CLASSIFYING A FAMILY OF SYMMETRIC GRAPHS

Sanming Zhou*

Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size v. For blocks B, C of $\mathcal{B}$ adjacent in the quotient graph $\Gamma_{\mathcal{B}}$, let k be the number of vertices in B adjacent to at least one vertex in C. In this paper we classify all possibilities for $(\Gamma, \Gamma_{\mathcal{B}}, G)$ in the case where $k = v - 1 \geq 2$ and $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ, where for a vertex of Γ, say $\gamma \in B$, $\mathcal{B}(\gamma)$ denotes the set of blocks C such that γ is the only vertex in B not adjacent to any vertex in C.

1 Introduction

A finite graph $\Gamma = (V(\Gamma), E(\Gamma))$ is said to *admit* a finite group G as a group of automorphisms if G acts on $V(\Gamma)$ in such a way that it preserves the adjacency of Γ. For such a pair (Γ, G), if G is transitive on $V(\Gamma)$ and, in its induced action, is transitive on the set $\mathrm{Arc}(\Gamma)$ of arcs of Γ, then Γ is said to be a *G-symmetric graph*, where an *arc* is an ordered pair of adjacent vertices. Roughly speaking, in most cases such a graph Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in \mathcal{B}$ for $B \in \mathcal{B}$ and $g \in G$, where $B^g := \{\alpha^g : \alpha \in B\}$. In this case Γ is said to be an *imprimitive G-symmetric graph.* From permutation group theory [3, Corollary 1.5A], this happens precisely when G_α is not a maximal subgroup of G, where $\alpha \in V(\Gamma)$ and G_α is the stabilizer of α in G. For such a graph Γ we have a natural *quotient graph* $\Gamma_{\mathcal{B}}$ with respect to $\mathcal{B}$, which is defined to have vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exists an edge $\{\alpha, \beta\} \in E(\Gamma)$ with $\alpha \in B$ and $\beta \in C$. In the following we will always assume that $\Gamma_{\mathcal{B}}$ has at least one edge, so each block of $\mathcal{B}$ is an independent set of Γ (see e.g. [1, Proposition 22.1] and [8]). This quotient graph $\Gamma_{\mathcal{B}}$ conveys a lot of information about the graph Γ, and in particular it inherits the G-symmetry from Γ (under the induced action of G on $\mathcal{B}$). For $B \in \mathcal{B}$, denote by $\Gamma_{\mathcal{B}}(B)$ the neighbourhood of B in $\Gamma_{\mathcal{B}}$. In introducing a geometric approach to imprimitive symmetric graphs, Gardiner and Praeger [4] suggested an analysis of this quotient graph $\Gamma_{\mathcal{B}}$ together with (i) the 1-design with point set B and "blocks" $\Gamma(C) \cap B$ (with possible repetitions), for all $C \in \Gamma_{\mathcal{B}}(B)$; and

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(ii) the induced bipartite subgraph $\Gamma[B, C]$ of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$, where $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$ with $\Gamma(\alpha)$ the neighbourhood of α in Γ. Since Γ is G-symmetric, $\Gamma[B, C]$ is, up to isomorphism, independent of the choice of adjacent blocks B, C of $\mathcal{B}$.

The purpose of this paper is to classify a family of imprimitive symmetric graphs and the corresponding quotients and groups. This makes partial contribution to our project of the study of G-symmetric graphs Γ with $k = v - 1 \geq 2$, where $v := |B|$ is the block size of $\mathcal{B}$ and $k := |\Gamma(C) \cap B|$ is the size of each part of the bipartition of $\Gamma[B, C]$. It seems that this case is rather rich in both theory and examples: In [7, Section 6] a natural construction of a subclass of such graphs was discovered, and this was further developed in [9, 10]. In [5] such graphs Γ with $\Gamma_{\mathcal{B}}$ a complete graph and G a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, q)$ were determined and characterized, for any prime power q. In [11] an intertwined relationship between G-symmetric graphs with $k = v - 1 \geq 2$ and certain kinds of G-point- and G-block-transitive 1-designs was revealed. For such a graph Γ and a vertex α of Γ, we denote by $B(\alpha)$ the unique block of $\mathcal{B}$ containing α. Since $k = v - 1 \geq 2$, we may define

$$\mathcal{B}(\alpha) := \{C \in \mathcal{B} : \Gamma(C) \cap B(\alpha) = B(\alpha) \setminus \{\alpha\}\} \tag{1}$$

and set

$$m := |\mathcal{B}(\alpha)|. \tag{2}$$

Thus $\mathcal{B}(\alpha)$ is the set of blocks of $\mathcal{B}$ which are adjacent to $B(\alpha)$ in $\Gamma_{\mathcal{B}}$ but contain no vertex adjacent to α in Γ. Since G is transitive on $V(\Gamma)$, the integer m does not depend on the choice of α. It seems that the size of $\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)$, for adjacent vertices α, β of Γ, influences a lot the structure of Γ. For example, we will see in Lemma 2.1(c) that, if it is greater than $m/2$, then Γ is forced to be an *almost cover* [9] of $\Gamma_{\mathcal{B}}$, that is, $\Gamma[B, C]$ is a matching of $v - 1$ edges. In this paper, we investigate the extreme case where $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ, and (without loss of generality) $\Gamma_{\mathcal{B}}$ is connected. In this case, we will prove that the group G is rather restrictive and all of Γ, $\Gamma_{\mathcal{B}}$ and $\Gamma[B, C]$ can be determined explicitly, namely $\Gamma \cong (v + 1) \cdot K_m^v$, $\Gamma_{\mathcal{B}} \cong K_m^{v+1}$, Γ is an almost cover of $\Gamma_{\mathcal{B}}$, and G is an extension of a group by any 3-transitive group of degree $v + 1$ (see Theorem 3.1 and Remark 3.2). Here we denote by K_m^n the complete n-partite graph with m vertices in each part of its n-partition, and by $n \cdot \Sigma$ the union of n vertex-disjoint copies of a given graph Σ.

2 Preliminary

For terminology and notation on graphs and permutation groups, the reader is referred to [1] and [3], respectively. Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition

$\mathcal{B}$ such that $k = v - 1 \geq 2$. For two vertices α, β of Γ, if $B(\alpha) \in \mathcal{B}(\beta)$ and $B(\beta) \in \mathcal{B}(\alpha)$ hold simultaneously, then we say that α, β are *mates*, and that α is the mate of β in $B(\alpha)$ (so β is the mate of α in $B(\beta)$ as well). Define Γ' to be the graph with vertex set $V(\Gamma)$ in which α, β are adjacent if and only if they are mate. Then Γ' is G-symmetric ([7, Proposition 3]). One can see that the set $\{\mathcal{B}(\alpha) : \alpha \in B\}$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$, and hence G_B induces an action on it, where G_B is the setwise stabilizer of B in G. Clearly, for $(\alpha, \beta) \in \mathrm{Arc}(\Gamma)$, the value of $|\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)|$ is between 0 and m, and is independent of the choice of such (α, β) since Γ is G-symmetric. Part (c) of the following lemma gives an upper bound for this integer in terms of m and the valency of $\Gamma[B, C]$.

Lemma 2.1 *Let (Γ, G) be as above, and let s be the valency of $\Gamma[B, C]$ (for adjacent blocks B, C of $\mathcal{B}$). Then the following (a)-(c) hold.*

(a) *The valency of Γ is equal to $ms(v-1)$, and the valency of $\Gamma_{\mathcal{B}}$ is equal to mv ([7, Theorem 5(a)]).*

(b) *G_B is doubly transitive on $\{\mathcal{B}(\alpha) : \alpha \in B\}$ ([7, Theorem 5(b)]).*

(c) *For $(\alpha, \beta) \in \mathrm{Arc}(\Gamma)$, we have $|\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)| \leq m/s$. In particular, if $|\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)| > m/2$, then $\Gamma[B, C] \cong (v-1) \cdot K_2$.*

Proof We need to prove (c) only. Let $n = |\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)|$ for $(\alpha, \beta) \in \mathrm{Arc}(\Gamma)$. Let $B = B(\alpha)$, $C \in \Gamma_{\mathcal{B}}(B) \setminus \mathcal{B}(\alpha)$, and set $\Gamma(\alpha) \cap C = \{\beta_1, \ldots, \beta_s\}$. Then $\mathcal{B}(\alpha) \cap \mathcal{B}(\beta_i)$, for $i = 1, \ldots, s$, are pairwise disjoint with each containing n blocks of $\mathcal{B}(\alpha)$. So we have $sn \leq m$, as required. In particular, if $n > m/2$, then we must have $s = 1$ and thus $\Gamma[B, C] \cong (v-1) \cdot K_2$. $\square$

The following example shows that the case where $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ can occur. For a finite set I, we denote by $I^{(2)}$ the set of ordered pairs of distinct elements of I.

Example 2.2 Let X be a finite group acting 3-transitively on a finite set I of degree $v+1 \geq 4$, and Y a finite group acting on a finite set J of degree $m \geq 1$. We require that Y is 2-transitive on J whenever $m \geq 2$. Then $G := X \times Y$ is transitive on $V := I^{(2)} \times J$ in its action defined by $(i, h, j)^{(x,y)} := (i^x, h^x, j^y)$ for $(i, h, j) \in V$ and $(x, y) \in G$. Define Γ to be the graph with vertex set V in which $(i, h, j), (i', h', j')$ are adjacent if and only if $i \neq i'$ and $h = h'$. Then $\Gamma \cong (v+1) \cdot K_m^v$, and the assumptions on X, Y imply that Γ is G-symmetric. Clearly, Γ admits $\mathcal{B} := \{[i, j] : i \in I, j \in J\}$ as a G-invariant partition, where $[i, j] := \{(i, h, j) : h \in I \setminus \{i\}\}$. We have $\Gamma_{\mathcal{B}} \cong K_m^{v+1}$ with $[i, j], [i', j']$ adjacent if and only if $i \neq i'$. Also, we have $\Gamma[B, C] \cong (v-1) \cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$ (hence $k = v - 1 \geq 2$). Moreover, for adjacent vertices $\alpha = (i, h, j)$, $\alpha' = (i', h, j')$ of Γ, we have $\mathcal{B}(\alpha) = \mathcal{B}(\alpha') = \{[h, \ell] : \ell \in J\}$, and hence $|\mathcal{B}(\alpha)| = m$.

3 Main result and the proof

Unexpectedly, the graphs Γ in Example 2.2 are the only G-symmetric graphs with $\Gamma_{\mathcal{B}}$ connected such that $k = v - 1 \geq 2$ and $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ, and $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ are as shown therein. More precisely, we have the following theorem, which is the main result of this paper.

Theorem 3.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 2$. Suppose further that $\Gamma_{\mathcal{B}}$ is connected and that $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ. Let $m = |\mathcal{B}(\alpha)|$. Then $\Gamma \cong (v+1) \cdot K_m^v$, $\Gamma_{\mathcal{B}} \cong K_m^{v+1}$, $\Gamma[B, C] \cong (v-1) \cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$, and the induced action of G on the natural $(v+1)$-partition $\mathbf{B}$ of $\Gamma_{\mathcal{B}}$ is 3-transitive. Moreover, the vertices of Γ can be labelled by ordered triples of integers such that the following (a)-(c) hold (where we set $I := \{0, 1, \ldots, v\}$ and $J := \{1, 2, \ldots, m\}$):*

(a) *$V(\Gamma) = I^{(2)} \times J$, and two vertices $(i, h, j), (i', h', j') \in V(\Gamma)$ are adjacent in Γ if and only if $i \neq i'$ and $h = h'$.*

(b) *$\mathcal{B} = \{[i, j] : i \in I, j \in J\}$, where $[i, j] := \{(i, h, j) : h \in I \setminus \{i\}\}$, and $[i, j], [i', j']$ are adjacent blocks if and only if $i \neq i'$.*

(c) *$\mathbf{B} = \{\mathbf{i} : i \in I\}$, where $\mathbf{i} = \{[i, j] : j \in J\}$.*

Conversely, the graph Γ defined in (a) together with the group $G = X \times Y$ satisfies all conditions of the theorem, where X is a group acting 3-transitively on I, Y is a group acting on J which is 2-transitive if $m \geq 2$, and the action of G on $V(\Gamma)$ is as defined in Example 2.2.

Proof By our assumption we have $|\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)| = m > m/2$ for $(\alpha, \beta) \in \mathrm{Arc}(\Gamma)$. Thus Lemma 2.1(c) implies

(i) $\Gamma[D, E] \cong (v-1) \cdot K_2$ for adjacent blocks D, E of $\mathcal{B}$.

Let B be a block of $\mathcal{B}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_v$ be vertices of B. For each $\alpha_i \in B$, we label (in an arbitrary way) the m blocks in $\mathcal{B}(\alpha_i)$ by $[i, j]$, $j \in J$. Also, we label the unique mate β_{ij} of α_i in the block $[i, j]$ by $(i, 0, j)$, $j \in J$. For each block $[i, j]$ and for each $h \in I \setminus \{0\}$ distinct from i, (i) implies that $[i, j]$ contains a unique vertex adjacent to α_h. We label such a vertex in $[i, j]$ by (i, h, j). In view of (i) one can see that each vertex in $[i, j]$ receives a unique label, and that the labels of distinct vertices in $[i, j]$ have distinct second coordinates. Therefore, for each $i \in I \setminus \{0\}$ and $j \in J$, we may identify the block $[i, j]$ with the set $\{(i, h, j) : h \in I \setminus \{i\}\}$. By our assumption, for $i, h \in I \setminus \{0\}$ with $i \neq h$ and $j \in J$, we have

(ii) $\mathcal{B}((i,h,j)) = \mathcal{B}(\alpha_h) = \{[h,1],[h,2],\ldots,[h,m]\}$.

In particular, this implies that

(iii) $[i,j],[i',j']$ are adjacent blocks, for distinct $i,i' \in I\setminus\{0\}$ and any $j,j' \in J$.

Moreover, if two vertices $(i,h,j),(i',h',j')$ are adjacent, then by (ii) and our assumption we must have $\mathcal{B}(\alpha_h) = \mathcal{B}((i,h,j)) = \mathcal{B}((i',h',j')) = \mathcal{B}(\alpha_{h'})$, which is true only when $h = h'$. This, together with (i) and (iii), implies the following assertion.

(iv) For distinct $i,i' \in I\setminus\{0\}$ and any $j,j' \in J$, two labelled vertices $(i,h,j),(i',h',j')$ of Γ are adjacent if and only if $h = h'$. In other words, for adjacent blocks $D = [i,j], E = [i',j']$ of $\mathcal{B}$, the bipartite subgraph $\Gamma[D,E]$ of Γ is the matching of $v-1$ edges joining (i,h,j) and (i',h,j'), for $h \in I\setminus\{i,i'\}$.

Therefore, (i,i',j) and (i',i,j') are mates and hence, for the graph Γ' defined at the beginning of the previous section, we have

(v) $\Gamma'((i,h,j)) = \{(h,i,j') : j' \in J\}$.

Now let us examine a particular labelled vertex, say (i,h,j). From Lemma 2.1(a) and (i) above, the valency of Γ is $m(v-1)$, and hence the neighbourhood $\Gamma((i,h,j))$ of (i,h,j) contains $m(v-1)$ vertices. From (iv) we have $\{(i',h,j') : i' \in I\setminus\{0,h,i\}, j' \in J\} \subseteq \Gamma((i,h,j))$ and this contributes $m(v-2)$ neighbours of (i,h,j). Note that α_h is also a neighbour of (i,h,j). Apart from these, there are $m-1$ remaining neighbours of (i,h,j), which we denote by $\delta_2,\ldots,\delta_m$, respectively. By (i) these vertices $\delta_2,\ldots,\delta_m$ belong to distinct blocks, say $B_2,\ldots,B_m$, of $\mathcal{B}$. For each δ_t, we have $\mathcal{B}(\delta_t) = \mathcal{B}((i,h,j)) = \mathcal{B}(\alpha_h) = \{[h,1],[h,2],\ldots,[h,m]\}$ by (ii) and our assumption. In particular, this implies that all the blocks $[h,\ell]$, for $\ell \in J$, are adjacent to the block B_t. On the other hand, from (v) we have $\Gamma'((h,h',\ell)) = \{(h',h,t) : t \in J\}$ for each vertex $(h,h',\ell) \in [h,\ell]\setminus\{\beta_{h\ell}\}$. In other words, the m mates of each vertex in $[h,\ell]\setminus\{\beta_{h\ell}\}$ are in $\bigcup_{h' \in I\setminus\{0,h\}, t\in J}\ [h',t]$. So the only possibility is that $\beta_{h\ell}$ is the mate of δ_t in $[h,\ell]$, for each $\ell \in J$. Consequently, we have

(vi) $\mathcal{B}(\beta_{h1}) = \cdots = \mathcal{B}(\beta_{hm}) = \{B, B_2,\ldots,B_m\}$, and hence none of $B, B_2,\ldots,B_m$ coincides with $[i,j]$ for any $i \in I\setminus\{0\}$ and $j \in J$.

We know from (iii) that the blocks $[i',j']$, for $i' \in I\setminus\{0,h\}$ and $j' \in J$, are all adjacent to $[h,\ell]$. Besides these $m(v-1)$ blocks, $B, B_2,\ldots,B_m$ are the only blocks of $\mathcal{B}$ adjacent to $[h,\ell]$ in $\Gamma_{\mathcal{B}}$ since $\Gamma_{\mathcal{B}}$ has valency mv (Lemma 2.1(a)). Therefore, if we apply the procedure above to another vertex (i',h,j'), we would get the same blocks $B_2,\ldots,B_m$. In other words, these blocks are independent of the choice of the vertex (i,h,j) (depending only on h), and hence they are adjacent to the block $[i,j]$ for any $i \in I \setminus \{0\}$ and $j \in J$. Moreover, since the mate δ_t of $\beta_{h\ell}$ in B_t is unique, the vertices $\delta_2, \ldots, \delta_m$ are also independent of the choice of (i,h,j) and thus they are common neighbours of all such vertices (i,h,j). Thus, since the valency of $\Gamma_{\mathcal{B}}$ is mv, $B, B_2, \ldots, B_m$ are the only unlabelled blocks of $\mathcal{B}$. From this and by a similar argument to that above, we see that for each $h \in I \setminus \{0\}$, all the vertices (i,h,j), $i \in I \setminus \{0,h\}$, $j \in J$, have a common neighbour in each B_t, which we now label by $(0,h,t)$. Since for distinct h, h' the vertices (i,h,j), (i,h',j) have different neighbours in B_t, the vertices of B_t receive pairwise distinct labels. Now let us label $B, B_2, \ldots, B_m$ with $[0,1], [0,2], \ldots, [0,m]$, respectively, and label each α_h with $(0,h,1)$. Then all the vertices of Γ and all the blocks of $\mathcal{B}$ have been labelled. From the labelling above, the validity of (a) and (b) follows immediately.

Since the valency of Γ is $m(v-1)$, the argument above also shows that for each $h \in I$ the connected component of Γ containing the vertex α_h is the complete v-partite graph K_m^v with v-partition $\{\{(i,h,j) : j \in J\} : i \in I\}$, where we set $\alpha_0 = \beta_{11}$. Hence we have $\Gamma \cong (v+1) \cdot K_m^v$. Also, $\Gamma_{\mathcal{B}}$ is the complete $(v+1)$-partite graph K_m^{v+1} with $(v+1)$-partition $\mathbf{B} := \{\mathbf{i} : i \in I\}$, where $\mathbf{i} := \mathcal{B}(\alpha_i) = \{[i,j] : j \in J\}$ for $i \in I$. Clearly, $(\Gamma_{\mathcal{B}})_{\mathbf{B}} \cong K_{v+1}$ and $\mathbf{B}$ is a G-invariant partition of $\mathcal{B}$. From Lemma 2.1(b), G_B is doubly transitive on $\{\mathcal{B}(\gamma) : \gamma \in B\}$. The setwise stabilizer in G of the block $\mathbf{0}$ contains G_B as a subgroup, and so is doubly transitive on the neighbourhood $\mathbf{B} \setminus \{\mathbf{0}\}$ of $\mathbf{0}$ in $(\Gamma_{\mathcal{B}})_{\mathbf{B}}$. Therefore, G is 3-transitive on $\mathbf{B}$.

Finally, for $G = X \times Y$ with X triply transitive on I and Y doubly transitive on J whenever $m \geq 2$, Example 2.2 shows that the graph Γ defined in (a) satisfies all the conditions in the theorem. $\square$

Remark 3.2 In Theorem 3.1, G may or may not be faithful on $\mathbf{B}$. (This can be seen from Example 2.2, where the action of G on $\mathbf{B}$ is permutationally isomorphic to the action of X on I which is not necessarily faithful.) Let K be the kernel of the action of G on $\mathbf{B}$, and set $H := G/K$. Then H is 3-transitive and faithful on $\mathbf{B}$ of degree $v+1$, and G is an extension of K by H. From the classification of finite highly transitive permutation groups (see e.g. [2, 6]), H is one of the following: S_{v+1} $(v \geq 3)$, A_{v+1} $(v \geq 4)$, M_{v+1} $(v = 10, 11, 21, 22, 23)$, M_{11} $(v = 11)$, $\mathrm{AGL}(d,2)$ $(v = 2^d - 1)$, $\mathbb{Z}_2^4.A_7$ $(v = 15)$, and $\mathrm{PSL}(2,v) \leq H \leq \mathrm{P\Gamma L}(2,v)$ (v a prime power). Example 2.2 shows that $m = |\mathcal{B}(\alpha)|$ defined in (2) can be any positive integer and H can be any group listed above.

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Department of Mathematics and Statistics
The University of Western Australia
Perth, WA 6907, Australia
Email: *smzhou@maths.uwa.edu.au*

Current address:

Department of Mathematics and Statistics

The University of Melbourne

Parkville, VIC 3052, Australia