta_{121}\Delta_{121} - \Delta_{121}K_{121} - \frac{\Delta_{12}}{\Gamma_{12}}\left(\Gamma_{121}\Gamma_{121} - \Gamma_{121}K_{121}\right)\right\} Z_1 \\
Z_{121}Z_{212} &= \frac{1}{\Gamma_{12}}\left\{\Gamma_{121}K_{212}\right\}Z_{12} + \frac{1}{\Gamma_{21}}\left\{\Gamma_{212}K_{121}\right\}Z_{21} \\
&\quad + \frac{1}{\Delta_1}\left\{\Delta_{121}\Gamma_{212} - K_{121}\Gamma_{212} - \frac{\Delta_{12}}{\Gamma_{12}}\left(\Gamma_{121}K_{212}\right)\right\} Z_1 \\
&\quad + \frac{1}{\Delta_2}\left\{\Delta_{212}\Gamma_{121} - K_{212}\Gamma_{121} - \frac{\Delta_{21}}{\Gamma_{21}}\left(\Gamma_{212}K_{121}\right)\right\} Z_2 \\
&\quad + \frac{1}{y_{R^-}}\left\{\Delta_{121}\Delta_{212} - \Delta_{121}\Gamma_{212} + K_{121}\Gamma_{212} + K_{212}\Gamma_{121} - \Gamma_{121}\Delta_{212}\right\} Z_{pt} \\
Z_{121}Z_{1212} &= K_{1212}Z_{121} + \frac{1}{\Gamma_{12}}\left\{\Gamma_{121}\Gamma_{1212} - \Gamma_{121}K_{1212}\right\}Z_{12} \\
&\quad + \frac{1}{\Delta_1}\left\{\Delta_{121}\Delta_{1212} - \Delta_{121}K_{1212} - \frac{\Delta_{12}}{\Gamma_{12}}\left(\Gamma_{121}\Gamma_{1212} - \Gamma_{121}K_{1212}\right)\right\} Z_1
\end{aligned}$$

$$
\begin{aligned}
Z_{121}Z_{2121} &= L_{2121}Z_{121} + \frac{1}{\Gamma_{12}}\Big\{\Gamma_{121}K_{2121} - \Gamma_{121}L_{2121}\Big\}Z_{12} + \frac{1}{\Gamma_{21}}\Big\{K_{121}\Gamma_{2121} - K_{121}L_{2121}\Big\}Z_{21} \\
&+ \frac{1}{\Delta_1}\Big\{\Delta_{121}\Gamma_{2121} - K_{121}\Gamma_{2121} - \Delta_{121}L_{2121} + K_{121}L_{2121} - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{121}K_{2121} - \Gamma_{121}L_{2121}\Big)\Big\}Z_1 \\
&+ \frac{1}{\Delta_2}\Big\{\Gamma_{121}\Delta_{2121} - \Gamma_{121}K_{2121} - \frac{\Delta_{21}}{\Gamma_{21}}\Big(K_{121}\Gamma_{2121} - K_{121}L_{2121}\Big)\Big\}Z_2 \\
&+ \frac{1}{y_{R^-}}\Big\{\Delta_{121}\Delta_{2121} - \Delta_{121}\Gamma_{2121} + K_{121}\Gamma_{2121} + K_{2121}\Gamma_{121} - K_{121}L_{2121} - \Gamma_{121}\Delta_{2121}\Big\}Z_{pt} \\
Z_{121}Z_{12121} &= K_{12121}Z_{121} + \frac{1}{\Gamma_{12}}\Big\{\Gamma_{121}\Gamma_{12121} - \Gamma_{121}K_{12121}\Big\}Z_{12} \\
&+ \frac{1}{\Delta_1}\Big\{\Delta_{121}\Delta_{12121} - \Delta_{121}K_{12121} - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{121}\Gamma_{12121} - \Gamma_{121}K_{12121}\Big)\Big\}Z_1 \\
Z_{121}Z_{21212} &= L_{21212}Z_{121} + \frac{1}{\Gamma_{12}}\Big\{\Gamma_{121}K_{21212} - \Gamma_{121}L_{21212}\Big\}Z_{12} + \frac{1}{\Gamma_{21}}\Big\{K_{121}\Gamma_{21212} - K_{121}L_{21212}\Big\}Z_{21} \\
&+ \frac{1}{\Delta_1}\Big\{\Delta_{121}\Gamma_{21212} - K_{121}\Gamma_{21212} - \Delta_{121}L_{21212} + K_{121}L_{21212} - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{121}K_{21212} - \Gamma_{121}L_{21212}\Big)\Big\}Z_1 \\
&+ \frac{1}{\Delta_2}\Big\{\Gamma_{121}\Delta_{21212} - \Gamma_{121}K_{21212} - \frac{\Delta_{21}}{\Gamma_{21}}\Big(K_{121}\Gamma_{21212} - K_{121}L_{21212}\Big)\Big\}Z_2 \\
&+ \frac{1}{y_{R^-}}\Big\{\Delta_{121}\Delta_{21212} - \Delta_{121}\Gamma_{21212} + K_{121}\Gamma_{21212} + K_{21212}\Gamma_{121} - K_{121}L_{21212} - \Gamma_{121}\Delta_{21212}\Big\}Z_{pt}
\end{aligned}
$$

$$
\begin{aligned}
Z_{1212}Z_{1212} &= L_{1212}Z_{1212} + \frac{1}{K_{121}}\Big\{K_{1212}K_{1212} - K_{1212}L_{1212}\Big\}Z_{121} \\
&+ \frac{1}{\Gamma_{12}}\Big\{\Gamma_{1212}\Gamma_{1212} - \Gamma_{1212}L_{1212} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}K_{1212} - K_{1212}L_{1212}\Big)\Big\}Z_{12} \\
&+ \frac{1}{\Delta_1}\Big\{\Delta_{1212}\Delta_{1212} - \Delta_{1212}L_{1212} - \frac{\Delta_{121}}{K_{121}}\Big(K_{1212}K_{1212} - K_{1212}L_{1212}\Big) \\
&\qquad - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{1212}\Gamma_{1212} - \Gamma_{1212}L_{1212} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}K_{1212} - K_{1212}L_{1212}\Big)\Big)\Big\}Z_1 \\
Z_{1212}Z_{2121} &= \frac{1}{K_{121}}\Big\{K_{1212}L_{2121}\Big\}Z_{121} + \frac{1}{K_{212}}\Big\{K_{2121}L_{1212}\Big\}Z_{212} \\
&+ \frac{1}{\Gamma_{12}}\Big\{K_{2121}\Gamma_{1212} - K_{2121}L_{1212} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}L_{2121}\Big)\Big\}Z_{12} \\
&+ \frac{1}{\Gamma_{21}}\Big\{K_{1212}\Gamma_{2121} - K_{1212}L_{2121} - \frac{\Gamma_{212}}{K_{212}}\Big(K_{2121}L_{1212}\Big)\Big\}Z_{21} \\
&+ \frac{1}{\Delta_1}\Big\{\Gamma_{2121}\Delta_{1212} - \Gamma_{2121}K_{1212} + K_{1212}L_{2121} - \frac{\Delta_{121}}{K_{121}}\Big(K_{1212}L_{2121}\Big) \\
&\qquad - \frac{\Delta_{12}}{\Gamma_{12}}\Big(K_{2121}\Gamma_{1212} - K_{2121}L_{1212} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}L_{2121}\Big)\Big)\Big\}Z_1 \\
&+ \frac{1}{\Delta_2}\Big\{\Gamma_{1212}\Delta_{2121} - \Gamma_{1212}K_{2121} + L_{1212}K_{2121} - \frac{\Delta_{212}}{K_{212}}\Big(K_{2121}L_{1212}\Big) \\
&\qquad - \frac{\Delta_{21}}{\Gamma_{21}}\Big(K_{1212}\Gamma_{2121} - K_{1212}L_{2121} - \frac{\Gamma_{212}}{K_{212}}\Big(K_{2121}L_{1212}\Big)\Big)\Big\}Z_2 \\
&+ \frac{1}{y_{R^-}}\Big\{\Delta_{1212}\Delta_{2121} - \Delta_{1212}\Gamma_{2121} + K_{1212}\Gamma_{2121} - K_{1212}L_{2121} \\
&\quad - \Gamma_{1212}\Delta_{2121} + \Gamma_{1212}K_{2121} - L_{1212}K_{2121}\Big\}Z_{pt} \\
Z_{1212}Z_{12121} &= L_{12121}Z_{1212} + \frac{1}{K_{121}}\Big\{K_{1212}K_{12121} - K_{1212}L_{12121}\Big\}Z_{121} \\
&+ \frac{1}{\Gamma_{12}}\Big\{\Gamma_{1212}\Gamma_{12121} - \Gamma_{1212}L_{12121} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}K_{12121} - K_{1212}L_{12121}\Big)\Big\}Z_{12} \\
&+ \frac{1}{\Delta_1}\Big\{\Delta_{1212}\Delta_{12121} - \Delta_{1212}L_{12121} - \frac{\Delta_{121}}{K_{121}}\Big(K_{1212}K_{12121} - K_{1212}L_{12121}\Big) \\
&\qquad - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{1212}\Gamma_{12121} - \Gamma_{1212}L_{12121} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}K_{12121} - K_{1212}L_{12121}\Big)\Big)\Big\}Z_1
\end{aligned}
$$

$$
\begin{aligned}
Z_{1212}Z_{21212} = {} & M_{21212}Z_{1212} + \frac{1}{K_{121}}\Big\{K_{1212}L_{21212} - K_{1212}M_{21212}\Big\}Z_{121} \\
& + \frac{1}{K_{212}}\Big\{L_{1212}K_{21212} - L_{1212}M_{21212}\Big\}Z_{212} + \frac{1}{\Gamma_{12}}\Big\{K_{21212}\Gamma_{1212} - K_{21212}L_{1212} - \Gamma_{1212}M_{21212} \\
& \quad + L_{1212}M_{21212} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}L_{21212} - K_{1212}M_{21212}\Big)\Big\}Z_{12} \\
& + \frac{1}{\Gamma_{21}}\Big\{K_{1212}\Gamma_{21212} - K_{1212}L_{21212} - \frac{\Gamma_{212}}{K_{212}}\Big(L_{1212}K_{21212} - L_{1212}M_{21212}\Big)\Big\}Z_{21} \\
& + \frac{1}{\Delta_1}\Big\{\Delta_{1212}\Gamma_{21212} - \Delta_{1212}M_{21212} - K_{1212}\Gamma_{21212} + K_{1212}L_{21212} - \frac{\Delta_{121}}{K_{121}}\Big(K_{1212}L_{21212} - K_{1212}M_{21212}\Big) \\
& \qquad - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{1212}K_{21212} - \Gamma_{1212}M_{21212} - L_{1212}K_{21212} + L_{1212}M_{21212} \\
& \qquad - \frac{\Gamma_{121}}{K_{121}}\Big(K_{1212}L_{21212} - K_{1212}M_{21212}\Big)\Big)\Big\}Z_1 \\
& + \frac{1}{\Delta_2}\Big\{\Gamma_{1212}\Delta_{21212} - \Gamma_{1212}K_{21212} + L_{1212}K_{21212} - L_{1212}M_{21212} - \frac{\Delta_{212}}{K_{212}}\Big(L_{1212}K_{21212} - L_{1212}M_{21212}\Big) \\
& \qquad - \frac{\Delta_{21}}{\Gamma_{21}}\Big(K_{1212}\Gamma_{21212} - K_{1212}L_{21212} - \frac{\Gamma_{212}}{K_{212}}\Big(L_{1212}K_{21212} - L_{1212}M_{21212}\Big)\Big)\Big\}Z_2 \\
& + \frac{1}{y_{R^-}}\Big\{\Delta_{1212}\Delta_{21212} - \Delta_{1212}\Gamma_{21212} + K_{1212}\Gamma_{21212} - K_{1212}L_{21212} - \Gamma_{1212}\Delta_{21212} + \Gamma_{1212}K_{21212} \\
& \qquad - L_{1212}K_{21212} + L_{1212}M_{21212}\Big\}Z_{pt}
\end{aligned}
$$

$$
\begin{aligned}
Z_{12121}Z_{12121} = {} & M_{12121}Z_{12121} + \frac{1}{L_{12121}}\Big\{L_{12121}L_{12121} - L_{12121}M_{12121}\Big\}Z_{1212} \\
& + \frac{1}{K_{121}}\Big\{K_{12121}K_{12121} - K_{12121}M_{12121} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big)\Big\}Z_{121} \\
& + \frac{1}{\Gamma_{12}}\Big\{\Gamma_{12121}\Gamma_{12121} - \Gamma_{12121}M_{12121} - \frac{\Gamma_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big) \\
& - \frac{\Gamma_{121}}{K_{121}}\Big(K_{12121}K_{12121} - K_{12121}M_{12121} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big)\Big)\Big\}Z_{12} \\
& + \frac{1}{\Delta_1}\Big\{\Delta_{12121}\Delta_{12121} - \Delta_{12121}M_{12121} - \frac{\Delta_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big) \\
& - \frac{\Delta_{121}}{K_{121}}\Big(K_{12121}K_{12121} - K_{12121}M_{12121} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big)\Big) \\
& - \frac{\Delta_{12}}{\Gamma_{12}}\Big(\Gamma_{12121}\Gamma_{12121} - \Gamma_{12121}M_{12121} - \frac{\Gamma_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big) \\
& - \frac{\Gamma_{121}}{K_{121}}\Big(K_{12121}K_{12121} - K_{12121}M_{12121} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}L_{12121} - L_{12121}M_{12121}\Big)\Big)\Big)\Big\}Z_1
\end{aligned}
$$

$$
\begin{aligned}
Z_{12121}Z_{21212} =& \frac{1}{y_{R^-}}\Big\{\Delta_{12121}\Delta_{21212} - \Delta_{12121}\Gamma_{21212} - \Gamma_{12121}\Delta_{21212} + \Gamma_{12121}K_{21212} + K_{12121}\Gamma_{21212} \\
&- K_{12121}L_{21212} - L_{12121}K_{21212} + L_{12121}M_{21212} + M_{12121}L_{21212}\Big\}Z_{pt} \\
&+\frac{1}{\Delta_1}\Big\{\Delta_{12121}\Gamma_{21212} - \frac{\Delta_{1212}}{L_{1212}}\Big(L_{12121}M_{21212}\Big) - \frac{\Delta_{121}}{K_{121}}\Big(K_{12121}L_{21212} - M_{12121}L_{21212} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}M_{21212}\Big)\Big) \\
&- K_{12121}\Gamma_{21212} + K_{12121}L_{21212} - M_{12121}L_{21212} - \frac{\Delta_{12}}{\Gamma_{12}}\Big[\Gamma_{12121}K_{21212} - \frac{\Gamma_{1212}}{L_{1212}}\Big(L_{12121}M_{21212}\Big) \\
&- L_{12121}K_{21212} + L_{12121}M_{21212} - \frac{\Gamma_{121}}{K_{121}}\Big(K_{12121}L_{21212} - M_{12121}L_{21212} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}M_{21212}\Big)\Big)\Big]\Big\}Z_1 \\
&+\frac{1}{\Delta_2}\Big\{\Gamma_{12121}\Delta_{21212} - \frac{\Delta_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big) - \frac{\Delta_{212}}{K_{212}}\Big(L_{12121}K_{21212} - L_{12121}M_{21212} - \frac{K_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big)\Big) \\
&- \Gamma_{12121}K_{21212} + L_{12121}K_{21212} - L_{12121}M_{21212} - \frac{\Delta_{21}}{\Gamma_{21}}\Big[K_{12121}\Gamma_{21212} - \frac{\Gamma_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big) \\
&- K_{12121}L_{21212} + M_{12121}L_{21212} - \frac{\Gamma_{212}}{K_{212}}\Big(L_{12121}K_{21212} - L_{12121}M_{21212} - \frac{K_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big)\Big)\Big]\Big\}Z_2 \\
&+\frac{1}{\Gamma_{12}}\Big\{\Gamma_{12121}K_{21212} - \frac{\Gamma_{1212}}{L_{1212}}\Big(L_{12121}M_{21212}\Big) - L_{12121}K_{21212} + L_{12121}M_{21212} \\
&- \frac{\Gamma_{121}}{K_{121}}\Big(K_{12121}L_{21212} - M_{12121}L_{21212} - \frac{K_{1212}}{L_{1212}}\Big(L_{12121}M_{21212}\Big)\Big)\Big\}Z_{12} \\
&+\frac{1}{\Gamma_{21}}\Big\{K_{12121}\Gamma_{21212} - \frac{\Gamma_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big) - K_{12121}L_{21212} + M_{12121}L_{21212} \\
&- \frac{\Gamma_{212}}{K_{212}}\Big(L_{12121}K_{21212} - L_{12121}M_{21212} - \frac{K_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big)\Big)\Big\}Z_{21} \\
&+\frac{1}{K_{121}}\Big\{K_{12121}L_{21212} - M_{12121}L_{21212} - \frac{K_{1212}}{L_{1212}}\Big(M_{12121}L_{21212}\Big)\Big\}Z_{121} \\
&+\frac{1}{K_{212}}\Big\{L_{12121}K_{21212} - L_{12121}M_{21212} - \frac{K_{2121}}{L_{2121}}\Big(M_{12121}L_{21212}\Big)\Big\}Z_{212} \\
&+\frac{1}{L_{2121}}\Big\{M_{12121}L_{21212}\Big\}Z_{2121} + \frac{1}{L_{1212}}\Big\{L_{12121}M_{21212}\Big\}Z_{1212}
\end{aligned}
$$

Appendix B

The Cancelling Expressions, $D\left[Z_\gamma Z_\mu\right]_w$

Listed here are the cancelling expressions which have not been re-expressed so that they are obviously well-defined. The expressions that are exceedingly long excluded.

$$\begin{aligned} D\left[Z_{pt}Z_{12121}\right]_{pt} = y_{(3,2)}\Bigg\{ &\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}}\Big(1 - p\left[y_{(1,1)}, y_{(1,1)}\right]\Big) + \frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(0,1)}, y_{(2,1)}\right]\Big) \\ &+ \left(y_{(1,1)}\right)^2\Big(p\left[y_{(1,1)}, y_{(1,1)}\right] - p\left[y_{-(1,0)}, y_{(1,0)}\right]\Big) + y_{(2,2)}\Big(1 - p\left[y_{(1,0)}, y_{(2,2)}\right]\Big) \\ &+ \frac{\left(y_{(1,1)}\right)^2 y_{(3,2)}}{y_{-(1,0)}}\Big(1 - p\left[y_{(1,1)}, y_{(3,1)}\right]\Big)\Bigg\} + \frac{D\left[Z_{pt}Z_{2121}\right]_{pt}}{y_{(1,0)}} + \frac{D\left[Z_1 Z_{2121}\right]_1}{y_{-(1,0)}} \end{aligned}$$

$$\begin{aligned} D\left[Z_1 Z_{2121}\right]_1 = y_{(1,1)}y_{(3,2)}\Bigg\{ &\frac{y_{(1,1)}y_{(2,1)}}{y_{(3,1)}}\Big(p\left[y_{(2,1)}, y_{(1,1)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) \\ &+ \frac{\left(y_{(2,1)}\right)^2}{y_{(3,1)}}\Big(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\Big) + y_{(0,1)}\Big(p\left[y_{-(3,1)}, y_{(3,1)}\right] - p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) \\ &+ \frac{y_{(0,1)}y_{(2,1)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,0)}, y_{(0,1)}\right]\Big)\Bigg\} \end{aligned}$$

$$\begin{aligned} D\left[Z_1 Z_{2121}\right]_{pt} = y_{(3,2)}\Bigg\{ &\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(1,1)}, y_{(3,1)}\right] - p\left[y_{(1,1)}, y_{(1,1)}\right]\Big) + \frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)}, y_{(2,1)}\right] - p\left[y_{(0,1)}, y_{(2,1)}\right]\Big)\Bigg\} \\ &+ \frac{D\left[Z_{pt}Z_{2121}\right]_{pt}}{y_{(1,0)}} - \frac{D\left[Z_1 Z_{2121}\right]_1}{y_{(1,0)}} \end{aligned}$$

$$D\left[Z_1 Z_{12121}\right]_1 = D\left[Z_{pt}Z_{12121}\right]_{pt}$$

$$D\left[Z_1 Z_{21212}\right]_{pt} = y_{(2,1)}\frac{y_{(2,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)}, y_{(2,1)}\right] - 1\Big) + \frac{D\left[Z_{pt}Z_{21212}\right]_{pt}}{y_{(1,0)}} - \frac{D\left[Z_1 Z_{21212}\right]_1}{y_{(1,0)}}$$

$$\begin{aligned} D\left[Z_2 Z_{12121}\right]_{pt} = \Bigg\{ &\frac{\left(y_{(2,1)}\right)^2}{y_{(0,1)}}\Big(p\left[y_{(1,1)}, y_{(2,1)}\right] - p\left[y_{(2,1)}, y_{(2,1)}\right]\Big) + \frac{y_{(2,1)}y_{(4,2)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)}, y_{(4,2)}\right] - p\left[y_{(2,1)}, y_{(4,2)}\right]\Big) \\ &+ \frac{\left(y_{(3,2)}\right)^2}{y_{(0,1)}}\Big(p\left[y_{(3,2)}, y_{(3,2)}\right] - p\left[y_{(1,0)}, y_{(3,2)}\right]\Big)\Bigg\} + \frac{D\left[Z_{pt}Z_{12121}\right]_{pt}}{y_{(0,1)}} - \frac{D\left[Z_2 Z_{12121}\right]_2}{y_{(0,1)}} \end{aligned}$$

$$
\begin{aligned}
D\left[Z_{12}Z_{212}\right]_{pt} &= y_{(2,1)}\Bigg\{\frac{y_{(1,0)}y_{(3,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,0)},y_{(1,1)}\right]-p\left[y_{(3,1)},y_{(1,0)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(3,2)},y_{(1,1)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\Big)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(3,1)},y_{(2,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(3,2)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(2,1)}\right]\Big)\Bigg\}+\frac{D\left[Z_1Z_{212}\right]_{pt}}{y_{(0,1)}}-\frac{D\left[Z_{12}Z_{212}\right]_2}{y_{(0,1)}}\\
D\left[Z_{12}Z_{2121}\right]_1 &= \Bigg\{\frac{\left(y_{(1,1)}\right)^2}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(1,1)}\right]-1\Big)+\frac{\left(y_{(1,1)}\right)^2y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(3,2)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)\\
&+\frac{2y_{(1,1)}y_{(3,2)}}{y_{(0,1)}}\Big(1-p\left[y_{(1,1)},y_{(3,2)}\right]\Big)+\frac{y_{(1,1)}y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(3,2)},y_{(1,1)}\right]-1\Big)\\
&+\frac{y_{(1,1)}y_{(4,3)}}{y_{(0,1)}}\Big(1-p\left[y_{(4,3)},y_{(1,1)}\right]\Big)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(0,1)}}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-1\Big)\Bigg\}+\frac{D\left[Z_1Z_{2121}\right]_1}{y_{(0,1)}}\\
D\left[Z_{12}Z_{2121}\right]_{pt} &= \Bigg\{\frac{\left(y_{(2,1)}\right)^2}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(2,1)},y_{(2,1)}\right]\Big)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(3,1)}\right]-p\left[y_{(2,1)},y_{(3,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(4,2)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(4,2)}\right]-p\left[y_{(2,1)},y_{(4,2)}\right]\Big)+\frac{y_{(2,2)}y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(3,2)},y_{(2,2)}\right]-p\left[y_{(1,0)},y_{(2,2)}\right]\Big)\\
&+\frac{\left(y_{(3,2)}\right)^2}{y_{(0,1)}}\Big(p\left[y_{(3,2)},y_{(3,2)}\right]-p\left[y_{(1,0)},y_{(3,2)}\right]\Big)\Bigg\}+\frac{D\left[Z_1Z_{2121}\right]_{pt}}{y_{(0,1)}}-\frac{D\left[Z_{12}Z_{2121}\right]_2}{y_{(0,1)}}\\
D\left[Z_{12}Z_{12121}\right]_{12} &= y_{(2,1)}\Bigg\{\frac{y_{(2,1)}y_{(3,2)}}{y_{(1,1)}}\Big(p\left[y_{(3,2)},y_{(2,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\Big)+y_{(3,1)}\Big(p\left[y_{-(1,1)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_2Z_{2121}\right]_2}{y_{(1,1)}}\\
D\left[Z_{12}Z_{12121}\right]_1 &= D\left[Z_2Z_{12121}\right]_{pt}\\
D\left[Z_{12}Z_{21212}\right]_{12} &= y_{(3,1)}\Bigg\{\frac{\left(y_{(2,1)}\right)^2}{y_{(3,2)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\Big)+y_{(1,0)}\Big(p\left[y_{-(3,2)},y_{(3,2)}\right]-p\left[y_{(1,0)},y_{(3,2)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_{12}Z_{1212}\right]_{12}}{y_{(3,2)}}\\
D\left[Z_{12}Z_{21212}\right]_2 &= \frac{\left(y_{(2,1)}\right)^2}{y_{(1,1)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-1\Big)+\frac{D\left[Z_2Z_{21212}\right]_2}{y_{(1,1)}}-\frac{D\left[Z_{12}Z_{21212}\right]_{12}}{y_{(1,1)}}\\
D\left[Z_{12}Z_{21212}\right]_1 &= \frac{D\left[Z_1Z_{21212}\right]_1}{y_{(0,1)}}-\frac{D\left[Z_{12}Z_{21212}\right]_{12}}{y_{(0,1)}}\\
D\left[Z_{12}Z_{21212}\right]_{pt} &= \Bigg\{\frac{y_{(3,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)},y_{(3,1)}\right]-1\Big)+\frac{y_{(3,2)}}{y_{(0,1)}}\Big(1-p\left[y_{(1,0)},y_{(3,2)}\right]\Big)\Bigg\}+\frac{D\left[Z_1Z_{21212}\right]_{pt}}{y_{(0,1)}}-\frac{D\left[Z_{12}Z_{21212}\right]_2}{y_{(0,1)}}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{21}Z_{121}\right]_1 &= \frac{y_{R^-}}{y_{(1,0)}y_{(0,1)}y_{(3,1)}}\left\{\frac{y_{(1,1)}}{y_{(3,1)}}\left(p\left[y_{(2,1)},y_{(1,1)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\right)\right.\\
&\left.+\frac{y_{(2,1)}}{y_{(3,1)}}\left(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(1,0)},y_{(2,1)}\right]\right)\right\}+\frac{D\left[Z_1Z_{121}\right]_1}{y_{(3,1)}}\\
D\left[Z_{21}Z_{121}\right]_{pt} &= y_{(3,2)}\left\{\frac{y_{(0,1)}y_{(1,1)}}{y_{(1,0)}}\left(p\left[y_{(0,1)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(0,1)}\right]\right)+\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}}\left(p\left[y_{(1,1)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(1,1)}\right]\right)\right.\\
&\left.+\frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\right)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\right)\right\}\\
&+\frac{D\left[Z_2Z_{121}\right]_{pt}}{y_{(1,0)}}-\frac{D\left[Z_{21}Z_{121}\right]_1}{y_{(1,0)}}\\
D\left[Z_{21}Z_{1212}\right]_2 &= \left\{\frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}}\left(p\left[y_{(1,1)},y_{(3,2)}\right]-1\right)+\frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(2,1)}\right]-1\right)\right.\\
&+\frac{\left(y_{(2,1)}\right)^2y_{(3,1)}}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\right)+\frac{2y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\left(1-p\left[y_{(2,1)},y_{(3,1)}\right]\right)\\
&\left.+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\left(p\left[y_{(3,1)},y_{(2,1)}\right]-1\right)+\frac{y_{(2,1)}y_{(5,2)}}{y_{(1,0)}}\left(1-p\left[y_{(5,2)},y_{(2,1)}\right]\right)\right\}+\frac{D\left[Z_2Z_{1212}\right]_2}{y_{(1,0)}}\\
D\left[Z_{21}Z_{1212}\right]_{pt} &= \left\{\frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}}\left(p\left[y_{(1,1)},y_{(3,1)}\right]-p\left[y_{(3,2)},y_{(1,1)}\right]\right)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\right)\right.\\
&\left.+\frac{y_{(2,1)}y_{(5,2)}}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(5,2)}\right]-p\left[y_{(0,1)},y_{(5,2)}\right]\right)\right\}+\frac{D\left[Z_2Z_{1212}\right]_{pt}}{y_{(1,0)}}-\frac{D\left[Z_{21}Z_{1212}\right]_1}{y_{(1,0)}}\\
D\left[Z_{21}Z_{2121}\right]_{21} &= D\left[Z_{21}Z_{121}\right]_1+y_{(1,1)}y_{(3,2)}y_{(0,1)}\left(p\left[y_{-(3,1)},y_{(3,1)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\right)\\
D\left[Z_{21}Z_{2121}\right]_2 &= D\left[Z_1Z_{2121}\right]_{pt}\\
D\left[Z_{21}Z_{12121}\right]_{21} &= y_{(1,1)}\left\{\frac{y_{(1,1)}y_{(3,2)}}{y_{(2,1)}}\left(p\left[y_{(3,2)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\right)+y_{(0,1)}\left(p\left[y_{-(2,1)},y_{(2,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\right)\right\}\\
&+\frac{D\left[Z_{21}Z_{2121}\right]_{21}}{y_{(2,1)}}\\
D\left[Z_{21}Z_{12121}\right]_2 &= \left\{\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}}\left(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(1,1)}\right]\right)+\frac{y_{(1,1)}y_{(2,2)}}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(2,2)}\right]-p\left[y_{(1,1)},y_{(2,2)}\right]\right)\right.\\
&\left.+\frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\right)+\frac{y_{(2,1)}y_{(4,2)}}{y_{(1,0)}}\left(p\left[y_{(2,1)},y_{(4,2)}\right]-p\left[y_{(1,1)},y_{(4,2)}\right]\right)\right\}\\
&\frac{D\left[Z_2Z_{12121}\right]_2}{y_{(1,0)}}-\frac{D\left[Z_{21}Z_{12121}\right]_{21}}{y_{(1,0)}}\\
D\left[Z_{21}Z_{12121}\right]_1 &= \left\{\frac{\left(y_{(1,1)}\right)^2}{y_{(3,1)}}\left(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(1,1)}\right]\right)+\frac{y_{(1,1)}y_{(2,2)}}{y_{(3,1)}}\left(p\left[y_{(2,1)},y_{(2,2)}\right]-p\left[y_{(1,1)},y_{(2,2)}\right]\right)\right.\\
&\left.+\frac{\left(y_{(3,2)}\right)^2}{y_{(3,1)}}\left(p\left[y_{(3,2)},y_{(3,2)}\right]-p\left[y_{(1,0)},y_{(3,2)}\right]\right)\right\}+\frac{D\left[Z_1Z_{12121}\right]_1}{y_{(3,1)}}-\frac{D\left[Z_{21}Z_{12121}\right]_{21}}{y_{(3,1)}}\\
D\left[Z_{21}Z_{12121}\right]_{pt} &= \left\{\frac{y_{(3,3)}}{y_{(1,0)}}\left(p\left[y_{(3,1)},y_{(3,3)}\right]-1\right)+\frac{y_{(6,3)}}{y_{(1,0)}}\left(1-p\left[y_{(0,1)},y_{(6,3)}\right]\right)\right\}+\frac{D\left[Z_2Z_{12121}\right]_{pt}}{y_{(1,0)}}-\frac{D\left[Z_{21}Z_{12121}\right]_1}{y_{(1,0)}}\\
D\left[Z_{21}Z_{21212}\right]_2 &= D\left[Z_1Z_{21212}\right]_{pt}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{121}Z_{121}\right]_1 &= y_{(3,2)}\Bigg\{\frac{y_{(0,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(3,1)}\right]-1\Big)+\frac{y_{(1,1)}}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(3,1)}\right]-1\Big)\\
&+\frac{y_{(1,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(1,2)},y_{(3,1)}\right]\Big)+\frac{\big(y_{(2,1)}\big)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\Big)-\frac{y_{(2,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(2,2)},y_{(3,1)}\right]\Big)\\
&+\frac{y_{(0,1)}y_{(1,1)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,1)},y_{(0,1)}\right]\Big)+\frac{\big(y_{(1,1)}\big)^2}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,1)},y_{(1,1)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(3,2)}\right]-1\Big)+\frac{y_{(1,1)}y_{(4,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(4,2)}\right]-1\Big)\\
&+\frac{y_{(0,1)}y_{(1,1)}}{y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(0,1)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)+\frac{\big(y_{(1,1)}\big)^2}{y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(1,1)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(1,1)},y_{(3,2)}\right]\Big)+\frac{y_{(1,1)}y_{(4,2)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(1,1)},y_{(4,2)}\right]\Big)\\
&+y_{(0,1)}\Big(p\left[y_{(0,1)},y_{(1,0)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\Big)-y_{(1,1)}\Big(p\left[y_{(1,0)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)\\
&+y_{(1,2)}\Big(p\left[y_{(1,2)},y_{(3,1)}\right]-p\left[y_{(1,0)},y_{(1,2)}\right]\Big)+y_{(2,2)}\Big(p\left[y_{(2,2)},y_{(3,1)}\right]-p\left[y_{(1,0)},y_{(2,2)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_{12}Z_{121}\right]_1}{y_{(1,0)}}+\frac{D\left[Z_1Z_{121}\right]_1}{y_{-(1,0)}y_{(3,1)}}\\
D\left[Z_{121}Z_{212}\right]_2 &= \Bigg\{\frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(3,2)}\right]-1\Big)+\frac{\big(y_{(2,1)}\big)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-1\Big)\\
&+\frac{\big(y_{(2,1)}\big)^2y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\Big)+\frac{2y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(1-p\left[y_{(2,1)},y_{(3,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(2,1)}\right]-1\Big)+\frac{y_{(2,1)}y_{(5,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(5,2)},y_{(2,1)}\right]\Big)\\
&+y_{(2,1)}y_{(3,1)}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(1,0)},y_{(1,1)}\right]\Big)\Bigg\}+\frac{D\left[Z_{12}Z_{212}\right]_2}{y_{(1,0)}}-\frac{D\left[Z_{121}Z_{212}\right]_{21}}{y_{(1,0)}}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{121}Z_{212}\right]_1 = &\Bigg\{ \frac{y_{(0,1)}}{y_{(1,0)}}\Big(1 - p\left[y_{(3,1)}, y_{(0,1)}\right]\Big) + \frac{y_{(0,2)}}{y_{(1,0)}}\Big(p\left[y_{(0,2)}, y_{(3,1)}\right] - 1\Big) \\
&+ \frac{y_{(1,2)}}{y_{(1,0)}}\Big(1 - p\left[y_{(1,2)}, y_{(3,1)}\right]\Big) + \frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)}, y_{(3,1)}\right] - p\left[y_{(0,1)}, y_{(3,1)}\right]\Big) \\
&+ \frac{y_{(2,1)}y_{(4,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)}, y_{(4,1)}\right] - p\left[y_{(2,1)}, y_{(4,1)}\right]\Big) + \frac{y_{(2,1)}y_{(4,2)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)}, y_{(4,2)}\right] - p\left[y_{(2,1)}, y_{(4,2)}\right]\Big) \\
&+ \frac{y_{(2,2)}}{y_{(1,0)}}\Big(1 - p\left[y_{(2,2)}, y_{(3,1)}\right]\Big) + \frac{y_{(3,3)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)}, y_{(3,3)}\right] - 1\Big) \\
&+ \frac{y_{(1,1)}y_{(2,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)}, y_{(2,1)}\right] - 1\Big) + \frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)}, y_{(3,2)}\right] - 1\Big) \\
&+ \frac{y_{(2,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(2,1)}, y_{(3,2)}\right] - 1\Big) + \frac{\left(y_{(3,2)}\right)^2}{y_{(1,0)}y_{(3,1)}}\Big(1 - p\left[y_{(3,2)}, y_{(3,2)}\right]\Big) \\
&+ \frac{y_{(0,1)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)}, y_{(0,1)}\right] - 1\Big) + \frac{y_{(0,2)}}{y_{(3,1)}}\Big(1 - p\left[y_{(1,0)}, y_{(0,2)}\right]\Big) \\
&+ \frac{y_{(1,1)}y_{(2,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)}, y_{(1,0)}\right] - p\left[y_{(1,1)}, y_{(2,1)}\right]\Big) + \frac{y_{(1,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(1,1)}, y_{(3,2)}\right] - 1\Big) \\
&+ \frac{y_{(1,1)}y_{(3,2)}}{y_{(3,1)}}\Big(1 - p\left[y_{(1,1)}, y_{(3,2)}\right]\Big) + \frac{y_{(2,1)}}{y_{(3,1)}}\Big(1 - p\left[y_{(1,0)}, y_{(2,1)}\right]\Big) \\
&+ \frac{y_{(2,1)}y_{(3,2)}}{y_{(3,1)}}\Big(1 - p\left[y_{(2,1)}, y_{(3,2)}\right]\Big) + \frac{y_{(2,2)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)}, y_{(2,2)}\right] - 1\Big) \\
&+ \frac{\left(y_{(3,2)}\right)^2}{y_{(3,1)}}\Big(p\left[y_{(3,2)}, y_{(3,2)}\right] - 1\Big) + y_{(0,1)}\Big(p\left[y_{(0,1)}, y_{(3,1)}\right] - p\left[y_{(0,1)}, y_{(1,0)}\right]\Big) \\
&+ y_{(0,2)}\Big(p\left[y_{(1,0)}, y_{(0,2)}\right] - p\left[y_{(0,2)}, y_{(3,1)}\right]\Big) + y_{(2,2)}\Big(p\left[y_{(2,2)}, y_{(3,1)}\right] - p\left[y_{(1,0)}, y_{(2,2)}\right]\Big) \\
&+ y_{(3,3)}\Big(p\left[y_{(1,0)}, y_{(3,3)}\right] - p\left[y_{(3,1)}, y_{(3,3)}\right]\Big)\Bigg\} + \frac{D\left[Z_{12}Z_{212}\right]_1}{y_{(1,0)}} + \frac{D\left[Z_1Z_{212}\right]_1}{y_{-(1,0)}y_{(3,1)}}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{121}Z_{212}\right]_{pt} = &\Bigg\{ \frac{y_{(0,1)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(0,1)}\right]-1\Big) + \frac{y_{(0,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,2)},y_{(3,1)}\right]\Big) \\
&+ \frac{y_{(1,1)}}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(3,1)}\right]-1\Big) + \frac{y_{(1,2)}}{y_{(1,0)}}\Big(p\left[y_{(1,2)},y_{(3,1)}\right]-1\Big) \\
&+ \frac{y_{(2,1)}y_{(4,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(4,1)}\right]-p\left[y_{(0,1)},y_{(4,1)}\right]\Big) + \frac{y_{(2,1)}y_{(4,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(4,2)}\right]-p\left[y_{(0,1)},y_{(4,2)}\right]\Big) \\
&+ \frac{y_{(2,1)}y_{(5,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(5,2)}\right]-p\left[y_{(0,1)},y_{(5,2)}\right]\Big) + \frac{y_{(2,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,2)},y_{(3,1)}\right]-1\Big) \\
&+ \frac{y_{(3,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(3,1)},y_{(3,3)}\right]\Big) - \frac{y_{(4,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(3,1)},y_{(4,3)}\right]\Big) \\
&+ \frac{y_{(1,1)}y_{(2,1)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(2,1)},y_{(1,1)}\right]\Big) + \frac{y_{(1,1)}y_{(2,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(2,1)},y_{(3,2)}\right]\Big) \\
&+ \frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,1)},y_{(3,2)}\right]\Big) + \frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(3,2)},y_{(1,1)}\right]\Big) \\
&+ \frac{y_{(1,1)}y_{(5,3)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(5,3)},y_{(1,1)}\right]-1\Big) + \frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(2,1)},y_{(2,1)}\right]\Big) \\
&+ \frac{\left(y_{(2,1)}\right)^2 y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(2,1)},y_{(3,2)}\right]\Big) + \frac{2y_{(2,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(2,1)},y_{(3,2)}\right]-1\Big) \\
&+ \frac{y_{(2,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(3,2)},y_{(2,1)}\right]\Big) + \frac{y_{(2,1)}y_{(5,3)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(5,3)},y_{(2,1)}\right]-1\Big) \\
&+ \frac{y_{(0,1)}}{y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(0,1)}\right]\Big) + \frac{y_{(0,2)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)},y_{(0,2)}\right]-1\Big) \\
&+ \frac{y_{(1,1)}y_{(2,1)}}{y_{(3,1)}}\Big(p\left[y_{(2,1)},y_{(1,1)}\right]-1\Big) + \frac{y_{(1,1)}y_{(2,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(2,1)},y_{(3,2)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big) \\
&+ \frac{y_{(1,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(3,2)},y_{(1,1)}\right]-1\Big) + \frac{y_{(1,1)}y_{(5,3)}}{y_{(3,1)}}\Big(1-p\left[y_{(5,3)},y_{(1,1)}\right]\Big) \\
&+ \frac{y_{(2,1)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)},y_{(2,1)}\right]-1\Big) + \frac{\left(y_{(2,1)}\right)^2}{y_{(3,1)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-1\Big) \\
&+ \frac{\left(y_{(2,1)}\right)^2 y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(2,1)},y_{(3,2)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big) + \frac{2y_{(2,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(2,1)},y_{(3,2)}\right]\Big) \\
&+ \frac{y_{(2,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(3,2)},y_{(2,1)}\right]-1\Big) + \frac{y_{(2,1)}y_{(5,3)}}{y_{(3,1)}}\Big(1-p\left[y_{(5,3)},y_{(2,1)}\right]\Big) \\
&+ \frac{y_{(2,2)}}{y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(2,2)}\right]\Big) + y_{(0,1)}\Big(p\left[y_{(0,1)},y_{(1,0)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\Big) \\
&+ y_{(0,2)}\Big(p\left[y_{(0,2)},y_{(3,1)}\right]-p\left[y_{(1,0)},y_{(0,2)}\right]\Big) - y_{(1,1)}\Big(p\left[y_{(1,0)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big) \\
&+ y_{(2,2)}\Big(p\left[y_{(1,0)},y_{(2,2)}\right]-p\left[y_{(2,2)},y_{(3,1)}\right]\Big) + y_{(3,3)}\Big(p\left[y_{(3,1)},y_{(3,3)}\right]-p\left[y_{(1,0)},y_{(3,3)}\right]\Big) \\
&+ y_{(4,3)}\Big(p\left[y_{(3,1)},y_{(4,3)}\right]-p\left[y_{(1,0)},y_{(4,3)}\right]\Big) + \frac{\left(y_{(2,1)}\right)^2 y_{(3,2)}}{y_{-(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(2,1)}\right]\Big) \\
&+ \frac{y_{(2,1)}y_{(3,2)}y_{(1,1)}}{y_{-(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(1,1)}\right]\Big)\Bigg\} + \frac{D\left[Z_{12}Z_{212}\right]_{pt}}{y_{(1,0)}} + \frac{D\left[Z_{1}Z_{212}\right]_{pt}}{y_{-(1,0)}y_{(3,1)}} + \frac{D\left[Z_{121}Z_{212}\right]_{21}}{y_{(1,0)}y_{(3,1)}}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{121}Z_{1212}\right]_{12} = &\Bigg\{ \frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(3,2)}\right]-1\Big)+\frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-1\Big)\\
&+\frac{\left(y_{(2,1)}\right)^2 y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\Big)+\frac{2y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(1-p\left[y_{(2,1)},y_{(3,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(2,1)}\right]-1\Big)+\frac{y_{(2,1)}y_{(5,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(5,2)},y_{(2,1)}\right]\Big)\Bigg\}+\frac{D\left[Z_{12}Z_{1212}\right]_{12}}{y_{(1,0)}}\\
D\left[Z_{121}Z_{1212}\right]_{1} = &\Bigg\{ \frac{y_{(1,1)}}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(3,1)}\right]-1\Big)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(5,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(5,2)}\right]-p\left[y_{(0,1)},y_{(5,2)}\right]\Big)+\frac{y_{(4,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(3,1)},y_{(4,3)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(3,2)},y_{(1,1)}\right]\Big)+\frac{y_{(3,2)}y_{(4,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(3,2)},y_{(4,2)}\right]-1\Big)\\
&+\frac{y_{(1,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(3,2)},y_{(1,1)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)+\frac{y_{(3,2)}y_{(4,2)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(3,2)},y_{(4,2)}\right]\Big)\\
&+y_{(1,1)}\Big(p\left[y_{(1,0)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)+y_{(4,3)}\Big(p\left[y_{(3,1)},y_{(4,3)}\right]-p\left[y_{(1,0)},y_{(4,3)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_{12}Z_{1212}\right]_{1}}{y_{(1,0)}}+\frac{D\left[Z_{1}Z_{1212}\right]_{1}}{y_{-(1,0)}y_{(3,1)}}-\frac{D\left[Z_{121}Z_{1212}\right]_{121}}{y_{-(1,0)}y_{(3,1)}}\\
D\left[Z_{121}Z_{2121}\right]_{21} = &\, y_{(1,1)}y_{(3,2)}\frac{y_{(1,1)}}{y_{(2,1)}}\Big(p\left[y_{(3,2)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)+\frac{D\left[Z_{21}Z_{2121}\right]_{21}}{y_{(2,1)}}\\
D\left[Z_{121}Z_{2121}\right]_{12} = &\Bigg\{ \frac{y_{(0,1)}y_{(1,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(1,1)}\right]-1\Big)+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(1-p\left[y_{(2,1)},y_{(3,1)}\right]\Big)\Bigg\}\\
D\left[Z_{121}Z_{2121}\right]_{2} = &\Bigg\{ \frac{y_{(0,1)}y_{(1,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(0,1)}\right]\Big)+\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(1,1)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(2,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,2)}\right]-p\left[y_{(1,1)},y_{(2,2)}\right]\Big)+\frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\Big)\\
&+\frac{y_{(2,1)}y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(3,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)+\frac{y_{(2,1)}y_{(4,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(4,2)}\right]-p\left[y_{(1,1)},y_{(4,2)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_{12}Z_{2121}\right]_{2}}{y_{(1,0)}}-\frac{D\left[Z_{121}Z_{2121}\right]_{21}}{y_{(1,0)}}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{121}Z_{2121}\right]_1 = &\Big\{ \frac{y_{(0,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(3,1)}\right]-1\Big)+\frac{y_{(0,1)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(0,1)}\right]-1\Big)\\
&+\frac{y_{(0,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(3,1)},y_{(0,2)}\right]\Big)+\frac{y_{(2,1)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,1)},y_{(2,1)}\right]\Big)\\
&+\frac{y_{(2,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(3,1)},y_{(2,3)}\right]\Big)+\frac{y_{(3,1)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,1)},y_{(3,1)}\right]\Big)\\
&+\frac{y_{(5,2)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(5,2)}\right]-1\Big)+\frac{y_{(5,3)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(5,3)}\right]-1\Big)\\
&+\frac{y_{(0,1)}y_{(1,1)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,1)},y_{(0,1)}\right]\Big)+\frac{\big(y_{(1,1)}\big)^2}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,1)},y_{(1,1)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(3,2)}\right]-1\Big)+\frac{y_{(1,1)}y_{(4,3)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(4,3)}\right]-1\Big)\\
&+\frac{y_{(0,1)}}{y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(0,1)}\right]\Big)+\frac{y_{(0,2)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)},y_{(0,2)}\right]-1\Big)\\
&+\frac{y_{(1,1)}y_{(2,2)}}{y_{(3,1)}}\Big(p\left[y_{(2,1)},y_{(2,2)}\right]-p\left[y_{(1,1)},y_{(2,2)}\right]\Big)+\frac{y_{(1,1)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(1,1)},y_{(3,2)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(4,3)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-p\left[y_{(1,1)},y_{(4,3)}\right]\Big)+\frac{y_{(2,2)}}{y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(2,2)}\right]\Big)\\
&+\frac{y_{(2,3)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)},y_{(2,3)}\right]-1\Big)-y_{(0,1)}\Big(p\left[y_{(0,1)},y_{(1,0)}\right]-p\left[y_{(0,1)},y_{(3,1)}\right]\Big)\\
&+y_{(0,2)}\Big(p\left[y_{(0,2)},y_{(3,1)}\right]-p\left[y_{(1,0)},y_{(0,2)}\right]\Big)+y_{(2,3)}\Big(p\left[y_{(3,1)},y_{(2,3)}\right]-p\left[y_{(1,0)},y_{(2,3)}\right]\Big)\\
&+\frac{\big(y_{(1,1)}\big)^2}{y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)+\frac{y_{(1,1)}y_{(0,1)}}{y_{(3,1)}}\Big(p\left[y_{(0,1)},y_{(2,1)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)\Big\}\\
&+\frac{D\left[Z_{12}Z_{2121}\right]_1}{y_{(1,0)}}+\frac{D\left[Z_1Z_{2121}\right]_1}{y_{-(1,0)}y_{(3,1)}}-\frac{D\left[Z_{121}Z_{2121}\right]_{21}}{y_{(3,1)}}
\end{aligned}
$$

$$
\begin{aligned}
D\left[Z_{121}Z_{2121}\right]_{pt} &= \Bigg\{ \frac{y_{(0,2)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(0,2)}\right]-1\Big)+\frac{y_{(2,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(2,1)}\right]-1\Big)\\
&+\frac{y_{(2,3)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(2,3)}\right]-1\Big)+\frac{y_{(3,1)}}{y_{(1,0)}}\Big(p\left[y_{(0,1)},y_{(3,1)}\right]-1\Big)\\
&+\frac{y_{(3,3)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(3,3)}\right]-1\Big)+\frac{y_{(5,2)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,1)},y_{(5,2)}\right]\Big)\\
&+\frac{y_{(5,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,1)},y_{(5,3)}\right]\Big)+\frac{y_{(6,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,1)},y_{(6,3)}\right]\Big)\\
&+\frac{y_{(0,1)}y_{(1,1)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(0,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\Big)+\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\Big)\\
&+\frac{y_{(1,1)}y_{(2,2)}}{y_{(1,0)}y_{(3,1)}}\Big(p\left[y_{(1,1)},y_{(2,2)}\right]-p\left[y_{(2,1)},y_{(2,2)}\right]\Big)+\frac{y_{(2,2)}y_{(3,2)}}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(3,2)},y_{(2,2)}\right]\Big)\\
&+\frac{\left(y_{(3,2)}\right)^2}{y_{(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(3,2)},y_{(3,2)}\right]\Big)+\frac{y_{(0,1)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)},y_{(0,1)}\right]-1\Big)\\
&+\frac{y_{(0,2)}}{y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(0,2)}\right]\Big)+\frac{y_{(2,2)}}{y_{(3,1)}}\Big(p\left[y_{(1,0)},y_{(2,2)}\right]-1\Big)\\
&+\frac{y_{(2,2)}y_{(3,2)}}{y_{(3,1)}}\Big(p\left[y_{(3,2)},y_{(2,2)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)+\frac{y_{(2,3)}}{y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(2,3)}\right]\Big)\\
&+\frac{\left(y_{(3,2)}\right)^2}{y_{(3,1)}}\Big(p\left[y_{(3,2)},y_{(3,2)}\right]-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)+y_{(0,1)}\Big(p\left[y_{(0,1)},y_{(3,1)}\right]-p\left[y_{(0,1)},y_{(1,0)}\right]\Big)\\
&+y_{(0,2)}\Big(p\left[y_{(1,0)},y_{(0,2)}\right]-p\left[y_{(0,2)},y_{(3,1)}\right]\Big)+y_{(2,3)}\Big(p\left[y_{(1,0)},y_{(2,3)}\right]-p\left[y_{(3,1)},y_{(2,3)}\right]\Big)\\
&-y_{(3,3)}\Big(p\left[y_{(1,0)},y_{(3,3)}\right]-p\left[y_{(3,1)},y_{(3,3)}\right]\Big)+\frac{\left(y_{(3,2)}\right)^2}{y_{-(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(3,2)}\right]\Big)\\
&+\frac{y_{(3,2)}y_{(2,2)}}{y_{-(1,0)}y_{(3,1)}}\Big(1-p\left[y_{(1,0)},y_{(2,2)}\right]\Big)\Bigg\}+\frac{D\left[Z_{12}Z_{2121}\right]_{pt}}{y_{(1,0)}}+\frac{D\left[Z_{1}Z_{2121}\right]_{pt}}{y_{-(1,0)}y_{(3,1)}}+\frac{D\left[Z_{121}Z_{2121}\right]_{21}}{y_{(1,0)}y_{(3,1)}}\\
D\left[Z_{121}Z_{12121}\right]_{121} &= y_{(1,1)}\Bigg\{\frac{y_{(1,1)}y_{(3,2)}}{y_{(2,1)}}\Big(p\left[y_{(3,2)},y_{(1,1)}\right]-p\left[y_{(1,1)},y_{(3,1)}\right]\Big)+y_{(0,1)}\Big(p\left[y_{-(2,1)},y_{(2,1)}\right]-p\left[y_{(0,1)},y_{(2,1)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_{21}Z_{2121}\right]_{21}}{y_{(2,1)}}\\
D\left[Z_{121}Z_{12121}\right]_{12} &= \Bigg\{\frac{\left(y_{(1,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(1,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(1,1)}\right]\Big)+\frac{y_{(1,1)}y_{(2,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,2)}\right]-p\left[y_{(1,1)},y_{(2,2)}\right]\Big)\\
&+\frac{\left(y_{(2,1)}\right)^2}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(2,1)}\right]-p\left[y_{(1,1)},y_{(2,1)}\right]\Big)+\frac{y_{(2,1)}y_{(4,2)}}{y_{(1,0)}}\Big(p\left[y_{(2,1)},y_{(4,2)}\right]-p\left[y_{(1,1)},y_{(4,2)}\right]\Big)\Bigg\}\\
&+\frac{D\left[Z_{12}Z_{12121}\right]_{12}}{y_{(1,0)}}-\frac{D\left[Z_{121}Z_{12121}\right]_{121}}{y_{(1,0)}}\\
D\left[Z_{121}Z_{12121}\right]_{1} &= \Bigg\{\frac{y_{(3,3)}}{y_{(1,0)}}\Big(p\left[y_{(3,1)},y_{(3,3)}\right]-1\Big)+\frac{y_{(6,3)}}{y_{(1,0)}}\Big(1-p\left[y_{(0,1)},y_{(6,3)}\right]\Big)\\
&+\frac{y_{(3,3)}}{y_{(3,1)}}\Big(1-p\left[y_{-(1,0)},y_{(1,0)}\right]\Big)+\frac{y_{(6,4)}}{y_{(3,1)}}\Big(p\left[y_{-(1,0)},y_{(1,0)}\right]-1\Big)\\
&+y_{(3,3)}\Big(p\left[y_{(1,0)},y_{(3,3)}\right]-p\left[y_{(3,1)},y_{(3,3)}\right]\Big)\Bigg\}+\frac{D\left[Z_{12}Z_{12121}\right]_{1}}{y_{(1,0)}}+\frac{D\left[Z_{1}Z_{12121}\right]_{1}}{y_{-(1,0)}y_{(3,1)}}\\
&-\frac{D\left[Z_{121}Z_{12121}\right]_{121}}{y_{-(1,0)}y_{(3,1)}}-\frac{D\left[Z_{121}Z_{12121}\right]_{121}}{y_{-(1,0)}y_{(3,1)}}
\end{aligned}
$$

$$D\left[Z_{121}Z_{21212}\right]_{21} = \frac{D\left[Z_{21}Z_{12121}\right]_{21}}{y_{(2,1)}} - \frac{D\left[Z_{121}Z_{12121}\right]_{121}}{y_{(2,1)}}$$

$$D\left[Z_{121}Z_{21212}\right]_{12} = \frac{D\left[Z_{12}Z_{21212}\right]_{12}}{y_{(1,0)}}$$

$$\begin{aligned}
D\left[Z_{212}Z_{212}\right]_{2} = y_{(2,1)}\Bigg\{ & \frac{y_{(1,0)}}{y_{(0,1)}}\Big(p\left[y_{(1,0)}, y_{(1,1)}\right] - 1\Big) + \frac{y_{(1,1)}y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(3,2)}, y_{(1,1)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) \\
& + \frac{y_{(2,1)}}{y_{(0,1)}}\Big(p\left[y_{(1,1)}, y_{(2,1)}\right] - 1\Big) + \frac{y_{(2,1)}y_{(3,2)}}{y_{(0,1)}}\Big(p\left[y_{(3,2)}, y_{(2,1)}\right] - p\left[y_{(1,0)}, y_{(2,1)}\right]\Big) \\
& + \frac{y_{(4,1)}}{y_{(0,1)}}\Big(1 - p\left[y_{(1,1)}, y_{(4,1)}\right]\Big) + \frac{y_{(5,2)}}{y_{(0,1)}}\Big(1 - p\left[y_{(1,1)}, y_{(5,2)}\right]\Big) \\
& + \frac{y_{(1,0)}y_{(3,1)}}{y_{(0,1)}y_{(1,1)}}\Big(1 - p\left[y_{(3,1)}, y_{(1,0)}\right]\Big) + \frac{y_{(3,1)}y_{(3,2)}}{y_{(0,1)}y_{(1,1)}}\Big(p\left[y_{(3,1)}, y_{(3,2)}\right] - 1\Big) \\
& + \frac{y_{(1,0)}y_{(3,1)}}{y_{(1,1)}}\Big(p\left[y_{(3,1)}, y_{(1,0)}\right] - p\left[y_{-(0,1)}, y_{(0,1)}\right]\Big) + \frac{y_{(3,1)}y_{(3,2)}}{y_{(1,1)}}\Big(p\left[y_{-(0,1)}, y_{(0,1)}\right] - p\left[y_{(3,1)}, y_{(3,2)}\right]\Big) \\
& + y_{(1,0)}\Big(p\left[y_{(1,0)}, y_{(0,1)}\right] - p\left[y_{(1,0)}, y_{(1,1)}\right]\Big) + y_{(4,1)}\Big(p\left[y_{(1,1)}, y_{(4,1)}\right] - p\left[y_{(0,1)}, y_{(4,1)}\right]\Big) \\
& + y_{(5,2)}\Big(p\left[y_{(1,1)}, y_{(5,2)}\right] - p\left[y_{(0,1)}, y_{(5,2)}\right]\Big)\Bigg\} + \frac{D\left[Z_{21}Z_{212}\right]_{2}}{y_{(0,1)}} + \frac{D\left[Z_{2}Z_{212}\right]_{2}}{y_{-(0,1)}y_{(1,1)}}
\end{aligned}$$

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