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SOLUTION TO A QUESTION ON A FAMILY OF IMPRIMITIVE SYMMETRIC GRAPHS

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Abstract

We answer a recent question posed by Li *et al*. ['Imprimitive symmetric graphs with cyclic blocks', *European J. Combin.* **31** (2010), 362–367] regarding a family of imprimitive symmetric graphs.

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A graph $\Gamma = (V, E)$ is called *G-symmetric* if Γ admits G as a group of automorphisms such that G is transitive on V and on the set of arcs of Γ, where an *arc* is an ordered pair of adjacent vertices. If in addition Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of V such that $1 < |B| < |V|$ and $B^g := \{\alpha^g : \alpha \in B\} \in \mathcal{B}$ for $B \in \mathcal{B}$ and $g \in G$, then Γ is called an *imprimitive G-symmetric graph*. In this case the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$ is defined to have vertex set $\mathcal{B}$ such that $B, C \in \mathcal{B}$ are adjacent if and only if there exists at least one edge of Γ between B and C. We assume that $\Gamma_{\mathcal{B}}$ contains at least one edge, so that each block of $\mathcal{B}$ is an independent set of Γ. Denote by $\Gamma(\alpha)$ the neighbourhood of $\alpha \in V$ in Γ, and define $\Gamma(X) = \bigcup_{\alpha \in X} \Gamma(\alpha)$ for $X \in \mathcal{B}$. For blocks $B, C \in \mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$, let $\Gamma[B, C]$ be the bipartite subgraph of Γ induced on $(B \cap \Gamma(C)) \cup (C \cap \Gamma(B))$. Then $\Gamma[B, C]$ is independent of the choice of (B, C) up to isomorphism. Define

$$v := |B| \quad \text{and} \quad k := |B \cap \Gamma(C)|$$

to be the block size of $\mathcal{B}$ and the size of each part of the bipartition of $\Gamma[B, C]$, respectively.

In line with a geometrical approach suggested in [1], various situations may occur for Γ, G, $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ and a certain 1-design with point set B; see, for example, [1, 3, 5–7]. The case where $k = v - 2 \geq 1$ was studied in [2, 4] and a necessary and sufficient condition for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc-transitive was given in [2]. In this case, the multigraph Γ^B [2] with vertex B and an edge joining the two vertices of $B \setminus \Gamma(C)$ for every $C \in \Gamma_{\mathcal{B}}(B)$ plays an important role in the structure of Γ and $\Gamma_{\mathcal{B}}$,

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where $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$. Since Γ is G-symmetric, up to isomorphism Γ^B is independent of the choice of B, and the multiplicity of each edge $\{\alpha, \beta\}$ of Γ^B, namely

$$m := |\{C \in \Gamma_{\mathcal{B}}(B) : B \setminus \Gamma(C) = \{\alpha, \beta\}\}|,$$

is independent of the choice of $\{\alpha, \beta\}$. Denote by $\mathtt{Simple}(\Gamma^B)$ the underlying simple graph of Γ^B and by G_B the setwise stabilizer of B in G. It has been proved [2, Theorem 2.1] that $\mathtt{Simple}(\Gamma^B)$ is G_B-vertex-transitive and G_B-edge-transitive, and either Γ^B is connected or v is even and $\mathtt{Simple}(\Gamma^B)$ is a perfect matching $(v/2) \cdot K_2$. In the latter case detailed information about Γ was obtained in [2, Theorem 1.3] when Γ^B is simple. In [4], Li *et al.* proved that, if $\mathtt{Simple}(\Gamma^B)$ is a cycle, then v must be small, namely v is equal to 3 or 4. Based on this they posed the following question.

QUESTION 1. In the case where $k = v - 2$ and Γ^B is connected, is v bounded by some function of the valency of $\mathtt{Simple}(\Gamma^B)$?

Define

$$b := \mathrm{val}(\Gamma_{\mathcal{B}}), \quad s := \mathrm{val}(\Gamma[B, C]), \quad r := |\{C \in \mathcal{B} : \alpha \in \Gamma(C)\}|$$

to be respectively the valency of $\Gamma_{\mathcal{B}}$, the valency of $\Gamma[B, C]$, and the number of blocks of $\mathcal{B}$ that contain at least one neighbour of a fixed vertex $\alpha \in V$ in Γ. Note that v, k, b, r and s all rely on the G-invariant partition $\mathcal{B}$.

In this paper we answer Question 1 by proving the following stronger result: there are only two possibilities for $\mathtt{Simple}(\Gamma^B)$ and v can take two values only.

THEOREM 2. *Suppose that Γ is a G-symmetric graph which admits a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geq 1$, $\Gamma_{\mathcal{B}}$ is connected and $\mathtt{Simple}(\Gamma^B)$ is connected with valency $d \geq 2$. Then one of the following occurs.*

(a) $\mathtt{Simple}(\Gamma^B) \cong K_v$, $v = d + 1$, $b = m(v - 1)v/2$, *and* G_B^B *is 2-homogeneous.*

(b) $\mathtt{Simple}(\Gamma^B) \cong K_{v/2, v/2}$, $v = 2d$, $b = mv^2/4$, *and the bipartition of* $\mathtt{Simple}(\Gamma^B)$ *induces a G-invariant partition $\mathcal{B}^*$ of the vertex set of Γ (which is a refinement of $\mathcal{B}$) such that one of the following holds for its parameters:*

  (i) $v^* = k^* + 1 = v/2$, $b^* = b$, $r^* = r$, $s^* = s$;

  (ii) $v^* = k^* + 1 = v/2$, $b^* = 2b$, $r^* = 2r$, $s^* = s/2$;

  (iii) $v^* = 2k^* + 1 = v/2$, $b^* = 2b$, $r^* = r$, $s^* = s$.

PROOF. Suppose that Γ, G and $\mathcal{B}$ satisfy the conditions in the theorem. Denote $\Omega := \mathtt{Simple}(\Gamma^B)$. Let B and C be two blocks of $\mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$, and let $\{\alpha, \beta\} = B \setminus \Gamma(C)$ be the corresponding edge of Ω. Define

$$U := (\Omega(\alpha) \cup \Omega(\beta)) \setminus \{\alpha, \beta\}$$

to be the neighbourhood of the subset $\{\alpha, \beta\}$ of B in Ω, and set

$$W := B \setminus (U \cup \{\alpha, \beta\}).$$

Since Ω has valency $d \geq 2$, we have $U \neq \emptyset$. Since every element of G_{BC} $(= (G_B)_C)$ fixes $\{\alpha, \beta\}$ setwise, it follows that every element of G_{BC} fixes each of U and W setwise. Thus $G_{BC} \leq G_U \cap G_W$.

Claim 1. $W = \emptyset$, that is, $U = B \setminus \{\alpha, \beta\}$, or every vertex in B is adjacent to at least one of α and β in Ω.

Suppose otherwise and let $\delta \in W$. Since $U \neq \emptyset$, we may take a vertex $\gamma \in U$. Since $\delta, \gamma \neq \alpha, \beta$, there exist $\delta_1, \gamma_1 \in C$ adjacent to δ, γ in Γ, respectively. (It may occur that $\delta_1 = \gamma_1$.) Since Γ is G-symmetric, there exists $g \in G$ such that $(\gamma, \gamma_1)^g = (\delta, \delta_1)$. Since g maps $\gamma \in B$ to $\delta \in B$ and $\gamma_1 \in C$ to $\delta_1 \in C$, it fixes B and C setwise. Hence $g \in G_{BC} \leq G_U \cap G_W$. However, this is a contradiction, because g maps $\gamma \in U$ to $\delta \in W$. Therefore $W = \emptyset$ as claimed.

Since Ω has valency d, by Claim 1, $d - 1 \leq |U| \leq 2(d - 1)$. Since $v = |U| + 2$ by Claim 1, it follows that

$$d + 1 \leq v \leq 2d.$$

Claim 2. In Ω any two adjacent vertices have $2d - v$ common neighbours, and two nonadjacent vertices have the same neighbourhood.

In fact, since Ω is G_B-edge-transitive [2, Theorem 2.1], the number λ of common neighbours of a pair of adjacent vertices in Ω is a constant. Consider the neighbourhood U of $\{\alpha, \beta\}$ in Ω, where α and β are as above. There are exactly $d - \lambda - 1$ vertices in B which are adjacent to α but not β (β but not α, respectively). Thus, by Claim 1, $2(d - \lambda - 1) + \lambda = v - 2$, which implies that $\lambda = 2d - v$.

Now let σ and τ be any two nonadjacent vertices of Ω. If $\gamma \in B$ is adjacent to σ in Ω, then by applying Claim 1 to the edge $\{\sigma, \gamma\}$, every vertex in B is adjacent to either σ or γ in Ω. Thus, since τ is not adjacent to σ, it must be adjacent to γ in Ω and so $\Omega(\sigma) \subseteq \Omega(\tau)$. Similarly, $\Omega(\tau) \subseteq \Omega(\sigma)$. Hence $\Omega(\sigma) = \Omega(\tau)$ and Claim 2 is proved.

Consider any maximal (with respect to set-theoretic inclusion) independent set X of Ω. By Claim 2 the vertices in X have the same neighbourhood in Ω. Denote this common neighbourhood by Y, so that $|Y| = d$. If $B \setminus (X \cup Y) \neq \emptyset$, then by the maximality of X, any vertex in $B \setminus (X \cup Y)$ must be adjacent to at least one vertex $\delta \in X$ in Ω, which implies that δ is adjacent to $d + 1$ vertices in Ω. This contradiction shows that $X \cup Y = B$ and consequently $|X| = v - d$. Since this holds for any maximal independent set of Ω and since Ω is G_B-vertex-transitive, we have the following claim.

Claim 3. $v - d$ divides d and Ω is a complete t-partite graph with each part containing $v - d$ vertices, where $t = v/(v - d)$.

Based on this we now prove the following claim.

Claim 4. $\Omega \cong K_v$ or $K_{v/2,v/2}$; that is, $t = v$ or 2.

Suppose to the contrary that $2 < t < v$. Denote by $B^1, B^2, \ldots, B^t$ the parts of the t-partition of Ω. Similarly, for any $D \in \mathcal{B}$, denote by $D^1, D^2, \ldots, D^t$ the parts of the t-partition of $\texttt{Simple}(\Gamma^D)$ ($\cong \Omega$). Set

$$\mathcal{B}^* := \{D^1, D^2, \ldots, D^t : D \in \mathcal{B}\}.$$

It is straightforward to verify that $\mathcal{B}^*$ is a nontrivial G-invariant partition of the vertex set of Γ and that $\mathcal{B}^*$ is a refinement of $\mathcal{B}$. For adjacent $B, C \in \mathcal{B}$ and $\{\alpha, \beta\} = B \setminus \Gamma(C)$ as above, α and β belong to different parts of Ω, and so we may assume that $\alpha \in B^1$ and $\beta \in B^2$ without loss of generality. Since $t < v$, each part of Ω has size at least two and hence we can take a vertex $\xi \in B^2 \setminus \{\beta\}$. Since $t > 2$, Ω has at least three parts and so we can take a vertex $\eta \in B^3$. Since $B \setminus \Gamma(C) = \{\alpha, \beta\}$ and $\xi, \eta \neq \alpha, \beta$, each of ξ and η has at least one neighbour in C. Let ξ be adjacent to $\gamma \in C$ and η adjacent to $\delta \in C$. Since Γ is G-symmetric, there exists an element $g \in G$ which maps (η, δ) to (ξ, γ). Thus $g \in G_{BC}$. Since $\mathcal{B}^*$ is G-invariant and g maps $\eta \in B^3$ to $\xi \in B^2$, g should map B^3 to B^2. Since every vertex in B^3 has a neighbour in C, it follows that every vertex in B^2 has a neighbour in C. However, this is a contradiction since $\beta \in B^2$ has no neighbour in C. Therefore we have proved Claim 4.

Obviously, if $\Omega \cong K_v$, then $d = v - 1$, $b = mdv/2 = m(v-1)v/2$, and moreover G_B is 2-homogeneous on B since Ω is G_B-edge-transitive by [2, Theorem 2.1].

In the case $\Omega \cong K_{v/2,v/2}$, we have $d = v/2$, $b = mdv/2 = mv^2/4$, and the G-invariant partition $\mathcal{B}^*$ above becomes $\mathcal{B}^* = \{D^1, D^2 : D \in \mathcal{B}\}$. Obviously, $\mathcal{B}^*$ is a nontrivial partition of the vertex set of Γ and is a refinement of $\mathcal{B}$. In the case where each of $\Gamma(B^1)$ and $\Gamma(B^2)$ has nonempty intersection with exactly one of C^1 and C^2, it is easy to see that $v^* = k^* + 1$, $b = b^*$, $r = r^*$ and $s = s^*$, and so case (b)(i) occurs. In the remaining case, each of $\Gamma(B^1)$ and $\Gamma(B^2)$ has nonempty intersection with both C^1 and C^2, and hence $b^* = 2b$. If further every vertex in $B^1 \setminus \{\alpha\}$ has neighbours in both C^1 and C^2, then $v^* = k^* + 1$, $r^* = 2r$ and $s^* = s/2$, and so case (b)(ii) occurs. If not every vertex in $B^1 \setminus \{\alpha\}$ has neighbours in both C^1 and C^2, then by symmetry the numbers of vertices in $B^1 \setminus \{\alpha\}$ having neighbours in C^1 and C^2 are equal. This implies that

$$k^* = (v^* - 1)/2, \quad r^* = b^*k^*/v^* = b(v-2)/v = r \quad \text{and} \quad s^* = rs/r^* = s,$$

and hence case (b)(iii) occurs. □

Example 2.4 in [2] can serve as an example for case (a) in Theorem 2 when $v = 3$. Examples for case (b)(i) when $v = 4$ can be obtained from [4, Construction 3.2]: let M be a regular map on a closed surface such that its underlying graph Σ has valency four. (A regular map is a 2-cell embedding of a connected (multi)graph on a closed surface such that its automorphism group is regular on incident vertex–edge–face triples.) For each edge $\{\sigma, \sigma'\}$ of Σ, let f and f' denote the faces of M with $\{\sigma, \sigma'\}$ as a common edge. Denote by f_σ and f'_σ the other two faces of M incident with σ and opposite to f and f' respectively, and define $f_{\sigma'}$ and $f'_{\sigma'}$ similarly. Let $\Gamma_1(M)$, $\Gamma_2(M)$, $\Gamma_3(M)$ and $\Gamma_4(M)$ be the graphs [4] with vertices the incident vertex–face pairs of M and adjacency defined as follows (where $\sim$ means adjacency): for each edge $\{\sigma, \sigma'\}$ of Σ, $(\sigma, f) \sim (\sigma', f)$ and $(\sigma, f') \sim (\sigma', f')$ in $\Gamma_1(M)$; $(\sigma, f) \sim (\sigma', f')$ and $(\sigma, f') \sim (\sigma', f)$ in $\Gamma_2(M)$; $(\sigma, f_\sigma) \sim (\sigma', f_{\sigma'})$ and $(\sigma, f'_\sigma) \sim (\sigma', f'_{\sigma'})$ in $\Gamma_3(M)$; $(\sigma, f_\sigma) \sim (\sigma', f'_{\sigma'})$ and $(\sigma, f'_\sigma) \sim (\sigma', f_{\sigma'})$ in $\Gamma_4(M)$. These graphs are G-symmetric [4, Lemma 3.3] and admit $\mathcal{B} := \{B(\sigma) : \sigma \in V(\Sigma)\}$ as a G-invariant partition, where $B(\sigma) = \{(\sigma, f) : \sigma$ incident with $f\}$, such that $k = v - 2 = 2$, $\Gamma_{\mathcal{B}} \cong \Sigma$, $\Gamma^{B(\sigma)} = K_{2,2}$ and $\Gamma[B(\sigma), B(\tau)] = 2 \cdot K_2$ for adjacent $B(\sigma), B(\tau) \in \mathcal{B}$. These four graphs fall into case (b)(i) in Theorem 2 and the G-invariant partition induced by the bipartition of $\Gamma^{B(\sigma)}$ is $\mathcal{B}^* := \{B^1(\sigma), B^2(\sigma) : \sigma \in V(\Sigma)\}$, where $B^1(\sigma) = \{(\sigma, f), (\sigma, f_\sigma)\}$ and $B^2(\sigma) = \{(\sigma, f'), (\sigma, f'_\sigma)\}$.

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