The diagonal correlations of the rectangular Ising Model at critical temperature in the scaling limit

Thesis for Masters of Science (Mathematics and Statistics)

The University of Melbourne

Allan Kaching Trinh

October 7, 2016

Contents

Abstract

The spin-spin correlation function $\langle \sigma_{0,0}\sigma_{N,N}\rangle$ of the two dimensional Ising model will be studied in the scaling limit $T \to T_C$ and $N \to \infty$ such that $N(T - T_C)$ is fixed. In this limit, the diagonal correlation is $\langle \sigma_{0,0}\sigma_{N,N}\rangle = \hat{G}_0^{\pm}(s)/N^{1/4} + \hat{G}_1^{\pm}/N^{5/4} + O(N^{-9/4})$ in the scaling variable s. For small and large s we find the asymptotic expressions for $\hat{G}_0^{\pm}(s)$ and $\hat{G}_1^{\pm}(s)$ using a Toeplitz determinant and a form factor expansion. We also characterise $\hat{G}_0^{\pm}(s)$ and $\hat{G}_1^{\pm}(s)$ as the solution of a Painlevé V equation and second order linear differential equation respectively and by analysing their solutions we can express $\hat{G}_0^{\pm}(s)$ and $\hat{G}_1^{\pm}(s)$ as a series about the critical temperature T_C.

Chapter 1

Introduction

1.1 Historical context

The Ising model in two dimensions is perhaps one of the most important integrable system studied in mathematical physics. Its history traces back to the physicist Ernst Ising [6] who first computed the free energy in one dimension in his PhD thesis in 1924 under the supervision of Wilhelm Lenz. Major developments in the two dimensional case were first done by Onsager [15] in 1944 where he first gave exact computations for the zero field free energy. Many important works then followed. Yang [19] gave a exact answer for the magnetisation in the absence of an external magnetic field in 1952. Montroll, Potts and Ward[14] then showed in 1963 that the two point spin-spin correlation function can be expressed by a single Toeplitz determinant which will become a major concern in this work. Moreover in 1980 Jimbo and Miwa [8] discovered that if this same spin-spin correlation function is specialised to the diagonal then the diagonal correlation can be characterised as a solution to a second order nonlinear Painlevé VI differential equation. The work of [8] extends the discovery of Wu et al. [18] that in the scaling limit the two point spin-spin correlation function is given in terms of a Painlevé III transcendent. The new work of this thesis is to use [8] to characterise, in terms of a differential equation, the leading correction to the scaling limit.

1.2 Defining the Ising model in two dimensions

To specify the Ising model in two dimensions, we start with a square lattice of size $(2N+1) \times (2N+1)$ where we use the usual Cartesian coordinates $(i, j) \in \{0, 1, \cdots, N\} \times \{0, 1, \cdots, N\}$ to label each node. The origin $(0, 0)$ is defined as the centre of the square lattice. On each node of the square lattice, there is an

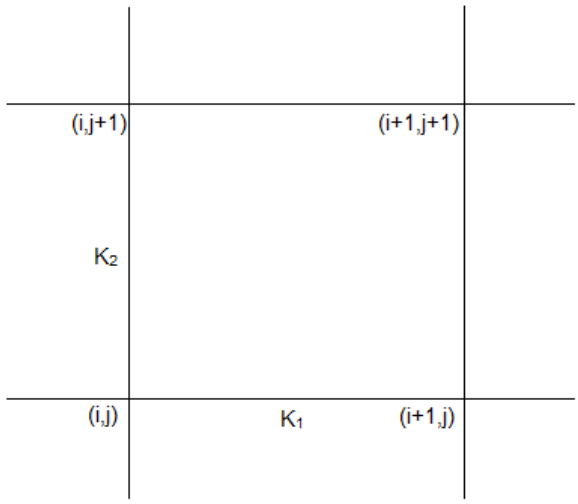


Figure 1: The square lattice and its couplings

associated spin $\sigma_{i,j} \in \{-1, 1\}$ where $+1$ and -1 denotes an up and down spin respectively. The spins interact with their nearest neighbours in the horizontal abd vertical directions according to the interaction energy

$$\beta\mathcal{E} = K_1 \sum_{i=-N}^{N-1} \sum_{j=-N}^{N} \sigma_{i,j}\sigma_{i+1,j} + K_2 \sum_{i=-N}^{N} \sum_{j=-N}^{N-1} \sigma_{i,j}\sigma_{i,j+1} \tag{1.2.1}$$

where

$$\beta = \frac{1}{k_B T}$$

is the inverse temperature and k_B is Boltzmann's constant. K_1 and K_2 are the dimensionless coupling constants in the horizontal and vertical directions respectively (see Figure 1). We remark that 1.2.1 applies in zero magnetic filed. With a magnetic field, there is an extra term $h \sum_{i,j=-N}^{N} \sigma_{i,j}$ on the right hand side.

An important feature of a statistical mechanical model like the Ising model is the free energy f which is computed from the partition function

$$Z_N = \sum_{\{\sigma_{i,j}\}} \exp(-\beta\mathcal{E}) \tag{1.2.2}$$

by taking the thermodynamic limit

$$\exp(-\beta f) = \lim_{N\to\infty} Z_N^{1/N} \tag{1.2.3}$$

Onsager [15] showed that

$$\begin{aligned} -\beta f = \log 2 + \frac{1}{8\pi^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \\ \times \log(\cosh 2K_1 \cosh 2K_2 - \sinh 2K_1 \cos\theta_1 - \sinh 2K_2 \cos\theta_2) \end{aligned} \tag{1.2.4}$$

Here there exists a critical value such that the argument in the logarithm in 1.2.4 vanishes. This is the critical temperature $T = T_C$ which satisfies the transcendental equation

$$\sinh 2K_1 \sinh 2K_2 = 1. \tag{1.2.5}$$

It is natural to define a new temperature parameter

$$k = \sinh 2K_1 \sinh 2K_2 \tag{1.2.6}$$

where $k > 1$ and $0 < k < 1$ denotes temperatures below and above criticality respectively. In contrast to the one dimensional Ising model, the two dimensional Ising model exhibits a phase transition at a nonzero critical temperature $T = T_C$.

The probability density function for a particular configuration $\{\sigma_{i,j}\}_{i,j=-N}^{N}$ is defined as the product of the Boltzmann weights in the horizontal and vertical axes

$$\mathbb{P}(\{\sigma_{i,j}\}_{i,j=-N}^{N}) = \frac{1}{Z_{2N+1}} \exp\left(K_1 \sum_{i=-N}^{N-1} \sum_{j=-N}^{N} \sigma_{i,j}\sigma_{i+1,j} + K_2 \sum_{i=-N}^{N} \sum_{j=-N}^{N-1} \sigma_{i,j}\sigma_{i,j+1} \right) \tag{1.2.7}$$

where Z_{2N+1} is the partition function.
Using this we can define the spontaneous magnetisation as

$$\mathcal{M} = \langle \sigma_{0,0} \rangle = \sum_{\{\sigma_{i,j}\} \in (-1,1)^{2N+1}} \sigma_{0,0} \mathbb{P}(\{\sigma_{i,j}\}_{i,j=-N}^{N}) \tag{1.2.8}$$

and the spin-spin correlation function as

$$\langle \sigma_{0,0}\sigma_{m,n} \rangle = \sum_{\{\sigma_{i,j}\} \in (-1,1)^{2N+1}} \sigma_{0,0}\sigma_{m,n} \mathbb{P}(\{\sigma_{i,j}\}_{i,j=-N}^{N}). \tag{1.2.9}$$

As first announced by Onsager and later proved by Yang [19], the precise result for the spontaneous magnetisation is

$$\mathcal{M} = \begin{cases} (1-k^2)^{1/8}, & \text{for } T < T_C \\ 0, & \text{for } T > T_C. \end{cases} \tag{1.2.10}$$

This is the evaluation of 1.2.10 in the limit $N \to \infty$. In this calculation the boundary spins can be specified to be all pointing up, to break the symmetry of the model

In the case of the spin-spin correlation, the challenge is to compute the limit

$N \to \infty$ of 1.2.9. Although this is now a function of the position of the spin (m, n) and the couplings, fortunately there are mathematical techniques that can be used without resorting to probabilistic simulations. Three main ways that will be discussed for the diagonal correlation will its determinantal expression, form factor expansion and solution to a Painlevé differential equation.

1.3 The diagonal correlation $\langle \sigma_{0,0}\sigma_{N,N} \rangle$

As first observed by Onsager in a draft paper [3] in the diagonal case, Montroll, Potts and Ward [14] showed that the spin-spin correlation $\langle \sigma_{0,0}\sigma_{M,N} \rangle$ has a determinantal expression. In particular, the diagonal correlation $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ can be written as a $N \times N$ Toeplitz determinant

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = \det[a_{i-j}]_{1 \leq i,j \leq N} \tag{1.3.1}$$

where the elements are given by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\theta}) e^{in\theta} d\theta \tag{1.3.2}$$

with the weight

$$a(\zeta) = \left[\frac{1 - k^{-1}\zeta^{-1}}{1 - k^{-1}\zeta} \right]^{\frac{1}{2}} \tag{1.3.3}$$

and k is given by 1.2.6. In 1.3.1, as is conventional, the position of the spin of the lattice has been denoted by (N, N), not to be confused with the use of N in 1.2.9 as relating to the size of the lattice. In particular in 1.3.1, the size of the lattice has been taken to infinity.

It is well known that the Toeplitz elements 1.3.2 have ${}_2F_1$ hypergeometric function representations [16], stemming from its integral form

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xz)^{-a} dx$$

defined for $\operatorname{Re} c > \operatorname{Re} b > 0$. In the low temperature regime, $1 < k < \infty$,

$$a_n = -\frac{\Gamma(n-1/2)\Gamma(3/2)}{\pi\Gamma(n+1)} k^{-n} \, {}_2F_1(1/2, n-1/2; n+1; k^{-2}), \tag{1.3.4}$$

$$a_{-n} = \frac{\Gamma(n+1/2)\Gamma(1/2)}{\pi\Gamma(n+1)} k^{-n} \, {}_2F_1(-1/2, n+1/2; n+1; k^{-2}), \quad \text{for } n \geq 0 \tag{1.3.5}$$

and in the high temperature regime, $0 \leq k < 1$,

$$a_n = -\frac{\Gamma(n-1/2)\Gamma(1/2)}{\pi\Gamma(n)} k^{n-1} {}_2F_1(-1/2, n-1/2; n; k^2), \tag{1.3.6}$$

$$a_{-n} = \frac{\Gamma(n+1/2)\Gamma(3/2)}{\pi\Gamma(n+2)} k^{n+1} {}_2F_1(1/2, n+1/2; n+2; k^2), \quad \text{for } n \geq 0. \tag{1.3.7}$$

Using the contiguous relations of the hypergeometric function,

$${}_2F_1(a,b;c;z) = \frac{(1-c)_m t^{-m}}{(b-c+1)_m} \sum_{j=0}^{m} \binom{m}{j} (z-1)^{m-j} {}_2F_1(a-j, b; c-m; z) \tag{1.3.8}$$

where

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} \tag{1.3.9}$$

the elements satisfy the difference equation

$$(2n+3)a_{n+1} - 2[(n+1)k^{-1} + nk]a_n + (2n-1)a_{n-1} = 0 \tag{1.3.10}$$

with initial conditions, for $0 \leq k < 1$

$$a_0 = \frac{2}{\pi k}[(k^2-1)K(k) + E(k)] \tag{1.3.11}$$

$$a_{-1} = -\frac{2}{\pi} E(k) \tag{1.3.12}$$

and for $1 < k < \infty$

$$a_0 = \frac{2}{\pi} E(k) \tag{1.3.13}$$

$$a_{-1} = \frac{2}{\pi k}[(k^2-1)K(k) - k^2 E(k)] \tag{1.3.14}$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}} = \frac{\pi}{2} {}_2F_1(1/2, 1/2; 1; k^2) \tag{1.3.15}$$

$$E(k) = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2} d\theta = \frac{\pi}{2} {}_2F_1(-1/2, 1/2; 1; k^2) \tag{1.3.16}$$

are the complete elliptic integrals of the first and second kind. This provides another way to generate the Toeplitz elements. Immediately from this for small N, there are exact expressions for the diagonal correlation in terms of special

functions. The first few are

$$\langle \sigma_{0,0}\sigma_{1,1}\rangle = \begin{cases} \frac{2}{\pi}E(k), & 1 < k < \infty \\ \frac{2}{\pi k}[(k^2-1)K(k)+E(k)], & 0 \le k < 1 \end{cases}$$

$$\langle \sigma_{0,0}\sigma_{2,2}\rangle = \begin{cases} -\dfrac{4}{3k^2\pi^2}[3(k^2-1)^2K(k)^2 - 2(k^2-1)^2K(k)E(k) \\ \qquad\qquad - k^2(k^2-5)E(k)^2], \quad 1 < k < \infty \\ \dfrac{4}{3k^2\pi^2}[3(k^2-1)^2K(k)^2 + 8(k^2-1)K(k)E(k) \\ \qquad\qquad - (k^2-5)E(k)^2], \quad 0 \le k < 1 \end{cases}$$

A significant fact is that 1.3.1 can be used to reclaim 1.2.10. Relevant for this purpose is the logarithm of the weight $a(\zeta)$ and in particular its Fourier series expansion

$$\log a(\zeta) = \sum_{p=-\infty}^{\infty} c_p \zeta^p, \quad |\zeta| = 1 \tag{1.3.17}$$

Assuming that

$$\sum_{p=-\infty}^{\infty} |pc_p| < \infty$$

We can envoke Szego's theorem [9] to obtain a large N expansion for $\langle \sigma_{0,0}\sigma_{N,N}\rangle$.

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle \sim \exp\left(nc_0 + \sum_{p=1}^{\infty} pc_p c_{-p} + \cdots\right) \quad \text{as } n \to \infty \tag{1.3.18}$$

For low temperature $k > 1$,

$$c_0 = 0$$
$$c_p = \frac{1}{2p}k^{-|p|}$$

giving the famous result

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle \sim (1-k^2)^{1/4} = \mathcal{M}^2 \tag{1.3.19}$$

as implicit in the result of Yang [19].

1.4 Form factor expansions of the diagonal correlations

The determinant 1.3.1 can be expressed as in an exponential form [10]. For $T < T_C$,

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{1/4}\exp\left(\sum_{p=1}^{\infty} F_N^{(2p)}\right) \tag{1.4.1}$$

where

$$F_N^{(2p)} = \frac{(-1)^{p+1}}{(2\pi)^{2p}p} \lim_{\epsilon\to 0} \prod_{j=1}^{2p} \oint_{|z_j|=1-\epsilon} dz_j \prod_{j=1}^{2p} \frac{z_j^N}{1-z_j z_{j+1}} \times \prod_{j=1}^{p} P(z_{2j})P(z_{2j}^{-1})Q(z_{2j-1})Q(z_{2j-1}^{-1}) \quad (1.4.2)$$

with $z_{2p+1} = z_1$ and

$$P(z) = \frac{1}{Q(z)} = (1-k^{-1}z)^{1/2} \quad (1.4.3)$$

For $T > T_C$,

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{1/4} \sum_{p=1}^{\infty} X_N^{(2p+1)} \exp\left(\sum_{p=1}^{\infty} \hat{F}_{N+1}^{(2p)}\right) \quad (1.4.4)$$

where

$$\hat{F}_N^{(2p)} = \frac{(-1)^{p+1}}{(2\pi)^{2p}p} \lim_{\epsilon\to 0} \prod_{j=1}^{2p} \oint_{|z_j|=1-\epsilon} dz_j \prod_{j=1}^{2p} \frac{z_j^N}{1-z_j z_{j+1}} \times \prod_{j=1}^{p} \hat{P}(z_{2j})\hat{P}(z_{2j}^{-1})\hat{Q}(z_{2j-1})\hat{Q}(z_{2j-1}^{-1}) \quad (1.4.5)$$

and

$$X_N^{(2p+1)} = \frac{1}{(2\pi i)^{2p+1}} \lim_{\epsilon\to 0} \prod_{j=1}^{2p+1} \oint_{|z_j|=1-\epsilon} dz_j z_j^{N+1} \frac{1}{1-z_1 z_{2p+1}} \prod_{j=1}^{2p} \frac{1}{1-z_j z_{j+1}} \times \prod_{j=1}^{p+1} \hat{P}(z_{2j-1})\hat{P}(z_{2j-1}^{-1}) \prod_{j=1}^{p} \hat{Q}(z_{2j})\hat{Q}(z_{2j}^{-1}) \quad (1.4.6)$$

with

$$\hat{P}(z) = \frac{1}{\hat{Q}(z)} = (1-kz)^{-1/2} \quad (1.4.7)$$

The exponential form 1.4.1 and 1.4.4 can be expanded to obtain the form factor expressions for the correlation functions. We first introduce the new variable

$$t = \begin{cases} k^{-2} & \text{for } T < T_C \\ k^2 & \text{for } T > T_C \end{cases} \quad (1.4.8)$$

For $T < T_C$,

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{\frac{1}{4}} \left(1 + \sum_{p=1}^{\infty} f_{N,N}^{(2p)}\right) \quad (1.4.9)$$

and for $T > T_C$

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{\frac{1}{4}} \sum_{p=0}^{\infty} f_{N,N}^{(2p+1)} \tag{1.4.10}$$

where the diagonal form factors $f_{N,N}^{(p)}(t)$ are given by

$$\begin{aligned} f_{N,N}^{(2p)} = \frac{t^{p(N+p)}}{(p!)^2\pi^{2p}} \int_0^1 dx_1 \cdots \int_0^1 dx_{2p} \prod_{k=1}^{2p} x_k^N \prod_{j=1}^{p} \left[\frac{(1-tx_{2j})(x_{2j}^{-1}-1)}{(1-tx_{2j-1})(x_{2j-1}^{-1}-1)} \right]^{\frac{1}{2}} \\ \times \prod_{j=1}^{p}\prod_{k=1}^{p} (1-tx_{2k-1}x_{2j})^{-2} \\ \times \prod_{1\le j<k\le p} (x_{2j-1}-x_{2k-1})^2 (x_{2j}-x_{2k})^2 \end{aligned} \tag{1.4.11}$$

and

$$\begin{aligned} f_{N,N}^{(2p+1)} =& \frac{t^{N(p+1/2)+p(p+1)}}{p!(p+1)!\pi^{2p+1}} \int_0^1 dx_1 \cdots \int_0^1 dx_{2p+1} \\ & \times \prod_{k=1}^{2p+1} x_k^N \prod_{j=1}^{p+1} \frac{1}{x_{2j-1}} (1-tx_{2j-1})^{-\frac{1}{2}} (x_{2j-1}^{-1}-1)^{-\frac{1}{2}} \\ & \times \prod_{j=1}^{p} x_{2j}(1-tx_{2j})^{\frac{1}{2}} (x_{2j}^{-1}-1)^{\frac{1}{2}} \prod_{j=1}^{p+1}\prod_{k=1}^{p} (1-tx_{2j-1}x_{2k})^{-2} \\ & \times \prod_{1\le j<k\le p+1} (x_{2j-1}-x_{2k-1})^2 \prod_{1\le j<k\le N} (x_{2j}-x_{2k})^2 \end{aligned} \tag{1.4.12}$$

1.5 Differential equation for $\langle \sigma_{0,0}\sigma_{N,N}\rangle$

On the finite square lattice, the spin-spin correlation $\langle \sigma_{0,0}\sigma_{N,N}\rangle$ can be characterised using the Painlevé VI differential equation [8]. For this purpose we introduce the σ function

$$\sigma_N(t) = \begin{cases} t(t-1)\frac{d}{dt}\log\langle \sigma_{0,0}\sigma_{N,N}\rangle - \frac{1}{4}t, & T < T_C \\ t(t-1)\frac{d}{dt}\log\langle \sigma_{0,0}\sigma_{N,N}\rangle - \frac{1}{4}, & T > T_C \end{cases} \tag{1.5.1}$$

The symbol $\sigma_N(t)$ on the left hand side of 1.5.1 is not to be confused with $\sigma_{N,N}$ on the left hand side of 1.3.1 which denotes the spin at position (N,N) of the lattice.

Jimbo and Miwa showed that $\sigma_N(t)$ satisfies the sigma form of the Painlevé

VI given by

$$\left[t(t-1)\frac{\mathrm{d}^2\sigma_N}{\mathrm{d}t^2}\right]^2 = N^2\left[(t-1)\frac{\mathrm{d}\sigma_N}{\mathrm{d}t} - \sigma_N\right]^2 - 4\frac{\mathrm{d}\sigma_N}{\mathrm{d}t}\left[(t-1)\frac{\mathrm{d}\sigma_N}{\mathrm{d}t} - \sigma_N - \tfrac{1}{4}\right]\left[t\frac{\mathrm{d}\sigma_N}{\mathrm{d}t} - \sigma_N\right] \tag{1.5.2}$$

consistent with the expansion as $t \to 0$ ($T < T_C$),

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle(t) = (1-t)^{\frac{1}{4}} + \frac{(1/2)_N(3/2)_N}{4[(N+1)!]^2}t^{N+1}(1+O(t)) \tag{1.5.3}$$

where $(a)_N = 1$ is the Pochhammer's symbol defined in 1.3.9. This follows by approximating the right hand side of 1.4.9 as $(1-t)^{1/4}(1+f^{(2)}_{N,N})$ then expanding $f^{(2)}_{N,N}$ as specified by 1.4.11 for small t. Some details of the general theory relating to the Painlevé equations in sigma form can be found in [4].

Substituting 1.5.3 into 1.5.1 we get the boundary condition for $\sigma_N(t)$ is

$$\sigma_N(t) = \frac{(n+1)(1/2)_N(3/2)_N}{4[(N+1)!]^2}\frac{t^N}{(1-t)^{1/4}} + O(t^{N+1}) \tag{1.5.4}$$

as $t \to 0$.

1.6 Large N solutions for $\langle \sigma_{0,0}\sigma_{N,N}\rangle$ at $T = T_C$

Much can be said about the diagonal correlation at the critical temperature $T = T_C$. Evaluating the Toeplitz element 1.3.2 at $k = 1$ gives

$$a^0_n = \frac{1}{\pi(n+1/2)} \tag{1.6.1}$$

Then 1.3.1 simplifies into an $N \times N$ Cauchy determinant. Using the determinantal formula for the Cauchy matrix [4]

$$\det\left[\frac{1}{x_j - y_k}\right]_{j,k=1,\cdots,N} = (-1)^{N(N-1)/2}\frac{\prod_{1\le j<k\le N}(x_k - x_j)(y_k - y_j)}{\prod_{j,k=1}^N(x_j - y_k)} \tag{1.6.2}$$

it follows from 1.6.2 that at $k =$, 1.3.1 has the exact evaluation [13]

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = \left(\frac{2}{\pi}\right)^N \prod_{p=1}^{N-1}\left(1 - \frac{1}{4p^2}\right)^{p-N} \tag{1.6.3}$$

Some exact values of the diagonal correlation at criticality can now be read off

$$\begin{aligned}
\langle \sigma_{0,0}\sigma_{1,1}\rangle_{T=T_C} &= \frac{2}{\pi} \\
\langle \sigma_{0,0}\sigma_{2,2}\rangle_{T=T_C} &= \frac{16}{3\pi^2} \\
\langle \sigma_{0,0}\sigma_{3,3}\rangle_{T=T_C} &= \frac{2048}{135\pi^3}
\end{aligned}$$

With further analysis we can write an asymptotic expansion for large N as [13]

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle \sim \frac{A}{N^{1/4}}\Big(1 - \frac{1}{64N^2} + O(N^{-4})\Big) \tag{1.6.4}$$

with

$$A = 2^{1/12}\exp\big(3\zeta'(-1)\big) \tag{1.6.5}$$

where $\zeta'(z)$ is the derivative of zeta function.

Chapter 2

Expansions of the diagonal correlation in the scaling limit

2.1 Introducing the scaling limit about criticality

The large N behaviour of the diagonal correlation function is mostly understood from the form factor expansions 1.4.9 and 1.4.10. Away from criticality, for $T < T_C$,

$$\begin{aligned}\langle \sigma_{0,0}\sigma_{N,N}\rangle &\sim (1-t)^{1/4}(1+f^{(2)}_{N,N}) \\ &= (1-t)^{1/4}\left(1+\frac{t^{N+1}}{2\pi N^2(1-t)^2}+\cdots\right)\end{aligned}$$

(compared with 1.5.3) and for $T > T_C$,

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle \sim (1-t)^{1/4} f^{(1)}_{N,N} = \frac{1}{(\pi N)^{1/2}}\frac{t^{N/2}}{(1-t)^{1/4}}+\cdots$$

However it is our interest to see find a similar expansion for the diagonal correlation near the critical temperature. In order to do this, we first seek an appropriate scaling limit for $\langle \sigma_{0,0}\sigma_{N,N}\rangle$. What is meant by this is that we will taking the limits

$$T \to T_C \quad \text{and} \quad N \to \infty \tag{2.1.1}$$

simultaneously such that $N(T-T_C)$ is a fixed quantity. We expect the correlation function to decay exponentially in N so if we define the correlation length $\xi_\pm$ on the diagonal such that

$$t^N = \exp\left(-\frac{\sqrt{2}N}{\xi_\pm}\right)$$

then as $T \to T_C$

$$\xi_{\pm}^{-1} = \frac{|\log t|}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}(1-t)$$

Introducing the scaled coordinate

$$n = N/(2\xi_{\pm})$$

and the length from the origin to the scaled coordinate (n, n)

$$s = \sqrt{2}n$$

Our scaling variable will be defined as

$$s = \frac{N(1-t)}{2} \tag{2.1.2}$$

where t is appropriately defined from 1.4.8.

The scaling limit captures features of the Ising model that distinguish its critical point as relating to a field theory; see e.g. [7]

2.2 Expansions for the diagonal correlation for large s

With the new variable s defined in 2.1.2, we hope to compute expressions for $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ about the critical temperature. It is however no easy task since the expressions $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ are well formulated for small t (temperatures away from criticality). So a good starting point is find asymptotic expressions for the diagonal correlation for large s. This can be achieved by scaling the form factor expansions. If we let, for $T < T_C$,

$$G_{-}(s) = 1 + \sum_{p=1}^{\infty} f_{N,N}^{(2p)} \tag{2.2.1}$$

and for $T > T_C$

$$G_{+}(s) = \sum_{p=0}^{\infty} f_{N,N}^{(2p+1)} \tag{2.2.2}$$

then for $T \gtrless T_C$,

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = (1-t)^{1/4} G_{\pm}(s) \tag{2.2.3}$$

In the limit $s \to \infty$ for the large N expansion if we seek an expansion

$$G_{\pm}(s) = G_0^{\pm}(s) + \frac{1}{N} G_1^{\pm}(s) + O(N^{-2}) \tag{2.2.4}$$

we have

$$G_0^{+}(s) \sim \frac{1}{\pi} K_0(s) \tag{2.2.5}$$

$$G_0^{-}(s) \sim 1 + \frac{1}{\pi^2}\left[s^2\left(K_1(s)^2 - K_0(s)^2\right) - sK_0(s)K_1(s) + \tfrac{1}{2}K_0(s)^2\right] \tag{2.2.6}$$

and

$$G_1^{+}(s) \sim \frac{s}{2\pi}[K_0(s) - 2sK_1(s)] \tag{2.2.7}$$

$$\begin{aligned} G_1^{-}(s) \sim \frac{e^{-2s}}{\pi^2} \int_0^{\infty} dx_1 \int_0^{\infty} dx_2 e^{-2sx_1} e^{-2sx_2} \left[\frac{x_2(1+x_2)}{x_1(1+x_1)}\right]^{\frac{1}{2}} (1+x_1+x_2)^{-2} \\ \times \Bigg[2s^2 + 2s - s(x_1 - x_2) + 2s^2(x_1^2 + x_2^2) \\ - \frac{sx_1}{1+x_1} + \frac{sx_2}{1+x_2} - 4s\frac{x_1x_2 + x_1 + x_2}{1+x_1+x_2}\Bigg] \end{aligned} \tag{2.2.8}$$

where $K_n(s)$ are the modified Bessel functions of the second kind of order n.

Proof. Case 1. For $T > T_C$ using the change of variables $x_j = 1 - (1-t)X_j = 1 - 2sX_j/N$, we note that

$$1 - tx_j = 1 - t(1 - X_j(1-t)) = \frac{2s(1+X_j)}{N}\left(1 - \frac{2sX_j}{N(1+X_j)}\right) \tag{2.2.9a}$$

$$x_j^{-1} - 1 = \frac{1}{1-(1-t)X_j} \sim \frac{2s}{N}X_j + \frac{4s^2}{N^2}X_j^2 \tag{2.2.9b}$$

$$\begin{aligned} 1 - tx_jx_k &= 1 - t(1-(1-t)X_j)(1-(1-t)X_k) \\ &\sim \frac{2s}{N}(1 + X_j + X_k) - \frac{4s^2}{N^2}(X_j + X_k + X_jX_k) \end{aligned} \tag{2.2.9c}$$

$$x_j - x_k = \frac{2s}{N}(X_k - X_j) \tag{2.2.9d}$$

$$t^{Np} = \left(1 - \frac{2s}{N}\right)^{Np} \sim e^{-2sp}\left(1 - \frac{2s^2p}{N}\right) \tag{2.2.9e}$$

$$x_j^N = \left(1 - \frac{2sX_j}{N}\right)^N \sim e^{-2sX_j}\left(1 - \frac{2s^2X_j^2}{N}\right) \tag{2.2.9f}$$

The last two expressions comes from Euler's limit with the correction term for large N

$$\left(1+\frac{x}{N}\right)^N = e^x\left(1-\frac{x^2}{2N}+O(N^{-2})\right). \tag{2.2.10}$$

Substituting 2.2.9a, 2.2.9b, 2.2.9c, 2.2.9d, 2.2.9e, 2.2.9f into the form factor expansion 1.4.12, we get

$$\begin{aligned}
f_{N,N}^{(2p+1)} \sim \quad & \frac{e^{-2s(p+1)}}{p!(p+1)!\pi^{2p+1}}\int_0^\infty dX_1\cdots\int_0^\infty dX_{2p+1}\prod_{j=1}^{2p+1}e^{-2sX_j} \\
& \times\prod_{j=1}^{p}[(1+X_{2j})X_{2j}]^{1/2}\prod_{j=1}^{p+1}[(1+X_{2j-1})X_{2j-1}]^{-1/2} \\
& \times\prod_{j=1}^{p+1}\prod_{k=1}^{p}(1+X_{2j-1}+X_{2k})^{-2} \\
& \times\prod_{1\le j<k\le p+1}(X_{2j-1}-X_{2k-1})^2\prod_{1\le j<k\le p}(X_{2j}-X_{2k})^2 \\
& -\frac{1}{N}\frac{e^{-2s(p+1)}}{p!(p+1)!\pi^{2p+1}}\int_0^\infty dX_1\cdots\int_0^\infty dX_{2p+1}\prod_{j=1}^{2p+1}e^{-2sX_j} \\
& \times\prod_{j=1}^{p}[(1+X_{2j})X_{2j}]^{1/2}\prod_{j=1}^{p+1}[(1+X_{2j-1})X_{2j-1}]^{-1/2} \\
& \times\prod_{j=1}^{p+1}\prod_{k=1}^{p}(1+X_{2j-1}+X_{2k})^{-2} \\
& \times\prod_{1\le j<k\le p+1}(X_{2j-1}-X_{2k-1})^2\prod_{1\le j<k\le p}(X_{2j}-X_{2k})^2 \\
& \times S^{(2p+1)}(X_1,\cdots,X_{2p+1})
\end{aligned} \tag{2.2.11}$$

where

$$\begin{aligned}
S^{(2p+1)}(X_1,\cdots,X_{2p+1}) = & -2s^2(p+1/2)-2sp(p+1)-2s^2\sum_{j=1}^{2p+1}X_j^2 \\
& +\sum_{j=1}^{p+1}\left(sX_{2j-1}+\frac{sX_{2j-1}}{1+X_{2j-1}}\right)+\sum_{j=1}^{p}\left(-sX_{2j}-\frac{sX_{2j}}{1+X_{2j}}\right) \\
& +4s\sum_{j=1}^{p}\sum_{k=1}^{p+1}\frac{X_{2j}+X_{2k-1}+X_{2j}X_{2k-1}}{X_{2j}+X_{2k-1}+1}
\end{aligned}$$

Here the first term in 2.2.11 $F^+_{0,p}$ say has the bound

$$\left|F^+_{0,p}\right| \leq \frac{e^{-2s(p+1}}{p!(p+1)!\pi^{2p+1}} \int_0^\infty dX_1 \cdots \int_0^\infty dX_{2p+1} \\ \times \prod_{j=1}^{2p+1} e^{-2sX_j} \prod_{j=1}^{p}(1+X_{2j})X_{2j}^{1/2} \prod_{j=1}^{p+1} X_{2j-1}^{-1/2} \\ \times \prod_{1\leq j<k\leq p+1}(X_{2j-1}-X_{2k-1})^2 \prod_{1\leq j<k\leq p}(X_{2j}-X_{2k})^2 \tag{2.2.12}$$

These are integrals which can be evaluated by the formula [5]

$$n!W_n(\alpha+1,\gamma) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \prod_{i=1}^{n} x_i^\alpha e^{-x_i} \prod_{1\leq j<k\leq n} \left|x_j - x_k\right|^{2\gamma}$$

where

$$W_n(\alpha,\gamma) = \frac{1}{n!}\prod_{j=0}^{n-1}\frac{\Gamma(\alpha+j\gamma)\Gamma((j+1)\gamma)}{\Gamma(\gamma)}$$

A similar bound for the second term $F^+_{1,p}$ in 2.2.11 can be found. These show

$$F^+_{0,p+1}(s) = o(F^+_{0,p}(s)) \\ F^+_{1,p+1}(s) = o(F^+_{1,p}(s))$$

as $s \to \infty$. So we can just consider

$$f^{(1)}_{N,N} \sim F^+_{0,0} + \frac{1}{N}F^+_{1,0} \tag{2.2.13}$$

to find the large s behaviour of $\langle \sigma_{0,0}\sigma_{N,N}\rangle$. It is now left to evaluate these integrals. Using the change of variables

$$x = (y-1)/2 \tag{2.2.14}$$

and the integral formula of the modified Bessel function of the second kind

$$K_\nu(z) = \frac{\pi^{1/2}z^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_1^\infty dt e^{-zt}(t^2-1)^{\nu-1/2} \tag{2.2.15}$$

where $\mathrm{Re}(\nu) > -\frac{1}{2}$ it follows

$$G^+_0(s) \sim F^+_{0,0} = \frac{e^{-s}}{\pi}\int_0^\infty \frac{dx}{[x(1+x)]^{1/2}}e^{-2sx} \\ = \frac{1}{\pi}\int_1^\infty \frac{dy}{(y^2-1)^{1/2}}e^{-sy} \\ = \frac{1}{\pi}K_0(s) \tag{2.2.16}$$

And using 2.2.14, 2.2.15 and

$$K_1(x) = -K_0'(x) = \int_1^\infty dt \frac{e^{-xt}t}{(t^2-1)^{1/2}} \tag{2.2.17}$$

we get the formulas

$$\begin{aligned}
\int_0^\infty dx \frac{e^{-2sx}x}{[x(1+x)]^{1/2}} &= \frac{e^s}{2}(-K_0(s) + K_1(s)) \\
\int_0^\infty dx \frac{e^{-2sx}x^2}{[x(1+x)]^{1/2}} &= \frac{e^s}{4s}(2sK_0(s) + (1-2s)K_1(s)) \\
\int_0^\infty dx \frac{e^{-2sx}x^{1/2}}{(1+x)^{3/2}} &= e^s((1+2s)K_0(s) - 2sK_1(s))
\end{aligned}$$

which means

$$\begin{aligned}
G_1^+(s) \sim F_{1,0}^+ &= -\frac{e^{-s}}{\pi}\int_0^\infty \frac{dx}{[x(1+x)]^{1/2}} e^{-2sx}\left(s^2 + 2s^2x^2 + sx - \frac{sx}{1+x}\right) \\
&= \frac{s}{2\pi}(K_0(s) - 2sK_1(s))
\end{aligned} \tag{2.2.18}$$

Case 2. for $T < T_C$, using the same change of variables $x_j = 1-(1-t)X_j = 1 - 2sX_j/N$ and substituting 2.2.9a, 2.2.9b, 2.2.9c, 2.2.9d, 2.2.9e, 2.2.9f into the form factor expansion 1.4.11, we get

$$\begin{aligned}
f_{N,N}^{(2p)} \sim 1 &+ \frac{e^{-2sp}}{(p!)^2\pi^{2p}}\int_0^\infty dX_1 \cdots \int_0^\infty dX_{2p} \prod_{j=1}^{2p} e^{-2sX_j} \\
&\times \prod_{j=1}^{p}[\frac{(1+X_{2j})X_{2j}}{(1+X_{2j-1})X_{2j-1}}]^{1/2} \\
&\times \prod_{j=1}^{p}\prod_{k=1}^{p}(1+X_{2j}+X_{2k-1})^{-2} \prod_{1\le j<k\le p}(X_{2j-1}-X_{2k-1})^2(X_{2j}-X_{2k})^2 \\
&- \frac{1}{N}\frac{e^{-2sp}}{(p!)^2\pi^{2p}}\int_0^\infty dX_1 \cdots \int_0^\infty dX_{2p} \prod_{j=1}^{2p} e^{-2sX_j} \\
&\times \prod_{j=1}^{p}[\frac{(1+X_{2j})X_{2j}}{(1+X_{2j-1})X_{2j-1}}]^{1/2} \\
&\times \prod_{j=1}^{p}\prod_{k=1}^{p}(1+X_{2j}+X_{2k-1})^{-2} \prod_{1\le j<k\le p}(X_{2j-1}-X_{2k-1})^2(X_{2j}-X_{2k})^2 \\
&\times S^{(2p)}(X_1, \cdots, X_{2p})
\end{aligned} \tag{2.2.19}$$

where

$$S^{(2p)}(X_1, \cdots, X_{2p}) = -2s^2 - 2sp^2 - 2s^2 \sum_{j=1}^{2p} X_j$$
$$+ \sum_{j=1}^{p} \Big(- \frac{sX_{2j}}{1+X_{2j}} + \frac{sX_{2j-1}}{1+X_{2j-1}} + sX_{2j} - sX_{2j-1} \Big)$$
$$+ 4s \sum_{j,k=1}^{p} \frac{X_{2j} + X_{2k-1} + X_{2j}X_{2k-1}}{X_{2j} + X_{2k-1} + 1}$$

Similar to 2.2.12, the first and the second term $F^-_{0,p}$ and $F^-_{1,p}$ of 2.2.19 can be written as

$$F^-_{0,p+1}(s) = o(F^-_{0,p}(s)) \tag{2.2.20}$$
$$F^-_{1,p+1}(s) = o(F^-_{1,p}(s)) \tag{2.2.21}$$

as $s \to \infty$. Consider only

$$f^{(2)}_{N,N} \sim F^-_{0,1} + \frac{1}{N} F^-_{1,1} \tag{2.2.22}$$

To evaluate the double integral $F^-_{0,1}$, we can use the technique

$$\frac{1}{a^2} = \int_0^\infty u e^{-au} du \tag{2.2.23}$$

on the coupled factors in the integrals in 2.2.19.

$$\begin{aligned} F^-_{0,1} &= \frac{e^{-2s}}{\pi^2} \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-2sx_1} e^{-2sx_2} \left[\frac{(1+x_2)x_2}{(1+x_1)x_1} \right]^{1/2} (1+x_1+x_2)^{-2} \\ &= \frac{e^{-2s}}{\pi^2} \int_0^\infty dt e^{-t} t \int_0^\infty dx_1 e^{-x_1(2s+t)} [(1+x_1)x_1]^{-1/2} \\ &\qquad \times \int_0^\infty dx_2 e^{-x_2(2s+t)} [(1+x_2)x_2]^{1/2} \end{aligned} \tag{2.2.24}$$

Again we use the change of variables $x_j = (y_j - 1)/2$ for $j = 1, 2$ and the integral representation of the modified Bessel function 2.2.15,

$$F^-_{0,1} = \frac{1}{\pi^2} \int_0^\infty \frac{t}{4s+2t} K_0(s+t/2) K_1(s+t/2) dt \tag{2.2.25}$$
$$= \frac{1}{\pi^2} \int_s^\infty \frac{u-s}{u} K_0(u) K_1(u) du \tag{2.2.26}$$

Using the integral formulas

$$\int K_0(x) K_1(x) dx = -\frac{1}{2} K_0(x)^2$$
$$\int \frac{K_0(x) K_1(x)}{x} dx = x[K_1(x)^2 - K_0(x)^2] - K_0(x) K_1(x)$$

allows us to conclude that

$$G_0^-(s) \sim 1 + F_{0,1}^- = 1 + \frac{1}{\pi^2}\left[s^2\left(K_1(s)^2 - K_0(s)^2\right) - sK_0(s)K_1(s) + \tfrac{1}{2}K_0(s)^2\right] \tag{2.2.27}$$

and

$$\begin{aligned} G_1^-(s) &\sim F_{1,1}^- \\ &= \frac{e^{-2s}}{\pi^2}\int_0^\infty dx_1 \int_0^\infty dx_2 e^{-2sx_1}e^{-2sx_2}\left[\frac{x_2(1+x_2)}{x_1(1+x_1)}\right]^{\frac{1}{2}}(1+x_1+x_2)^{-2} \\ &\times\Bigg[2s^2 + 2s - s(x_1 - x_2) + 2s^2(x_1^2 + x_2^2) - \frac{sx_1}{1+x_1} + \frac{sx_2}{1+x_2} \\ &\quad - 4s\frac{x_1x_2 + x_1 + x_2}{1+x_1+x_2}\Bigg] \end{aligned} \tag{2.2.28}$$

□

2.3 Expansions for the diagonal correlation for small s

From the works from Wu et al. [18], we can find a systematic way of expressing the diagonal correlation $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ in the scaling limit. Manipulating the Toeplitz determinantal form for the diagonal correlation 1.3.1 we have

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = \det(A_0)\det\left(1 + \Delta A_0^{-1}\right) \tag{2.3.1}$$

where

$$A_0 = [a_{j-k}^0]_{j,k=1,\cdots,N} \tag{2.3.2}$$

and

$$\Delta = [a_{j-k} - a_{j-k}^0]_{j,k=1,\cdots,N} \tag{2.3.3}$$

The elements in 2.3.2 and 2.3.3 are defined by 1.3.2 and 1.6.1. We should recall that the explicit expression and its asymptotic expansion of $\det A_0$ is given in 1.6.3 and 1.6.4. If $T \sim T_C$ such that Δ is small then the second factor in 2.3.1 has the expansion

$$\det\left(1 + \Delta A_0^{-1}\right) = \exp\left(\sum_{p=1}^{\infty}\frac{(-1)^p}{p}\operatorname{Tr}\left[(\Delta A_0^{-1})^p\right]\right) \tag{2.3.4}$$

To analyse 2.3.4 we first consider the leading term (when $p = 1$). Recalling that A_0 is a Cauchy matrix, we can make use of the fact that its inverse can be computed explicitly [17] to give

$$(A_0^{-1})_{k,l} = \frac{f(k)g(l)}{\pi(k-l+1/2)} \tag{2.3.5}$$

where

$$f(k) = \frac{\Gamma(N-k+\frac{1}{2})\Gamma(k+\frac{3}{2})}{\Gamma(N-k+1)\Gamma(k+1)} \tag{2.3.6}$$

$$g(l) = \frac{\Gamma(N-1+\frac{3}{2})\Gamma(l+\frac{1}{2})}{\Gamma(N-l+1)\Gamma(l+1)} \tag{2.3.7}$$

for $k, l = 0, 1, \cdots, N-1$. Then consequently

$$\mathrm{Tr}\left(\Delta A_0^{-1}\right) = \sum_{k,l=0}^{N-1} \frac{f(k)g(l)}{\pi(k-l+1/2)} \Delta_{k-l} \tag{2.3.8}$$

where Δ_n are the elements of 2.3.3. To summarise the results of the analysis of this double sum from [18], if we let $r = k(1-t)/2$, $r' = l(1-t)/2$ and $\kappa = (1-t)/2$ then in the limit $N \to \infty$ and $T \to T_C^{\pm}$, the sum 2.3.8 can be approximated by the integral

$$\mathrm{Tr}\left(\Delta A_0^{-1}\right) = \frac{1}{\pi^2} \int_0^s dr \int_0^s dr' \frac{f(r/\kappa)g(r'/\kappa)}{r - r' + \frac{1}{2}\kappa} \Delta_{\pm}(r - r') + E \tag{2.3.9}$$

where E is the error term. Here

$$\Delta_n \sim \frac{1}{2\pi}(1-t)\Delta_{\pm}(r) \tag{2.3.10}$$

in the limit $n \to \pm\infty$ and $T \to T_C^{\pm}$, with

$$\Delta_{\pm}(r) = \theta(r)K_1(|r|) \mp K_0(|r|) - \frac{1}{r} \tag{2.3.11}$$

$$\theta(r) = \begin{cases} +1 & \text{if } r \geq 0 \\ -1 & \text{if } r < 0 \end{cases} \tag{2.3.12}$$

2.3.11 comes from the limit $n \to \infty$ and $T \to T_C^{\pm}$ of the Toeplitz elements 1.3.2 and 1.6.1 after the transformation $r = n(1-t)/2$. Using Stirling's formula for small κ

$$f(r/\kappa)g(r/\kappa) \sim \left[\frac{u(1-u')}{u'(1-u)}\right]^{1/2}$$

then to leading order in the limit $T \to T_C$ and $N \to \infty$, 2.3.9 can be expressed as

$$\mathrm{Tr}\Big(\Delta A_0^{-1}\Big) \sim \frac{1}{\pi^2} P \int_0^s dr P \int_0^s dr' \left[\frac{r(s-r')}{r'(s-r)}\right]^{1/2} \frac{\Delta_\pm(r-r')}{r-r'}$$

where P denotes the Cauchy principal value. From the change of variables $r = su$ and $r' = su'$, we get

$$\mathrm{Tr}\Big(\Delta A_0^{-1}\Big) \sim \frac{s}{\pi^2} P \int_0^1 du P \int_0^1 du' \left[\frac{u(1-u')}{u'(1-u)}\right]^{1/2} \frac{\Delta_\pm[s(u-u')]}{u-u'} \tag{2.3.13}$$

If we assume the diagonal correlation admits the form

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = \frac{\hat{G}_0^\pm(s)}{N^{1/4}} + \frac{\hat{G}_1^\pm(s)}{N^{5/4}} + O(N^{-9/4}) \tag{2.3.14}$$

for $T \gtrless T_C$, then from 2.3.4 and 2.3.13

$$\hat{G}_0^\pm(s) \sim A \exp\Big(\mathrm{Tr}\Big(\Delta A_0^{-1}\Big)\Big). \tag{2.3.15}$$

If we use the small r expansion following from 2.3.11

$$\Delta_+(r) \sim \log\Big(\frac{r}{2}\Big) + \gamma_E \tag{2.3.16}$$

$$\Delta_-(r) \sim -\log\Big(\frac{r}{2}\Big) - \gamma_E \tag{2.3.17}$$

where $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant, then from 2.3.13

$$\begin{aligned}
\mathrm{Tr}\Big(\Delta A_0^{-1}\Big) &\sim \pm \frac{s}{\pi^2} P \int_0^1 du P \int_0^1 du' \left[\frac{u(1-u')}{u'(1-u)}\right]^{1/2} \frac{1}{u-u'}\Big[\log\frac{|s(u-u')|}{2} + \gamma_E\Big] \\
&= \pm \frac{s\log s}{\pi^2} P \int_0^1 du P \int_0^1 du' \left[\frac{u(1-u')}{u'(1-u)}\right]^{1/2} \frac{\log|u-u'|}{u-u'} \\
&\quad \pm \frac{s(\gamma_E - \log 2)}{\pi^2} P \int_0^1 du P \int_0^1 du' \left[\frac{u(1-u')}{u'(1-u)}\right]^{1/2} \frac{1}{u-u'} \\
&= \pm \frac{1}{2} s(\log(s/8) + \gamma_E)
\end{aligned} \tag{2.3.18}$$

Combining 2.3.18 and 2.3.15 gives

$$\hat{G}_0^\pm(s) \sim A\Big(1 \pm \frac{s}{2}(\log(s/8) + \gamma_E)\Big) \tag{2.3.19}$$

Chapter 3

The Painlevé VI in the scaling limit

The first two terms of the Painlevé VI characteristion 1.5.2 of σ_N also permit differential equation characterisations in the scaling limit.

The following result holds true. Introduce the scaling $s = N(1-t)/2$ and suppose that the solution to 1.5.2 can be written in the form $\sigma_N = \hat{\sigma}_0(s) + \frac{1}{N}\hat{\sigma}_1(s)$. Then the leading order function $\hat{\sigma}_0(s)$ satisfy the Painlevé V equation

$$(s\hat{\sigma}_0''(s))^2 = 4(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s))^2 - 4(\hat{\sigma}_0'(s))^2(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)) + (\hat{\sigma}_0'(s))^2 \tag{3.0.1}$$

and $\hat{\sigma}_1(s)$ satisfy the second order linear differential equation

$$A(s)\hat{\sigma}_1''(s) + B(s)\hat{\sigma}_1'(s) + C(s)\hat{\sigma}_1(s) = D(s) \tag{3.0.2}$$

where

$$A(s) = \frac{1}{2}s^2\hat{\sigma}_0''(s) \tag{3.0.3}$$

$$B(s) = s(\hat{\sigma}_0'(s))^2 - 2s(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)) + 2\hat{\sigma}_0'(s)(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s) - 1/4) \tag{3.0.4}$$

$$C(s) = 2(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)) - (\hat{\sigma}_0'(s))^2 \tag{3.0.5}$$

$$D(s) = s^3(\hat{\sigma}_0''(s))^2 + 2\hat{\sigma}_0'(s)(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s))(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s) - 1/4). \tag{3.0.6}$$

Proof. We begin with the Painlevé VI 1.5.2 and substitute the proposed form $\sigma_N = \hat{\sigma}_0(s) + \frac{1}{N}\hat{\sigma}_1(s)$ and the scaled variable $s = N(1-t)/2$ to replace t. Expanding

the first, second and third terms respectively gives

$$
\begin{aligned}
&[t(t-1)\sigma_N''(t)]^2 = [s\hat{\sigma}_0''(s)]^2 N^2 + [2s^2\hat{\sigma}_0''(s)\hat{\sigma}_1''(s) - 2s^3\hat{\sigma}_0''(s)^2]N + O(1) \\
&N^2[(t-1)\sigma_N'(t) - \sigma_N(t)]^2 = [\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)]^2 N^2 \\
&\qquad\qquad + 2[\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)][\hat{\sigma}_1'(s) - \hat{\sigma}_1(s)]N + O(1) \\
&\sigma_N'(t)[(t-1)\sigma_N'(t) - \sigma_N(t) - 1/4][t\sigma_N'(t) - \sigma_N(t)] = \\
&\quad \hat{\sigma}_0'(s)^2[s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s) - 1/4]N^2 + (2\hat{\sigma}_0'(s)\hat{\sigma}_1'(s)[s\hat{\sigma}_0'(s) - \hat{\sigma}_0 - 1/4] \\
&\quad + \hat{\sigma}_0'(s)^2[s\hat{\sigma}_1'(s) - \hat{\sigma}_1(s)] - \hat{\sigma}_0'(s)[s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)][s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s) - 1/4])N \\
&\quad + O(1)
\end{aligned}
$$

Comparing the coefficients of $O(N^2)$ and $O(N)$ produces 3.0.1 and 3.0.2 respectively. □

The first equation equation 3.0.1 is a known result and the second 3.0.2 is original to the present work. These differential equations and their characterisations of the spin-spin correlation will the basis of the study for the rest of this thesis. To relate this back to the Ising model's diagonal correlation, we also need to scale 1.5.1. We assume the form of the diagonal correlation in 2.2.3 and 2.2.4 and again use the scaled variable s then for $T < T_C$,

$$
\begin{aligned}
\sigma_N(s) = \frac{s\frac{\mathrm{d}G_0^-(s)}{\mathrm{d}s}}{G_0^-(s)} + \frac{1}{N}\frac{-2s^2\frac{\mathrm{d}G_0^-(s)}{\mathrm{d}s}G_0^-(s) + sG_0^-(s)\frac{\mathrm{d}G_1^-(s)}{\mathrm{d}s} - s\frac{\mathrm{d}G_0^-(s)}{\mathrm{d}s}G_1^-(s)}{G_0^-(s)^2} \\
+ O(N^{-2})
\end{aligned}
$$

and for $T > T_C$,

$$
\begin{aligned}
\sigma_N(s) = \frac{s\frac{\mathrm{d}G_0^+(s)}{\mathrm{d}s}}{G_0^+(s)} + \frac{1}{N}\left(\frac{-2s^2\frac{\mathrm{d}G_0^+(s)}{\mathrm{d}s}G_0^+(s) + sG_0^+(s)\frac{\mathrm{d}G_1^+(s)}{\mathrm{d}s} - s\frac{\mathrm{d}G_0^+(s)}{\mathrm{d}s}G_1^+(s)}{G_0^+(s)^2} - \frac{s}{2}\right) \\
+ O(N^{-2})
\end{aligned}
$$

So we have

$$
\hat{\sigma}_0(s) = \frac{s\frac{\mathrm{d}G_0^\pm(s)}{\mathrm{d}s}}{G_0^\pm(s)} \quad \text{for } T \gtrless T_C \tag{3.0.7}
$$

$$
\hat{\sigma}_1(s) = \begin{cases} \frac{-2s^2\frac{\mathrm{d}G_0^-(s)}{\mathrm{d}s}G_0^-(s) + sG_0^-(s)\frac{\mathrm{d}G_1^-(s)}{\mathrm{d}s} - s\frac{\mathrm{d}G_0^-(s)}{\mathrm{d}s}G_1^-(s)}{G_0^-(s)^2} & \text{for } T < T_C \\ \frac{-2s^2\frac{\mathrm{d}G_0^+(s)}{\mathrm{d}s}G_0^+(s) + sG_0^+(s)\frac{\mathrm{d}G_1^+(s)}{\mathrm{d}s} - s\frac{\mathrm{d}G_0^+(s)}{\mathrm{d}s}G_1^+(s)}{G_0^+(s)^2} - \frac{s}{2} & \text{for } T > T_C \end{cases} \tag{3.0.8}
$$

Expressions for $G_0^\pm$ and $G_1^\pm$ have been given for $s \to 0$ and $s \to \infty$ from the form factor expansions and the Toeplitz determinant discussed above. So 3.0.7 and 3.0.8 will be used to provide the boundary conditions to the differential equations 3.0.1 and 3.0.2.

3.1 The Painlevé V as a Painlevé III

The scaled function $G_0^{\pm}(s)$ was shown to satisfy another nonlinear differential equation [2] [18]. If one expresses the scaled function as [11]

$$G_0^{\pm}(s) = \frac{1}{2}\eta(s/2)^{-1/2}[1 \pm \eta(s/2] \exp \int_{s/2}^{\infty} \frac{1}{4}\theta\eta^{-2}[(1-\eta^2)^2 - (\eta')^2]d\theta \tag{3.1.1}$$

then from 3.0.1, $\eta(\theta)$ satisfies the Painlevé III differential equation

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\theta^2} = \frac{1}{\eta}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\theta}\right)^2 - \frac{1}{\theta}\frac{\mathrm{d}\eta}{\mathrm{d}\theta} + \eta^3 - \eta^{-1} \tag{3.1.2}$$

with the boundary condition

$$\eta(\theta) \sim 1 - \frac{2}{\pi}K_0(2\theta) \tag{3.1.3}$$

as $\theta \to \infty$.

By studying the Painlevé III [12], a local expansion of $\eta(s/2)$ can be found. However instead of musing in the technicalities of the Painlevé III, we will instead solve for $G_0^{\pm}(s)$ by directly computing a series solution for the Painlevé V by assuming an ansatz, as first done in [1], but without the details of the working. Numerical data for $\eta(s)$ presented in [18] can be used to check consistency between the methods.

3.2 Series solution of Painlevé V about $s \sim 0$

If we recall back from 3.2.1, the expansion about $s \sim 0$ for the leading behaviour of 2.3.19 can be written as sum of products of s and $\log s$. Combining the assumed forms of the diagonal correlation $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ 2.2.3 and 2.3.14 under the scaled variable s shows that

$$G_0^{\pm}(s) = \frac{1}{(2s)^{1/4}}\hat{G}_0^{\pm}(s) \tag{3.2.1}$$

Substituting 3.2.1 into the 3.0.7 to relate this back to the Painlevé V shows that for $T \gtrless T_C$,

$$\hat{\sigma}_0(s) \sim -\frac{1}{4} \pm \left(\frac{1}{2} + \frac{1}{2}L(s)\right)s \tag{3.2.2}$$

as $s \to 0$. In order to generate a series solution for 3.0.1, suppose we seek the solution for $\hat{\sigma}_0(s)$ with the form that is consistent with 3.2.2

$$\hat{\sigma}_0(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{m,n}L(s)^m s^n \tag{3.2.3}$$

with

$$L(s) = \log\left(\frac{s}{8}\right) + \gamma_E. \tag{3.2.4}$$

Firstly we note that

$$\hat{\sigma}_0'(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{m,n}(nL(s)+m)L(s)^{m-1}s^{n-1}$$
$$\hat{\sigma}_0''(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{m,n}[(n(n-1)L(s)^2 + m(2n-1)L(s) + m(m-1)]L(s)^{m-2}s^{n-2}$$

Substituting the ansatz 3.2.3 and its derivatives into the Painlevé differential equation 3.0.1, we hope to extract a recurrence relation for the coefficients $c_{m,n}$. By employing the Cauchy multiplication rule for power series,

$$\Big(\sum_{n=0}^{\infty} a_n s^n\Big) \times \Big(\sum_{n=0}^{\infty} b_n s^n\Big) = \sum_{n=0}^{\infty} c_n s^n$$

with

$$c_n = \sum_{p=0}^{n} a_p b_{n-p}$$

we will calculate each term in 3.0.1. For the first term, let

$$\alpha_n(s) = \sum_{m=0}^{n} c_{m,n}[(n(n-1)L(s)^2 + m(2n-1)L(s) + m(m-1)]L(s)^{m-2} \tag{3.2.5}$$

then

$$[s\hat{\sigma}_0''(s)]^2 = \frac{1}{s^2}\sum_{n=0}^{\infty} A_n(s)s^n \tag{3.2.6}$$

where

$$A_n(s) = \sum_{m=0}^{n} \alpha_m(s)\alpha_{n-m}(s) \tag{3.2.7}$$

For the second term, let

$$\beta_n(s) = \sum_{m=0}^{n} c_{m,n}[(n-1)L(s)+m]L(s)^{m-1} \tag{3.2.8}$$

then

$$[s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)]^2 = \sum_{n=0}^{\infty} B_n s^n \tag{3.2.9}$$

where

$$B_n(s) = \sum_{m=0}^{n} \beta_m(s)\beta_{n-m}(s) \tag{3.2.10}$$

For the final terms, let

$$\gamma_n(s) = \sum_{m=0}^{n} c_{m,n}[nL(s) + m]L(s)^{m-1} \tag{3.2.11}$$

then

$$[\hat{\sigma}_0'(s)]^2 = \frac{1}{s^2}\sum_{n=0}^{\infty} C_n(s)s^n \tag{3.2.12}$$

$$[\hat{\sigma}_0'(s)]^2[s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)] = \frac{1}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)C_{n-m}(s)\Big)s^n \tag{3.2.13}$$

where

$$C_n(s) = \sum_{m=0}^{n}\gamma_m\gamma_{n-m} \tag{3.2.14}$$

Putting 3.2.6, 3.2.9, 3.2.12, 3.2.13 into 3.0.1 gives

$$\frac{1}{s^2}\sum_{n=0}^{\infty}A_n(s)s^n = 4\sum_{n=0}^{\infty}B_n(s)s^n - \frac{4}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)C_{n-m}(s)\Big)s^n + \frac{1}{s^2}\sum_{n=0}^{\infty}C_n(s)s^n \tag{3.2.15}$$

Equating the powers of s gives the recurrence

$$A_n(s) - 4B_{n-2}(s) + 4\sum_{m=0}^{n}\beta_m(s)C_{n-m}(s) - C_n(s) = 0 \tag{3.2.16}$$

The expression of the left hand side of 3.2.16 is polynomial with respect to the variable $L(s)$. Equating powers of $L(s)$ finally gives a system of equations that allows us to determine $c_{m,n}$.
Now that we have a recurrence that solves 3.0.1, we need some boundary conditions to initiate values for $c_{m,n}$. This means that the choice of initial values for the recurrence 3.2.16 are

$$c_{0,0} = \frac{1}{4}$$
$$c_{1,1} = \pm\frac{1}{2} \quad \text{for } T \gtrless T_C$$

Some of the few equations for $c_{m,n}$ are

$$
\begin{aligned}
&c_{1,1}^2 - \frac{1}{4} = 0 \\
&-2c_{1,1} + 4c_{0,1}^2 c_{1,1} + 4c_{0,2}c_{1,1} + 8c_{0,1}c_{1,1}^2 + 4c_{1,1}^3 + 6c_{1,1}c_{1,2} + 4c_{1,1}c_{2,2} = 0 \\
&8c_{0,1}c_{1,1}^2 + 8c_{1,1}^3 + 4c_{1,1}c_{1,2} + 12c_{1,1}c_{2,2} = 0 \\
&4c_{1,1}^3 + c_{1,1}c_{2,2} = 0 \\
&-2c_{0,2} + 4c_{0,1}^2 c_{0,2} + 4c_{0,2}^2 + 24c_{0,1}c_{0,2}c_{1,1} + 12c_{0,3}c_{1,1} - 4c_{1,1}^2 + 20c_{0,2}c_{1,1}^2 - 2c_{1,2} \\
&\qquad + 4c_{0,1}^2 c_{1,2} + 12c_{0,2}c_{1,2} + 16c_{0,1}c_{1,1}c_{1,2} + 12c_{1,1}^2 c_{1,2} + 9c_{1,2}^2 + 10c_{1,1}c_{1,3} \\
&\qquad + 8c_{0,2}c_{2,2} + 12c_{1,2}c_{2,2} + 4c_{2,2}^2 + 4c_{1,1}c_{2,3} = 0 \\
&4c_{0,1}c_{0,2}c_{1,1} + 12c_{0,2}c_{1,1}^2 - c_{1,2} + 2c_{0,1}^2 c_{1,2} + 4c_{0,2}c_{1,2} + 16c_{0,1}c_{1,1}c_{1,2} + 18c_{1,1}^2 c_{1,2} \\
&\qquad + 6c_{1,2}^2 + 6c_{1,1}c_{1,3} - 2c_{2,2} + 4c_{0,1}^2 c_{2,2} + 12c_{0,1}c_{2,2} + 16c_{0,1}c_{1,1}c_{2,2} \\
&\qquad + 12c_{1,1}^2 c_{2,2} + 22c_{1,2}c_{2,2} + 12c_{2,2}^2 + 10c_{1,1}c_{2,3} + 6c_{1,1}c_{3,3} = 0 \\
&2c_{0,2}c_{1,1}^2 + 4c_{0,1}c_{1,1}c_{1,2} + 14c_{1,1}^2 c_{1,2} + 2c_{1,2}^2 - c_{2,2} + 2c_{0,1}^2 c_{2,2} + 4c_{0,2}c_{2,2} \\
&\qquad + 20c_{0,1}c_{1,1}c_{2,2} + 26c_{1,1}^2 c_{2,2} + 18c_{1,2}c_{2,2} + 22c_{2,2}^2 + 6c_{1,1}c_{2,3} \\
&\qquad + 15c_{1,1}c_{3,3} = 0 \\
&c_{1,1}^2 c_{1,2} + 2c_{0,1}c_{1,1}c_{2,2} + 8c_{1,1}^2 c_{2,2} + 2c_{1,2}c_{2,2} + 6c_{2,2}^2 + 3c_{1,1}c_{3,3} = 0 \\
&c_{1,1}^2 c_{2,2} + c_{2,2}^2 = 0
\end{aligned}
$$

Up to the order of s^{10}, the solution to 3.0.1 in the high temperature regime is

$$
\begin{aligned}
\hat{\sigma}_0^+(s) = -\frac{1}{4} &+ \Big(\frac{1}{2} + \frac{1}{2}L(s)\Big)s + \Big(\frac{1}{8} - \frac{1}{4}L(s) - \frac{1}{4}L(s)^2\Big)s^2 + \Big(\frac{1}{8}L(s)^2 + \frac{1}{8}L(s)^3\Big)s^3 \\
&+ \Big(-\frac{1}{512} + \frac{1}{128}L(s) - \frac{1}{64}L(s)^2 - \frac{1}{16}L(s)^3 - \frac{1}{16}L(s)^4\Big)s^4 \\
&+ \Big(-\frac{1}{4096} + \frac{1}{1024}L(s) - \frac{1}{256}L(s)^2 + \frac{5}{512}L(s)^3 \\
&\quad + \frac{1}{32}L(s)^4 + \frac{1}{32}L(s)^5\Big)s^5 \\
&+ \Big(\frac{1}{8192} - \frac{1}{4096}L(s) + \frac{1}{512}L(s)^3 - \frac{3}{512}L(s)^4 - \frac{1}{64}L(s)^5 - \frac{1}{64}L(s)^6\Big)s^6 \\
&+ \Big(-\frac{1}{8192}L(s) + \frac{5}{16384}L(s)^2 - \frac{1}{4096}L(s)^3 - \frac{1}{1024}L(s)^4 + \frac{7}{2048}L(s)^5 \\
&\quad + \frac{1}{128}L(s)^6 + \frac{1}{128}L(s)^7\Big)s^7 \\
&+ \Big(-\frac{25}{8388608} + \frac{25}{1048576}L(s) + \frac{7}{262144}L(s)^2 - \frac{5}{32768}L(s)^3 \\
&\quad + \frac{3}{16384}L(s)^4 + \frac{1}{2048}L(s)^5 - \frac{1}{512}L(s)^6 - \frac{1}{256}L(s)^7 - \frac{1}{256}L(s)^8\Big)s^8 \\
&+ \Big(-\frac{145}{536870912} + \frac{273}{67108864}L(s) - \frac{435}{16777216}L(s)^2 + \frac{65}{8388608}L(s)^3 \\
&\quad + \frac{9}{131072}L(s)^4 - \frac{15}{131072}L(s)^5 - \frac{1}{4096}L(s)^6 + \frac{9}{8192}L(s)^7 \\
&\quad + \frac{1}{512}L(s)^8 + \frac{1}{512}L(s)^9\Big)s^9 \\
&+ \Big(\frac{81}{1073741824} - \frac{179}{536870912}L(s) - \frac{111}{67108864}L(s)^2 + \frac{305}{16777216}L(s)^3 \\
&\quad - \frac{125}{8388608}L(s)^4 - \frac{7}{262144}L(s)^5 + \frac{17}{262144}L(s)^6 + \frac{1}{8192}L(s)^7 \\
&\quad - \frac{5}{8192}L(s)^8 - \frac{1}{1024}L(s)^9 - \frac{1}{1024}L(s)^{10}\Big)s^{10} \\
&+ O(s^{11}L(s)^{11})
\end{aligned}
\tag{3.2.17}
$$

and in the low temperature

$$
\begin{aligned}
\hat{\sigma}_0^-(s) = -\frac{1}{4} &- \Big(\frac{1}{2}+\frac{1}{2}L(s)\Big)s + \Big(\frac{1}{8}-\frac{1}{4}L(s)-\frac{1}{4}L(s)^2\Big)s^2 - \Big(\frac{1}{8}L(s)^2+\frac{1}{8}L(s)^3\Big)s^3 \\
&+\Big(-\frac{1}{512}+\frac{1}{128}L(s)-\frac{1}{64}L(s)^2-\frac{1}{16}L(s)^3-\frac{1}{16}L(s)^4\Big)s^4 \\
&-\Big(-\frac{1}{4096}+\frac{1}{1024}L(s)-\frac{1}{256}L(s)^2+\frac{5}{512}L(s)^3 \\
&\quad+\frac{1}{32}L(s)^4+\frac{1}{32}L(s)^5\Big)s^5 \\
&+\Big(\frac{1}{8192}-\frac{1}{4096}L(s)+\frac{1}{512}L(s)^3-\frac{3}{512}L(s)^4 \\
&\quad-\frac{1}{64}L(s)^5-\frac{1}{64}L(s)^6\Big)s^6 \\
&-\Big(-\frac{1}{8192}L(s)+\frac{5}{16384}L(s)^2-\frac{1}{4096}L(s)^3-\frac{1}{1024}L(s)^4 \\
&\quad+\frac{7}{2048}L(s)^5+\frac{1}{128}L(s)^6+\frac{1}{128}L(s)^7\Big)s^7 \\
&+\Big(-\frac{25}{8388608}+\frac{25}{1048576}L(s)+\frac{7}{262144}L(s)^2-\frac{5}{32768}L(s)^3 \\
&\quad+\frac{3}{16384}L(s)^4+\frac{1}{2048}L(s)^5-\frac{1}{512}L(s)^6-\frac{1}{256}L(s)^7-\frac{1}{256}L(s)^8\Big)s^8 \\
&-\Big(-\frac{145}{536870912}+\frac{273}{67108864}L(s)-\frac{435}{16777216}L(s)^2+\frac{65}{8388608}L(s)^3 \\
&\quad+\frac{9}{131072}L(s)^4-\frac{15}{131072}L(s)^5-\frac{1}{4096}L(s)^6+\frac{9}{8192}L(s)^7 \\
&\quad+\frac{1}{512}L(s)^8+\frac{1}{512}L(s)^9\Big)s^9 \\
&+\Big(\frac{81}{1073741824}-\frac{179}{536870912}L(s)-\frac{111}{67108864}L(s)^2+\frac{305}{16777216}L(s)^3 \\
&\quad-\frac{125}{8388608}L(s)^4-\frac{7}{262144}L(s)^5+\frac{17}{262144}L(s)^6+\frac{1}{8192}L(s)^7 \\
&\quad-\frac{5}{8192}L(s)^8-\frac{1}{1024}L(s)^9-\frac{1}{1024}L(s)^{10}\Big)s^{10} \\
&+O(s^{11}L(s)^{11}) \qquad (3.2.18)
\end{aligned}
$$

These expansions were generated by solving the recurrences using computer algebra.

3.3 The leading term $G_0^{\pm}(s)$

We can compute the small s expansion of the diagonal correlation by using the preceding results. Integrating 3.0.7 shows that

$$G_0^{\pm}(s) = \exp\int_0^s \frac{1}{x}\hat{\sigma}_0^{\pm}(x)dx$$

Substituting 3.2.17 and 3.2.18, we then find that

$$
\begin{aligned}
G_0^{\pm}(s) = \frac{\tilde{A}}{s^{1/4}} + \Bigg(& 1 + \frac{1}{2}L(s)(\pm s) + \frac{1}{16}s^2 + \frac{1}{32}L(s)(\pm s)^3 \\
& + \Big(\frac{1}{2048} + \frac{1}{256}L(s) - \frac{1}{256}L(s)^2\Big)s^4 \\
& + \Big(-\frac{1}{4096} + \frac{5}{4096}L(s)\Big)(\pm s)^5 \\
& + \Big(-\frac{5}{98304} + \frac{1}{4096}L(s) - \frac{1}{4096}L(s)^2\Big)s^6 \\
& + \Big(-\frac{1}{65536} + \frac{7}{196608}L(s)\Big)(\pm s)^7 \\
& + \Big(-\frac{469}{201326592} + \frac{35}{4194304}L(s) - \frac{17}{2097153}L(s)^2\Big)s^8 \\
& + \Big(-\frac{209}{536870912} + \frac{209}{4026531}L(s) + \frac{5}{16777216}L(s)^2 \\
& \quad - \frac{1}{8388608}L(s)^3\Big)(\pm s)^9 \\
& + \Big(-\frac{937}{16106127360} + \frac{41}{201326592}L(s) - \frac{19}{100663296}L(s)^2\Big)s^{10} \\
& + O(L(s)^3 s^{11})\Bigg)
\end{aligned}
\tag{3.3.1}
$$

In the limit $s \to 0$, applying the known result 1.6.4 shows

$$\tilde{A} = 2^{-1/4}A$$

where A is the constant in 1.6.5. This result is consistent to the series solution presented by Au-Yang and Perk [1].

3.4 Series solution of the second order differential equation about $s \sim 0$

The expressions 3.2.17 and 3.2.18 give the means to compute the coefficients 3.0.3, 3.0.4, 3.0.5 and 3.0.6 in 3.0.2. Up to their sub-leading terms,

$$
\begin{aligned}
A(s) &\sim \pm\frac{1}{4}s \quad \text{for } T \gtrless T_C \\
B(s) &\sim \Big(\frac{3}{2} + \frac{3}{2}L(s) + \frac{1}{4}L(s)^2\Big)s \\
C(s) &\sim -\frac{1}{2} - L(s) - \frac{1}{4}L(s)^2 \\
D(s) &\sim \Big(\frac{1}{2} + \frac{1}{8}L(s)\Big)s
\end{aligned}
$$

If we however reuse the definitions 3.2.5, 3.2.8, 3.2.10, 3.2.11 and 3.2.14 then 3.0.3, 3.0.4, 3.0.5 and 3.0.6 can be rewritten as

$$A(s) = \frac{1}{2}s^2\hat{\sigma}_0''(s) = \frac{1}{2}\sum_{n=0}^{\infty}\alpha_n(s)s^n \tag{3.4.1}$$

$$\begin{aligned} B(s) &= s(\hat{\sigma}_0'(s))^2 - 2s(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)) + 2\hat{\sigma}_0'(s)(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s) - 1/4) \\ &= \frac{1}{s}\sum_{n=0}^{\infty}C_n(s)s^n - 2s\sum_{n=0}^{\infty}\beta_n(s)s^n \\ &\qquad + \frac{2}{s}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\gamma_{n-m}(s)\Big)s^n - \frac{1}{2s}\sum_{n=0}^{\infty}\gamma_n(s)s^n \end{aligned} \tag{3.4.2}$$

$$\begin{aligned} C(s) &= 2(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s)) - (\hat{\sigma}_0'(s))^2 \\ &= 2\sum_{n=0}^{\infty}\beta_n(s)s^n - \frac{1}{s^2}\sum_{n=0}^{\infty}C_n(s)s^n \end{aligned} \tag{3.4.3}$$

$$\begin{aligned} D(s) &= s^3(\hat{\sigma}_0''(s))^2 + 2\hat{\sigma}_0'(s)(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s))(s\hat{\sigma}_0'(s) - \hat{\sigma}_0(s) - 1/4) \\ &= \frac{1}{s}\sum_{n=0}^{\infty}A_n(s)s^n + \frac{2}{s}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}B_m(s)\gamma_{n-m}(s)\Big)s^n \\ &\qquad - \frac{1}{2s}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\gamma_{n-m}(s)\Big)s^n \end{aligned} \tag{3.4.4}$$

We seek the solution to 3.0.2 with the form

$$\hat{\sigma}_1(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}k_{m,n}L(s)^m s^n \tag{3.4.5}$$

Differentiating 3.4.5 gives

$$\hat{\sigma}_1'(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}k_{m,n}(nL(s)+m)L(s)^{m-1}s^{n-1}$$

$$\hat{\sigma}_1''(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}k_{m,n}[n(n-1)L(s)^2 + m(2n-1)L(s) + m(m-1)]L(s)^{m-2}s^{n-2}$$

If we introduce the new variables

$$\lambda_n(s) = \sum_{m=0}^{n}k_{m,n}L(s)^m \tag{3.4.6}$$

$$\mu_n(s) = \sum_{m=0}^{n}k_{m,n}(nL(s)+m)L(s)^{m-1} \tag{3.4.7}$$

$$\nu_n(s) = \sum_{m=0}^{n}k_{m,n}[n(n-1)L(s)^2 + m(2n-1)L(s) + m(m-1)]L(s)^{m-2} \tag{3.4.8}$$

The first term in 3.0.2 is

$$A(s)\hat{\sigma}_1''(s) = \frac{1}{2s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\alpha_m(s)\nu_{n-m}(s)\Big)s^n \tag{3.4.9}$$

the second term is

$$B(s)\hat{\sigma}_1'(s) = \frac{1}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} C_m(s)\mu_{n-m}(s)\Big)s^n - 2\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\mu_{n-m}(s)\Big)s^n + \frac{2}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} D_n(s)\mu_{n-m}(s)\Big)s^n - \frac{1}{2s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\gamma_m(s)\mu_{n-m}(s)\Big)s^n \tag{3.4.10}$$

where

$$D_n(s) = \sum_{m=0}^{n}\beta_m(s)\gamma_{n-m}(s) \tag{3.4.11}$$

and the third term is

$$C(s)\hat{\sigma}_1(s) = 2\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\lambda_{n-m}(s)\Big)s^n - \frac{1}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} C_n(s)\lambda_{n-m}(s)\Big)s^n \tag{3.4.12}$$

Substituting 3.4.9, 3.4.10, 3.4.12 and 3.0.6 into 3.0.2 gives

$$\begin{aligned}
&\frac{1}{2s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\alpha_m(s)\nu_{n-m}(s)\Big)s^n \\
&+\frac{1}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} C_m(s)\mu_{n-m}(s)\Big)s^n - 2\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\mu_{n-m}(s)\Big)s^n \\
&+\frac{2}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} D_m(s)\mu_{n-m}(s)\Big)s^n - \frac{1}{2s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\gamma_m(s)\mu_{n-m}(s)\Big)s^n \\
&+2\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\lambda_{n-m}(s)\Big)s^n - \frac{1}{s^2}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} C_m(s)\lambda_{n-m}(s)\Big)s^n \\
&=\frac{1}{s}\sum_{n=0}^{\infty}A_n(s)s^n + \frac{2}{s}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} B_m(s)\gamma_{n-m}(s)\Big)s^n - \frac{1}{2s}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\beta_m(s)\gamma_{n-m}(s)\Big)s^n
\end{aligned}$$

Equating powers of s gives the recurrence

$$\begin{aligned}
\frac{1}{2}\sum_{m=0}^{n}\alpha_m(s)\nu_{n-m}(s) &+ \sum_{m=0}^{n} C_m(s)\mu_{n-m}(s) - 2\sum_{m=0}^{n-2}\beta_m(s)\mu_{n-m-2}(s) \\
&+2\sum_{m=0}^{n} D_m(s)\mu_{n-m}(s) - \frac{1}{2}\sum_{m=0}^{n}\gamma_m(s)\mu_{n-m}(s) \\
&+2\sum_{m=0}^{n-2}\beta_m(s)\lambda_{n-m-2}(s) - \sum_{m=0}^{n} C_m(s)\lambda_{n-m}(s) \\
&= A_{n-1}(s) + 2\sum_{m=0}^{n-1} B_m(s)\gamma_{n-m-1}(s) - \frac{1}{2}\sum_{m=0}^{n-1}\beta_m(s)\gamma_{n-m-1}(s)
\end{aligned} \tag{3.4.13}$$

3.4.13 is polynomial of order n in $L(s)$. Equating powers of $L(s)$ gives a system of linear equations that can determine $k_{m,n}$. Some of the first few equations for $k_{m,n}$ are

$$
\begin{aligned}
&\frac{1}{2}k_{0,0} - (c_{0,1}+c_{1,1})^2 k_{0,0} + \frac{1}{2}c_{1,1}k_{1,1} = 0 \\
&-2c_{1,1}(c_{0,1}+c_{1,1})k_{0,0} = 0 \\
&-c_{1,1}^2 k_{0,0} = 0 \\
&-c_{0,1}c_{1,1} - 3c_{1,1}^2 - 8c_{0,1}c_{0,2}k_{0,0} + 4c_{1,1}k_{0,0} - 8c_{0,2}c_{1,1}k_{0,0} - 4c_{0,1}c_{1,2}k_{0,0} \\
&\qquad - 4c_{1,1}c_{1,2}k_{0,0} + 4c_{0,1}c_{1,1}k_{0,1} + 4c_{1,1}^2 k_{0,1} + 2c_{1,1}k_{0,2} - k_{1,1} \\
&\qquad + 2c_{0,1}^2 k_{1,1} + 2c_{0,2}k_{1,1} + 8c_{0,1}c_{1,1}k_{1,1} + 6c_{1,1}^2 k_{1,1} + 3c_{1,2}k_{1,1} + 2c_{2,2}k_{1,1} \\
&\qquad + 3c_{1,1}k_{1,2} + 2c_{1,1}k_{2,2} = 0 \\
&-c_{1,1}^2 - 8c_{0,2}c_{1,1}k_{0,0} - 8c_{0,1}c_{1,2}k_{0,0} - 12c_{1,1}c_{1,2}k_{0,0} - 8c_{0,1}c_{2,2}k_{0,0} - 8c_{1,1}c_{2,2}k_{0,0} \\
&\quad + 4c_{1,1}^2 k_{0,1} + 8c_{0,1}c_{1,1}k_{1,1} + 12c_{1,1}^2 k_{1,1} + 2c_{1,2}k_{1,1} + 6c_{2,2}k_{1,1} + 2c_{1,1}k_{1,2} \\
&\quad + 6c_{1,1}k_{2,2} = 0 \\
&-4c_{1,1}c_{1,2}k_{0,0} - 4c_{0,1}c_{2,2}k_{0,0} - 8c_{1,1}c_{2,2}k_{0,0} + 3c_{1,1}^2 k_{1,1} + c_{2,2}k_{1,1} + c_{1,1}k_{2,2} = 0
\end{aligned}
$$

Now we need to know the boundary conditions to the differential equation 3.0.2 to initiate the recurrence. First we must look revisit the $\hat{G}_1^{\pm}(s)$ which was defined in 2.3.19. From the known large N expansion of the diagonal correlation at $T = T_C$ given in 1.6.4, we can see that

$$\hat{G}_1^{\pm}(s) \to 0 \quad \text{as } s \to 0$$

This is consistent with the recurrence relations given above for $k_{m,n}$ which required $k_{0,0} = 0$.

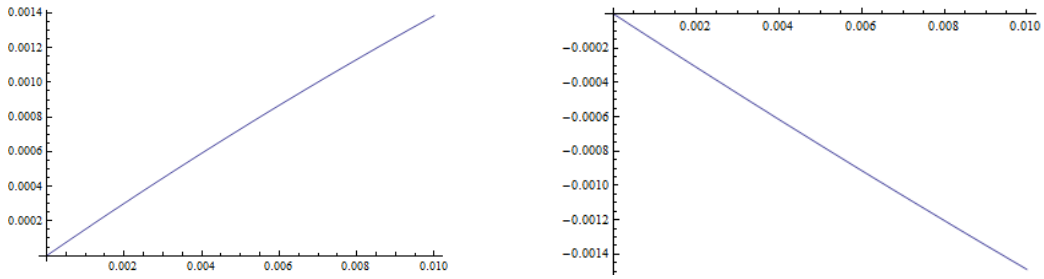
The next leading term of $\hat{G}_1^{\pm}(s)$ can be systematically calculated in much the same fashion described in 2.3. For our current purposes we will however use a numerical extrapolation of the data of $\hat{G}_1^{\pm}(s)$ computed from the Toeplitz determinant 1.3.1.

The data from in Figure 1, $\hat{G}_1^{\pm}(s)$ behaves like a straight line for small s with gradient ~ 0.19176. This suggests

$$\hat{G}_1^{\pm}(s) \sim \pm\frac{\tilde{A}}{4}s \approx \pm 0.19176s \tag{3.4.14}$$

Then the small s approximation for $G_1^{\pm}(s)$ defined in 2.2.3 and 2.2.4 is

$$G_1^{\pm}(s) = (2s)^{1/4}\hat{G}_1^{\pm}(s) \sim \pm\frac{A}{4}s^{3/4} \tag{3.4.15}$$



(a) $T > T_C$ (b) $T < T_C$

Figure 1: Small s extrapolation for $\hat{G}_1^{\pm}(s)$

Substituting 3.4.15 into 3.0.8 shows that

$$\hat{\sigma}_1(s) \sim \frac{1}{4}s \tag{3.4.16}$$

So we will take

$$k_{0,0} = 0$$
$$k_{0,1} = \frac{1}{4}$$

as the initial conditions for the recurrence 3.4.13. Solving then gives

$$\begin{aligned}
\hat{\sigma}_1^{+}(s) =& \frac{1}{4}s + \frac{1}{2}s^2 + \Big(-\frac{1}{8} - \frac{3}{4}L(s) - \frac{1}{4}L(s)^2\Big)s^3 + \Big(\frac{1}{4}L(s) + \frac{5}{8}L(s)^2 + \frac{1}{4}L(s)^3\Big)s^4 \\
&+\Big(\frac{1}{512} - \frac{1}{128}L(s) - \frac{15}{64}L(s)^2 - \frac{7}{16}L(s)^3 - \frac{3}{16}L(s)^4\Big)s^5 \\
&+\Big(-\frac{1}{256}L(s) + \frac{7}{512}L(s)^2 + \frac{21}{128}L(s)^3 + \frac{9}{32}L(s)^4 + \frac{1}{8}L(s)^5\Big)s^6 \\
&+\Big(\frac{3}{8192} - \frac{5}{4096}L(s) + \frac{3}{512}L(s)^2 - \frac{7}{512}L(s)^3 - \frac{55}{512}L(s)^4 \\
&\quad - \frac{11}{64}L(s)^5 - \frac{5}{64}L(s)^6\Big)s^7 \\
&+\Big(-\frac{3}{8192} - \frac{1}{8192}L(s) + \frac{9}{8192}L(s)^2 - \frac{11}{2048}L(s)^3 + \frac{23}{2048}L(s)^4 \\
&\quad + \frac{69}{1024}L(s)^5 + \frac{13}{128}L(s)^6 + \frac{3}{64}L(s)^7\Big)s^8 \\
&+\Big(\frac{25}{8388608} + \frac{231}{1048576}L(s) - \frac{71}{262144}L(s)^2 - \frac{11}{32768}L(s)^3 \\
&\quad + \frac{61}{16384}L(s)^4 - \frac{17}{2048}L(s)^5 - \frac{21}{512}L(s)^6 - \frac{15}{256}L(s)^7 - \frac{7}{256}L(s)^8\Big)s^9 \\
&+\Big(\frac{1}{524288} - \frac{81}{4194304}L(s) - \frac{1545}{8388608}L(s)^2 + \frac{353}{1048576}L(s)^3 - \frac{3}{131072}L(s)^4 \\
&\quad - \frac{39}{16384}L(s)^5 + \frac{47}{8192}L(s)^6 + \frac{25}{1024}L(s)^7 + \frac{17}{512}L(s)^8 + \frac{1}{64}L(s)^9\Big)s^{10} \\
&+O(L(s)^{10}s^{11})
\end{aligned} \tag{3.4.17}$$

$$
\begin{aligned}
\hat{\sigma}_1^-(s) =& \frac{1}{4}s - \frac{1}{2}s^2 + \Big(-\frac{1}{8} - \frac{3}{4}L(s) - \frac{1}{4}L(s)^2\Big)s^3 + \Big(-\frac{1}{4}L(s) - \frac{5}{8}L(s)^2 - \frac{1}{4}L(s)^3\Big)s^4 \\
&+\Big(\frac{1}{512} - \frac{1}{128}L(s) - \frac{15}{64}L(s)^2 - \frac{7}{16}L(s)^3 - \frac{3}{16}L(s)^4\Big)s^5 \\
&+\Big(\frac{1}{256}L(s) - \frac{7}{512}L(s)^2 - \frac{21}{128}L(s)^3 - \frac{9}{32}L(s)^4 - \frac{1}{8}L(s)^5\Big)s^6 \\
&+\Big(\frac{3}{8192} - \frac{5}{4096}L(s) + \frac{3}{512}L(s)^2 - \frac{7}{512}L(s)^3 - \frac{55}{512}L(s)^4 \\
&\quad - \frac{11}{64}L(s)^5 - \frac{5}{64}L(s)^6\Big)s^7 \\
&+\Big(\frac{3}{8192} + \frac{1}{8192}L(s) - \frac{9}{8192}L(s)^2 + \frac{11}{2048}L(s)^3 - \frac{23}{2048}L(s)^4 \\
&\quad - \frac{69}{1024}L(s)^5 - \frac{13}{128}L(s)^6 - \frac{3}{64}L(s)^7\Big)s^8 \\
&+\Big(\frac{25}{8388608} + \frac{231}{1048576}L(s) - \frac{71}{262144}L(s)^2 - \frac{11}{32768}L(s)^3 + \frac{61}{16384}L(s)^4 \\
&\quad - \frac{17}{2048}L(s)^5 - \frac{21}{512}L(s)^6 - \frac{15}{256}L(s)^7 - \frac{7}{256}L(s)^8\Big)s^9 \\
&+\Big(-\frac{1}{524288} + \frac{81}{4194304}L(s) + \frac{1545}{8388608}L(s)^2 - \frac{353}{1048576}L(s)^3 \\
&\quad + \frac{3}{131072}L(s)^4 + \frac{39}{16384}L(s)^5 - \frac{47}{8192}L(s)^6 - \frac{25}{1024}L(s)^7 \\
&\quad - \frac{17}{512}L(s)^8 - \frac{1}{64}L(s)^9\Big)s^{10} \\
&+O(L(s)^{10}s^{11})
\end{aligned}
\tag{3.4.18}
$$

3.5 The correction term $G_1^{\pm}(s)$

Now it is left to compute the sub-leading term of the diagonal correlation $G_1^{\pm}(s)$. Rearranging 3.0.8 gives a order linear differential equation for $G_1^{\pm}(s)$ for low and high temperature regimes. For $T < T_C$,

$$
sG_0^-(s)\frac{\mathrm{d}G_1^-}{\mathrm{d}s} - s\frac{\mathrm{d}G_0^-}{\mathrm{d}s}G_1^-(s) = \hat{\sigma}_1^-(s)G_0^-(s)^2 + 2s^2\frac{\mathrm{d}G_0^-}{\mathrm{d}s}G_0^-(s) \tag{3.5.1}
$$

and for $T > T_C$,

$$
sG_0^+(s)\frac{\mathrm{d}G_1^+}{\mathrm{d}s} - s\frac{\mathrm{d}G_0^+}{\mathrm{d}s}G_1^+(s) = \Big(\hat{\sigma}_1^+(s) + \frac{s}{2}\Big)G_0^+(s)^2 + 2s^2\frac{\mathrm{d}G_0^+}{\mathrm{d}s}G_0^+(s) \tag{3.5.2}
$$

Introduce the integrating factor

$$
\begin{aligned}
I^{\pm}(s) &= \exp\left(-\int \frac{1}{G_0^{\pm}(s)} \frac{\mathrm{d}G_0^{\pm}}{\mathrm{d}s} ds\right) \\
&= \exp\left(-\int \frac{1}{s}\hat{\sigma}_0^{\pm}(s) ds\right) \\
&= \frac{1}{G_0^{\pm}(s)}
\end{aligned}
$$

where the last two steps comes from the formulas 3.0.7 and 3.3.1. The differential equations 3.5.1 and 3.5.2 becomes for $T < T_C$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{G_0^{-}(s)} G_1^{-}(s)\right) &= \frac{1}{s}\hat{\sigma}_1^{-}(s) + 2s\frac{1}{G_0^{-}(s)}\frac{\mathrm{d}G_0^{-}}{\mathrm{d}s} \\
&= \frac{1}{s}\hat{\sigma}_1^{-}(s) + 2\hat{\sigma}_0^{-}(s) \\
G_1^{-}(s) &= G_0^{-}(s)\int_0^s \frac{1}{x}\hat{\sigma}_1^{-}(x) + 2\hat{\sigma}_0^{-}(x) dx
\end{aligned} \tag{3.5.3}
$$

and for $T > T_C$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{G_0^{+}(s)} G_1^{+}(s)\right) &= \frac{1}{2} + \frac{1}{s}\hat{\sigma}_1^{+}(s) + 2s\frac{1}{G_0^{+}(s)}\frac{\mathrm{d}G_0^{+}}{\mathrm{d}s} \\
&= \frac{1}{2} + \frac{1}{s}\hat{\sigma}_1^{+}(s) + 2s\hat{\sigma}_0^{+}(s) \\
G_1^{+}(s) &= G_0^{-}(s)\left(\frac{1}{2}s + \int_0^s \frac{1}{x}\hat{\sigma}_1^{+}(x) + 2x\hat{\sigma}_0^{+}(x) dx\right)
\end{aligned} \tag{3.5.4}
$$

The last step of 3.5.3 and 3.5.4 applied the known limit from 2.3.19 and 3.4.15

$$
\lim_{s\to 0} \frac{G_1^{\pm}(s)}{G_0^{\pm}(s)} = 0
$$

The series solutions to 3.5.3 and 3.5.4, deduced from 3.2.17 and 3.2.18, are

$$
\begin{aligned}
G_1^-(s) = \frac{\tilde{A}}{s^{1/4}}\Bigg(& -\frac{1}{4}s + \Big(-\frac{1}{2} - \frac{3}{8}L(s)\Big)s^2 + \frac{7}{64}s^3 + \Big(-\frac{1}{32} - \frac{11}{128}L(s)\Big)s^4 \\
& + \Big(\frac{47}{8192} + \frac{7}{1024}L(s) - \frac{15}{1024}L(s)^2\Big)s^5 \\
& + \Big(-\frac{1}{16384} - \frac{95}{16384}L(s)\Big)s^6 \\
& + \Big(-\frac{19}{393216} + \frac{15}{16384}L(s) - \frac{23}{16384}L(s)^2\Big)s^7 \\
& + \Big(\frac{53}{786432} - \frac{63}{262144}L(s)\Big)s^8 \\
& + \Big(-\frac{7819}{805306368} + \frac{813}{16777216}L(s) - \frac{527}{8388608}L(s)^2\Big)s^9 \\
& + \Big(\frac{18601}{6442450944} - \frac{8275}{1610612736}L(s) - \frac{151}{67108864}L(s)^2 \\
& \quad + \frac{35}{33554432}L(s)^3\Big)s^{10} + O(L(s)^3 s^{11})\Bigg)
\end{aligned}
\tag{3.5.5}
$$

and

$$
\begin{aligned}
G_1^+(s) = \frac{\tilde{A}}{s^{1/4}}\Bigg(& \frac{1}{4}s + \Big(\frac{1}{2} + \frac{5}{8}L(s)\Big)s^2 + \frac{9}{64}s^3 + \Big(\frac{1}{32} + \frac{13}{128}L(s)\Big)s^4 \\
& + \Big(\frac{49}{8192} + \frac{9}{1024}L(s) - \frac{17}{1024}L(s)^2\Big)s^5 \\
& + \Big(-\frac{1}{16384} + \frac{105}{16384}L(s)\Big)s^6 \\
& + \Big(-\frac{29}{393216} + \frac{17}{16384}L(s) - \frac{25}{16384}L(s)^2\Big)s^7 \\
& + \Big(-\frac{59}{786432} + \frac{203}{786432}L(s)\Big)s^8 \\
& + \Big(-\frac{2919}{268435456} + \frac{883}{16777216}L(s) - \frac{561}{8388608}L(s)^2\Big)s^9 \\
& + \Big(-\frac{19855}{6442450944} + \frac{8693}{1610612736}L(s) + \frac{161}{67108864}L(s)^2 \\
& \quad - \frac{37}{33554432}L(s)^3\Big)s^{10} + O(L(s)^3 s^{11})\Bigg).
\end{aligned}
\tag{3.5.6}
$$

Chapter 4

Comparison between series solutions for the diagonal correlation to numerical data

All that is left is to show numerical consistency of the asymptotic expansions and series solutions for the diagonal correlation $\langle \sigma_{0,0}\sigma_{N,N} \rangle$.

4.1 Numerical extrapolation of the diagonal correlation

The diagonal correlation $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ can be readily computed for a fixed s as a Toeplitz determinant 1.3.1, with its elements the hypergeometric functions defined in 1.3.4, 1.3.5, 1.3.6 and 1.3.7, by using numerical software. For purposes of numerically calculate its leading and correction term, similar to 2.3.19, we assume the diagonal correlation has the form

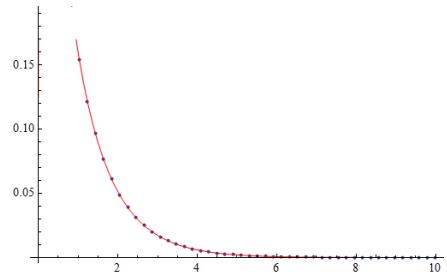
$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = \frac{a(s)}{N^{1/4}} + \frac{b(s)}{N^{5/4}} + \frac{c(s)}{N^{9/4}} \tag{4.1.1}$$

To extrapolate the $\langle \sigma_{0,0}\sigma_{N,N} \rangle$, we sample fixed values of s from some interval (a, b) and choose three values large of N and its corresponding k values. This creates a set of linear equations for the variables a, b and c from 4.1.1 for a fixed s which can be solved using computer algebra. This gives us a set of values for $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ over a discrete domain (a, b) in the s variable. In this work, the points for s are sampled from the $(0, 1)$ and $(1, 10)$ for small and large s behaviour and N is chosen up to 50. $a(s)$, $b(s)$ and $c(s)$ can be extrapolated continuous functions over these regions. These extrapolations are not the exact values of the leading terms of the leading terms of the diagonal correlation for large N. Nonetheless

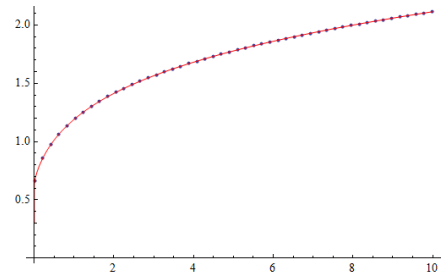
they are excellent approximations that will be used to show consistency of the series solutions derived above.

4.2 Numerical comparisons for large s

Figure 1 and 2 shows the asymptotic behaviour of $G_0^{\pm}(s)$ and $G_1^{\pm}(s)$ derived from the form factor expansions compared with its corresponding numerical data from extrapolation. We see an excellent fit for both $G_0^{\pm}(s)$ and $G_1^{\pm}(s)$ above and below criticality for large s. In fact, $G_{0,aysm}^{\pm}(s)$ is reasonable for fit to the data for $s > 0.4$. Surprisingly $G_{1,aysm}^{+}(s)$ gives a good qualitative bahaviour for almost all s and is a reasonable fit for $s > 0.05$. $G_{1,aysm}^{-}(s)$ has a good fit for $s > 0.5$ but it should be noted that numerical computation of the double integral 2.2.8 behaves badly as $s \to 0$.

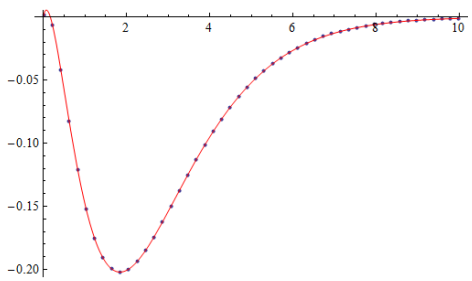


(a) $T > T_C$

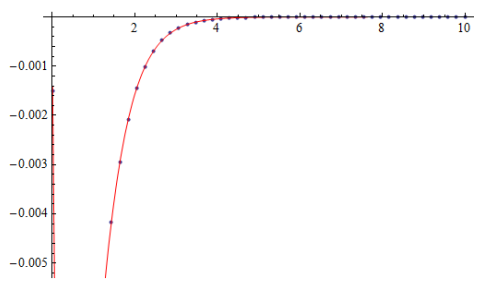


(b) $T < T_C$

Figure 1: Small s expansion for $(2s)^{1/4}G_{0,aysm}^{\pm}(s)$ (Red) v Numerical data (Blue dots)



(a) $T > T_C$



(b) $T < T_C$

Figure 2: Small s expansion for $(2s)^{1/4}G_{1,aysm}^{\pm}(s)$ (Red) v Numerical data (Blue dots)

4.3 Numerical comparisons for small s

To compare the data presented in 4.1.1 with the sigma functions $\hat{\sigma}_0(s)$ and $\hat{\sigma}_1(s)$, we need to write the Painlevé VI solution as a function of the extrapolated data $a(s)$, $b(s)$ and $c(s)$. Substituting 4.1.1 into 1.5.1 and expanding the function in large N after the change of variables from t to s gives

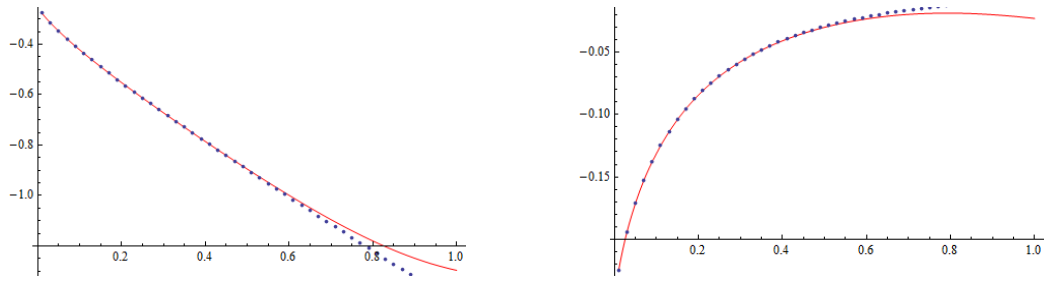
$$\hat{\sigma}_0(s) = s\frac{a'(s)}{a(s)} - \frac{1}{4}$$

and

$$\hat{\sigma}_1(s) = \frac{s}{2} + \frac{-2s^2a(s)a'(s) + sa(s)b'(s) - sa'(s)b(s)}{a(s)^2} \quad \text{for } T < T_C$$

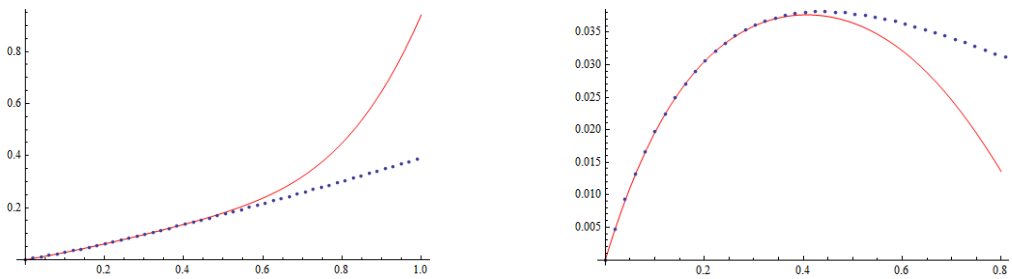
$$\hat{\sigma}_1(s) = \frac{-2s^2a(s)a'(s) + sa(s)b'(s) - sa'(s)b(s)}{a(s)^2} \quad \text{for } T > T_C$$

These expressions used in Figure 3 and 4 are presented as the 'numerical data'.



(a) $T > T_C$ (b) $T < T_C$

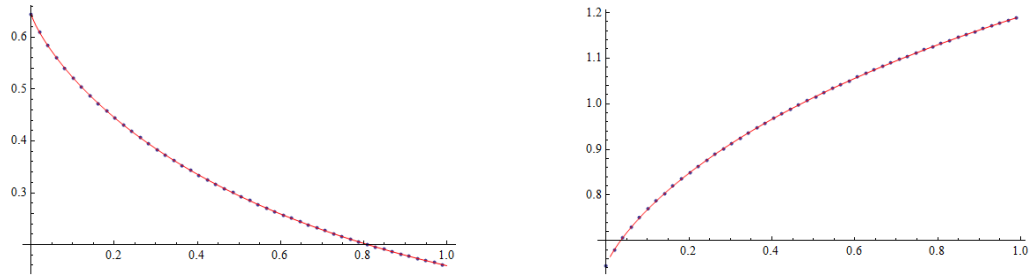
Figure 3: Small s expansion for $\hat{\sigma}_0(s)$ (Red) v Numerical data (Blue dots)



(a) $T > T_C$ (b) $T < T_C$

Figure 4: Small s expansion for $\hat{\sigma}_1(s)$ (Red) v Numerical data (Blue dots)

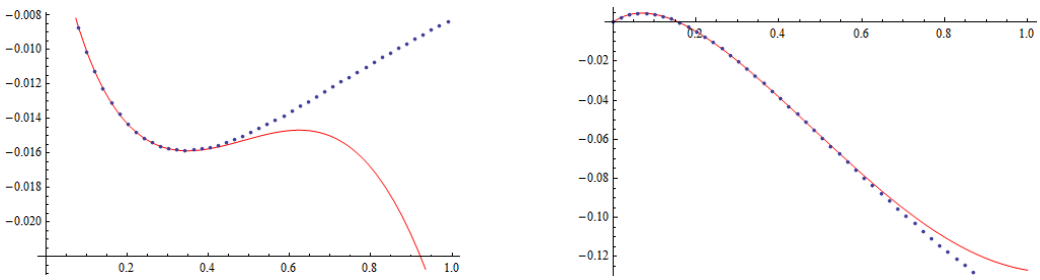
The series solutions for $\hat{\sigma}_0(s)$ and $\hat{\sigma}_1(s)$ used in Figure 3 and 3.4.5 are computed up to $O(s^{10})$. For approximately up to $s = 0.4$, this offers a good approximation for $\hat{\sigma}_0(s)$. Of course computing more terms for the series will increase the range of s where the approximation becomes reasonable.



(a) $T > T_C$ (b) $T < T_C$

Figure 5: Small s expansion for $(2s)^{1/4}G_0^{\pm}(s)$ (Red) v Numerical data (Blue dots)

In contrast to Figure 3 and 4, 5 shows that the series for $G_0^{\pm}(s)$ converges much more rapidly for larger values of s despite being $O(s^{10})$ in the plots. We can see graphically that as $s \to 0$ then $G_0^{\pm}(s) \sim A \approx 0.6545$ in Figure 5 for both below and above T_C. This is consistent with the known result at the critical temperature 1.6.4.



(a) $T > T_C$ (b) $T < T_C$

Figure 6: Small s expansion for $(2s)^{1/4}G_1^{\pm}(s)$ (Red) v Numerical data (Blue dots)

In Figure 6, we see a slower convergence to the solution than 5. Nonetheless it is good approximation for $s < 0.4$.

4.4 Conclusion

It follows from the characterisation of $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ as a Painlevé VI differential equation by Jimbo and Miwa [8] that $G_0^{\pm}(s)$ as specified by 2.2.3 and 2.2.4 can be expressed in terms of the Painlevé V equation 3.0.1. The solution of this equation must a small s expansion of the form 3.2.3, which is a consequence of the analysis of the Toeplitz determinant in the neighbourhood of $T = T_C$ undertaken in section 2.3. Alternatively, the differential equation characterisation of $G_0^{\pm}(s)$ could be made explicit by specifying the large s behaviour deduced in section 2.2. These characteristics involving the Painlevé V transcendent have previously been

given in [1].

Our main finding is the previously unknown characterisation of $G_1^{\pm}(s)$. This 'correction' term to the leading behaviour $G_0^{\pm}(s)$ can be characterised by a second order linear differential equation with functional coefficients that depends on the solution to the Painlevé V equation. By considering a similar small s expansion of the form 3.4.5, it is possible to generate a series expansion that characterises $G_1^{\pm}(s)$ for small s. In the same way, $G_1^{\pm}(s)$ has a large s expansion deduced from the form factor expansions.

Putting the expansions for small and large s for $G_0^{\pm}(s)$ and $G_1^{\pm}(s)$ together completes a picture of the behaviour of the diagonal correlation in the variable s. The consistency of the numerical data presented in Chapter 4 supports this claim.

Acknowledgements

I would like to express my gratitude to Peter Forrester, Nicholas Witte and Anthony Mays for their helpful advice and endearing patience over the past year.

Appendices

Mathematica Code

Solving the system of equations for $c_{m,n}$ and $k_{m,n}$

```
(*Recurrencerelationsfor\sigma_0*)
α[n_]:=Sum[c[m,n](n(n-1)L^2+m(2n-1)L+m(m-1))L^(m-2),
{m,0,n}]
β[n_]:=Sum[c[m,n]((n-1)L+m)L^(m-1),{m,0,n}]
γ[n_]:=Sum[c[m,n](nL+m)L^(m-1),{m,0,n}]
A1[n_]:=Sum[α[m]α[n-m],{m,0,n}]
B[n_]:=Sum[β[m]β[n-m],{m,0,n}]
C1[n_]:=Sum[γ[m]γ[n-m],{m,0,n}]
c[0,0]=-1/4;
rec1[n_]:=A1[n]-4B[n-2]+4Sum[β[m]*C1[n-m],{m,0,n}]-C1[n]
(*SolvesforT > T_Cuptoordero(s^{n+1})*)
rec1solveh[n_]:=
Solve[
Flatten[Table[Table[SeriesCoefficient[rec1[k],{L,0,m}]==0,
{m,0,k}],{k,2,n}]]/.c[0,1]→1/2,
Flatten[Table[Table[c[i,j],{i,0,n}],{j,1,n}]]]
(*SolvesforT < T_Cuptoordero(s^{n+1})*)
rec1solvel[n_]:=
Solve[
Flatten[Table[Table[SeriesCoefficient[rec1[k],{L,0,m}]==0,
{m,0,k}],{k,2,n}]]/.c[0,1]→-1/2,
Flatten[Table[Table[c[i,j],{i,0,n}],{j,1,n}]]]
rec1coeffh=rec1solveh[10][[2]];
rec1coeffl=rec1solvel[10][[1]];
(*Recurrencerelationsfor\sigma_1*)
λ[n_]:=Sum[k[m,n]L^m,{m,0,n}]
μ[n_]:=Sum[k[m,n](nL+m)L^(m-1),{m,0,n}]
ν[n_]:=Sum[k[m,n](n(n-1)L^2+m(2n-1)L+m(m-1))L^(m-2),
```

```
{m,0,n}]
D1[n_]:=Sum[β[m]γ[n − m], {m,0,n}]
rec2[n_]:=1/2Sum[α[m]ν[n − m], {m,0,n}]+
Sum[C1[m]μ[n − m], {m,0,n}] − 2Sum[β[m]μ[n − m − 2], {m,0,n − 2}]+
2Sum[D1[m]μ[n − m], {m,0,n}] − 1/2Sum[γ[m]μ[n − m], {m,0,n}]+
2Sum[β[m]λ[n − m − 2], {m,0,n − 2}] − Sum[C1[m]λ[n − m], {m,0,n}]−
A1[n − 1] − 2Sum[B[m]γ[n − m − 1], {m,0,n − 1}]+
1/2Sum[β[m]γ[n − m − 1], {m,0,n − 1}]
rec2solveh[n_]:=
Solve[
Flatten[
Table[
Table[SeriesCoefficient[rec2[p]/.rec1coeffh/.k[0,1] → 1/4,
{L,0,m}] == 0, {m,0,p}], {p,2,n}]],
Flatten[Table[Table[k[i,j], {i,0,n}], {j,0,n}]]]
rec2solvel[n_]:=
Solve[
Flatten[
Table[
Table[SeriesCoefficient[rec2[p]/.rec1coeffl/.k[0,1] → 1/4,
{L,0,m}] == 0, {m,0,p}], {p,2,n}]],
Flatten[Table[Table[k[i,j], {i,0,n}], {j,0,n}]]]
rec2coeffh = rec2solveh[10];
rec2coeffl = rec2solvel[10];
```

Calculating the series expansion for $G_0^{\pm}(s)$ and $G_1^{\pm}(s)$

```
(*FromAu − YangandPerk, σ0satisfiesthePVequation*)
A = Exp[3Zeta′[−1]]2^(−1/6);
h[r_, L_]:=1 + 1/2Lr + 1/16r^2 + 1/32Lr^3 + (−1/256L^2 + 1/256L + 1/2048) r^4+
(5/4096L − 1/4096) r^5 + (−1/4096L^2 + 1/4096L − 5/98304) r^6+
(7/196608L − 1/65536) r^7 + (−17/2097152L^2 + 35/4194304L − 469/201326592) r^8+
(−1/8388608L^3 + 5/16777216L^2 + 209/402653184L − 209/536870912) r^9+
(−19/100663296L^2 + 41/201326592L − 937/161106127360) r^10
Fh[r_]:=A/r^(1/4) h[r, Log[r] − Log[8] + EulerGamma]
Fl[r_]:=A/r^(1/4) h[−r, Log[r] − Log[8] + EulerGamma]
σh[r_]:=rFh′[r]/Fh[r]
```

```
σl[r_]:=rFl′[r]/Fl[r]

(* Series for Go and G1 *)
ζ1h[r_]:=
−1/4 + Sum[Sum[c[m,n]L[r]^mr^n, {m,0,n}], {n,1,10}]/.
c[0,1] → 1/2/.reccoeffh[[2]]
ζ1l[r_]:=
−1/4 + Sum[Sum[c[m,n]L[r]^m(−r)^n, {m,0,n}], {n,1,10}]/.
c[0,1] → 1/2/.reccoeffh[[2]]
q[r_]:=Log[r/8] + EulerGamma
a1h[s_]:=1/2s^2ζ1h″[s]
b1h[s_]:= − 2s(sζ1h′[s] − ζ1h[s]) + 2ζ1h′[s](sζ1h′[s] − ζ1h[s] − 1/4)+
s(ζ1h′[s])^2
c1h[s_]:=2(sζ1h′[s] − ζ1h[s]) − (ζ1h′[s])^2
d1h[s_]:=
s^3(ζ1h″[s])^2 + 2ζ1h′[s](sζ1h′[s] − ζ1h[s])
(sζ1h′[s] − ζ1h[s] − 1/4)
a1l[s_]:=1/2s^2ζ1l″[s]
b1l[s_]:= − 2s(sζ1l′[s] − ζ1l[s]) + 2ζ1l′[s](sζ1l′[s] − ζ1l[s] − 1/4)+
s(ζ1l′[s])^2
c1l[s_]:=2(sζ1l′[s] − ζ1l[s]) − (ζ1l′[s])^2
d1l[s_]:=
s^3(ζ1l″[s])^2 + 2ζ1l′[s](sζ1l′[s] − ζ1l[s])
(sζ1l′[s] − ζ1l[s] − 1/4)
ζ2[s_]:=Sum[Sum[k[m,n]L[s]^ms^(n + 1), {m,0,n}], {n,0,15}]
σhi1[r_]:=
Sum[Sum[k[m,n]L[r]^mr^n, {m,0,n}], {n,0,10}]/.rec2coeffh/.
k[0,1] → 1/4
σlow1[r_]:=
Sum[Sum[k[m,n]L[r]^mr^n, {m,0,n}], {n,0,10}]/.rec2coeffh/.
k[0,1] → 1/4
(* Integrating for Go and G1 *)
inthigh =
Integrate [1/r σhi1[r] + 2ζ1h[r]/.L[r] → Log[r], {r,0,s},
Assumptions → {s > 0}]/.Log[s] → L
G1h[r_]:=Fh[s] (1/2 s + inthigh) /.L → (Log[s/8] + EulerGamma)
intlow =
Integrate [1/r σlow1[r] + 2ζ1l[r]/.L[r] → Log[r], {r,0,s},
```

```
Assumptions → {s > 0}]/.Log[s] → L
G1l[r_]:=Fl[s](intlow/.L → (Log[s/8] + EulerGamma)
```

Plots for $\hat{\sigma}_0(s)$, $\hat{\sigma}_1(s)$, $G_0^{\pm}(s)$ and $G_1^{\pm}(s)$

```
ClearAll[linspace]
linspace[s_, f_, 1]:=(f + s)/2
linspace[s_, f_, n_]:=Range[s, f, (f − s)/(n − 1)]
(* Extrapolation of correlation function *)
whigh[k_, n_Integer/;n < 0]:=
−Gamma[1/2]Gamma[−n−1/2] / PiGamma[−n] k^(−n − 1)
Hypergeometric2F1[−1/2, −n − 1/2, −n, k^2]//N
whigh[k_, n_Integer/;n ≥ 0]:=
Gamma[3/2]Gamma[n+1/2] / PiGamma[n+2] k^(n + 1)
Hypergeometric2F1[1/2, n + 1/2, n + 2, k^2]//N
toeplitzh[k_, n_]:=ToeplitzMatrix[Array[whigh[k, # − 1]&, {n}],
Join[{whigh[k, 0]}, Array[whigh[k, −#]&, {n − 1}]]]
wlow[k_, n_Integer/;n < 0]:=
−Gamma[3/2]Gamma[−n−1/2] / PiGamma[−n+1] k^n
Hypergeometric2F1[1/2, −n − 1/2, −n + 1, k^ − 2]//N
wlow[k_, n_Integer/;n ≥ 0]:=
Gamma[1/2]Gamma[n+1/2] / PiGamma[n+1] k^ − n
Hypergeometric2F1[−1/2, n + 1/2, n + 1, k^ − 2]//N
toeplitzl[k_, n_]:=ToeplitzMatrix[Array[wlow[k, # − 1]&, {n}],
Join[{wlow[k, 0]}, Array[wlow[k, −#]&, {n − 1}]]]
G1hasym[r_]:=1/Pi BesselK[0, r]
G1lasym[r_]:=
1+
1/Pi^2 (r^2(BesselK[1, r]^2 − BesselK[0, r]^2)−
rBesselK[0, r]BesselK[1, r] + 1/2 BesselK[0, r]^2)
ζhasym[r_]:=rG1hasym′[r]/G1hasym[r]
ζlasym[r_]:=rG1lasym′[r]/G1lasym[r]
G2hasym[r_]:=r/2Pi (BesselK[0, r] − 2rBesselK[1, r])
σ2hasym[r_]:=
(−2r^2G1hasym′[r]G1hasym[r] − rG1hasym′[r]G2hasym[r]+
rG1hasym[r]G2hasym′[r])/(G1hasym[r]^2) − r/2
mesh = 50;
sfixed = linspace[0.01, 0.99, mesh];
```

```
sizeN = 50;
σhigh = Map[σh, sfixed];
PVh = ListPlot[MapThread[{#1, #2}&, {sfixed, σhigh}]];
σlow = Map[σl, sfixed];
PVl = ListPlot[MapThread[{#1, #2}&, {sfixed, σlow}]];
dnsh =
Grid[
Table[MapThread[{#1, #2}&,
{sfixed, Det[toeplitzh[#, n]]&/@ ((1 - 2*sfixed/n) ^(1/2))}],
{n, 2, sizeN}]];
abchigh =
Table[
NSolve[
Flatten[
Table [{a/n^(1/4) + b/n^(5/4) + c/n^(9/4) == dnsh[[1, n - 1]][[k, 2]]},
{n, sizeN - 2, sizeN}]], {a, b, c}], {k, 1, mesh}];
ahighlist = Table[({a, b, c}/.abchigh[[k, 1]])[[1]], {k, 1, mesh}];
intahigh = Interpolation[MapThread[{#1, #2}&, {sfixed, ahighlist}]];
intah[r_]:=rintahigh'[r]/intahigh[r] - 1/4
plotah1 =
ListPlot[MapThread[{#1, #2}&, {sfixed, Map[intahigh, sfixed]}]];
plotah2 = Plot[(2r)^(1/4)G1hasym[r], {r, 0.01, 0.99}, PlotStyle → Red];
plotah3 = Plot[(2r)^(1/4)Fh[r], {r, 0, 1}, PlotStyle → Red];
Show[plotah1, plotah2](* Large s *)
Show[plotah1, plotah3] (* small s *)
bhighlist = Table[({a, b, c}/.abchigh[[k, 1]])[[2]], {k, 1, mesh}];
intbhigh = Interpolation[MapThread[{#1, #2}&, {sfixed, bhighlist}]];
intbh[r_]:=
(-2r^2intahigh'[r]intahigh[r] - rintahigh'[r]intbhigh[r]+
rintahigh[r]intbhigh'[r])/(intahigh[r]^2)
plotbh1 =
ListPlot[MapThread[{#1, #2}&, {sfixed, Map[intbhigh, sfixed]}]];
plotbh2 = Plot[(2r)^(1/4)G2hasym[r], {r, 0.01, 0.99}, PlotStyle → Red];
Show[plotbh1, plotbh2]

analyGbh =
Plot [{(2s)^(1/4)Fh[s] (1/2 s + inthigh) /.L → (Log[s/8] + EulerGamma)},
{s, 0, 1}, PlotStyle → Red];
```

```
Show[analyGbh, plotbh1]

sigmaplotbh1 =
ListPlot[MapThread[{#1, #2}&, {sfixed, Map[intbh, sfixed]}]];
sigmaplotbh2 = Plot[σ2hasym[r], {r, 0.01, 0.99}, PlotStyle → Red];
Show[sigmaplotbh1, sigmaplotbh2]
Show[Plot[{ζ1h[r]/.L[r] → q[r]}, {r, 0.01, 1}, PlotStyle → Red], PVh]

dnsl =
Grid[
Table[MapThread[{#1, #2}&,
{sfixed, Det[toeplitzl[#, n]]&/@ ((1 − 2*sfixed/n) ^(−1/2))}],
{n, 2, sizeN}]];
abclow =
Table[
NSolve[
Flatten[
Table [{a/(n^(1/4)) + b/(n^(5/4)) + c/(n^(9/4)) == dnsl[[1, n − 1]][[k, 2]]},
{n, 48, 50}]], {a, b, c}], {k, 1, mesh}];
alowlist = Table[({a, b, c}/.abclow[[k, 1]])[[1]], {k, 1, mesh}];
intalow = Interpolation[MapThread[{#1, #2}&, {sfixed, alowlist}]];
intal[r_]:=rintalow′[r]/intalow[r] − 1/4
plotal1 =
ListPlot[MapThread[{#1, #2}&, {sfixed, Map[intalow, sfixed]}]];
plotal2 = Plot[(2r)^(1/4)G1lasym[r], {r, 0.01, 0.99}, PlotStyle → Red];
plotal3 = Plot[(2r)^(1/4)Fl[r], {r, 0.01, 0.99}, PlotStyle → Red]
Show[plotal1, plotal2](* Large s *)
Show[plotal1, plotal3](* small s *)

sigmaplotal1 =
ListPlot[MapThread[{#1, #2}&, {sfixed, Map[intal, sfixed]}]];
sigmaplotal2 = Plot[ζlasym[r], {r, 0, 0.99}, PlotStyle → Red];
Show[sigmaplotal1, sigmaplotal2]
Show[Plot[{ζ1l[r]/.L[r] → q[r]}, {r, 0.01, 1}, PlotStyle → Red], PVl]

sfixedLarge = linspace[0.01, 9.99, meshL];
doubleasym[n_]:=(Exp[−2sfixedLarge[[n]]]sfixedLarge[[n]])/(Pi^2) *
NIntegrate[Exp[−2sfixedLarge[[n]]x]Exp[−2sfixedLarge[[n]]y]
```

```
((1+y)y/((1+x)x))^(1/2) * (1 + x + y)^(-2)
(-2 - 2sfixedLarge[[n]] - x - 2sfixedLarge[[n]]x^2 + x/(1+x) +
y - 2sfixedLarge[[n]]y^2 - y/(1+y) + 4(x+y+x*y)/(x+y+1)),
{x, 0, Infinity}, {y, 0, Infinity}]
asymlow = MapThread[{#1, #2}&,
{sfixedLarge, Table[doubleasym[n], {n, 1, Length[sfixedLarge]}]}]
gasymlow = Interpolation[asymlow];

blowlist = Table[({a, b, c}/.abclow[[k, 1]])[[2]], {k, 1, mesh}];
intblow = Interpolation[MapThread[{#1, #2}&, {sfixed, blowlist}]];
intbl[r_]:=
(-2r^2intalow'[r]intalow[r] - rintblow[r]intalow'[r]+
rintalow[r]intblow'[r])/(intalow[r]^2) + r/2
plotbl1 =
ListPlot[MapThread[{#1, #2}&, {sfixed, Map[intblow, sfixed]}]];
plotbl2 = Plot[(2r)^(1/4)gasymlow[r], {r, 0.01, 0.99},
PlotStyle → Red];
Show[plotbl1, plotbl2]

analyGbl =
Plot[{(2s)^(1/4)Fl[s](intlow/.L → (Log[s/8] + EulerGamma))},
{s, 0, 1}, PlotStyle → Red];
Show[analyGbl, intalow]

(* Asymptoptic data *)
sizeNL = 50;
meshL = 50;
sfixedL = linspace[0.01, 9.99, meshL];
dnshL =
Grid[
Table[MapThread[{#1, #2}&,
{sfixedL, Det[toeplitzh[#, n]]&/@ ((1 - 2*sfixedL/n)^(1/2))}],
{n, 2sizeNL, 3sizeNL}]];
abchighL =
Table[
NSolve[
Flatten[
Table[
```

```
{a/(3*sizeNL-n)^(1/4) + b/(3*sizeNL-n)^(5/4) +
c/(3*sizeNL-n)^(9/4) == dnshL[[1, sizeNL - n + 1]][[k, 2]]},
{n, 0, 2}]], {a, b, c}], {k, 1, meshL}];
ahighlistL = Table[({a, b, c}/.abchighL[[k, 1]])[[1]], {k, 1, meshL}];
intahighL = Interpolation[MapThread[{#1, #2}&, {sfixedL, ahighlistL}]];
intahL[r_]:=rintahighL'[r]/intahighL[r] - 1/4
plotahL1 =
ListPlot[MapThread[{#1, #2}&, {sfixedL, Map[intahighL, sfixedL]}]];
plotahL2 = Plot[(2r)^(1/4)G1hasym[r], {r, 0, 10}, PlotStyle → Red];
Show[plotahL1, plotahL2]
bhighlistL = Table[({a, b, c}/.abchighL[[k, 1]])[[2]], {k, 1, meshL}];
intbhighL = Interpolation[MapThread[{#1, #2}&, {sfixedL, bhighlistL}]];
intbhL[r_]:=
(-2r^2intahighL'[r]intahighL[r] - rintahighL'[r]intbhighL[r]+
rintahighL[r]intbhighL'[r])/(intahighL[r]^2)
plotbhL1 =
ListPlot[MapThread[{#1, #2}&, {sfixedL, Map[intbhighL, sfixedL]}]];
plotbhL2 = Plot[(2r)^(1/4)G2hasym[r], {r, 0, 10}, PlotStyle → Red];
Show[plotbhL1, plotbhL2]
dnslL =
Grid[
Table[MapThread[{#1, #2}&,
{sfixedL, Det[toeplitzl[#, n]]&/@ ((1 - (2*sfixedL)/n)^(-1/2))}],
{n, 2 * sizeNL, 3 * sizeNL}]];
abclowL =
Table[
NSolve[
Flatten[
Table[
{a/(3*sizeNL-n)^(1/4) + b/(3*sizeNL-n)^(5/4) +
c/(3*sizeNL-n)^(9/4) == dnslL[[1, sizeNL - n + 1]][[k, 2]]},
{n, 0, 2}]], {a, b, c}], {k, 1, meshL}];
alowlistL = Table[({a, b, c}/.abclowL[[k, 1]])[[1]], {k, 1, meshL}]
intalowL = Interpolation[MapThread[{#1, #2}&, {sfixedL, alowlistL}]];
intalL[r_]:=rintalowL'[r]/intalowL[r] - 1/4
plotalL1 =
ListPlot[MapThread[{#1, #2}&, {sfixedL, Map[intalowL, sfixedL]}]];
plotalL2 = Plot[(2r)^(1/4)G1lasym[r], {r, 0, 10}, PlotStyle → Red];
```

```
Show[plotalL1, plotalL2]
blowlistL = Table[({a, b, c}/.abclowL[[k, 1]])[[2]], {k, 1, meshL}];
intblowL = Interpolation[MapThread[{#1, #2}&, {sfixedL, blowlistL}]];
intblL[r_]:=
(−2r^2intalowL′[r]intalowL[r] − rintalowL′[r]intblowL[r]+
rintalowL[r]intblowL′[r])/(intalowL[r]^2) + r/2
plotblL1 =
ListPlot[MapThread[{#1, #2}&, {sfixedL, Map[intblowL, sfixedL]}]];
plotblL2 = Plot[(2r)^(1/4)gasymlow[r], {r, 0.01, 9.99},
PlotStyle → Red];
Show[plotblL1, plotblL2]
```

Bibliography

[1] Helen Au-Yang and Jacques H. H. Perk. Correlation functions and susceptibility in the Z-invariant Ising model. In *MathPhys odyssey, 2001*, volume 23 of *Prog. Math. Phys.*, pages 23–48. Birkhäuser Boston, Boston, MA, 2002. 23, 29, 41

[2] Eytan Barouch, Barry M. McCoy, and Tai Tsun Wu. Zero-field susceptibility of the two-dimensional ising model near T_c. *Phys. Rev. Lett.*, 31:1409–1411, Dec 1973. 23

[3] R. J. Baxter. Onsager and Kaufman's Calculation of the Spontaneous Magnetization of the Ising Model. *Journal of Statistical Physics*, 145:518–548, November 2011. 4

[4] P. J. Forrester. *Log-gases and random matrices*, volume 34 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2010. 9

[5] P. J. Forrester and S. O. Warnaar. The importance of the Selberg integral. *ArXiv e-prints*, October 2007. 15

[6] E. Ising. Beitrag zur Theorie des Ferromagnetismus. *Zeitschrift fur Physik*, 31:253–258, February 1925. 1

[7] Claude Itzykson and Jean-Michel Drouffe. *Statistical field theory. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1989. From Brownian motion to renormalization and lattice gauge theory. 12

[8] Michio Jimbo and Tetsuji Miwa. Studies on holonomic quantum fields. XVII. *Proc. Japan Acad. Ser. A Math. Sci.*, 56(9):405–410, 1980. 1, 8, 40

[9] Kurt Ove Johansson. *On Szego's asymptotic formula for Toeplitz determinants and generalizations*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Fil.Dr.)–Uppsala Universitet (Sweden). 6

[10] I. Lyberg and B. M. McCoy. Form factor expansion of the row and diagonal correlation functions of the two-dimensional Ising model. *J. Phys. A*, 40(13):3329–3346, 2007. 6

[11] Barry M. McCoy. *Advanced statistical mechanics*, volume 146 of *International Series of Monographs on Physics*. Oxford University Press, Oxford, 2010. 23

[12] Barry M. McCoy, Craig A. Tracy, and Tai Tsun Wu. Painlevé functions of the third kind. *J. Mathematical Phys.*, 18(5):1058–1092, 1977. 23

[13] B.M. McCoy and T.T. Wu. *The two-dimensional Ising model*. Harvard University Press, 1973. 9, 10

[14] Elliott W. Montroll, Renfrey B. Potts, and John C. Ward. Correlations and spontaneous magnetization of the two-dimensional Ising model. *J. Mathematical Phys.*, 4:308–322, 1963. 1, 4

[15] Lars Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)*, 65:117–149, 1944. 1, 2

[16] N. S. Witte. Isomonodromic deformation theory and the next-to-diagonal correlations of the anisotropic square lattice Ising model. *Journal of Physics A Mathematical General*, 40:F491–F501, June 2007. 4

[17] Tai Tsun Wu. Theory of toeplitz determinants and the spin correlations of the two-dimensional ising model. i. *Phys. Rev.*, 149:380–401, Sep 1966. 19

[18] Tai Tsun Wu, Barry M. McCoy, Craig A. Tracy, and Eytan Barouch. Spin-spin correlation functions for the two-dimensional ising model: Exact theory in the scaling region. *Phys. Rev. B*, 13:316–374, Jan 1976. 1, 18, 19, 23

[19] C. N. Yang. The spontaneous magnetization of a two-dimensional ising model. *Phys. Rev.*, 85:808–816, Mar 1952. 1, 3, 6