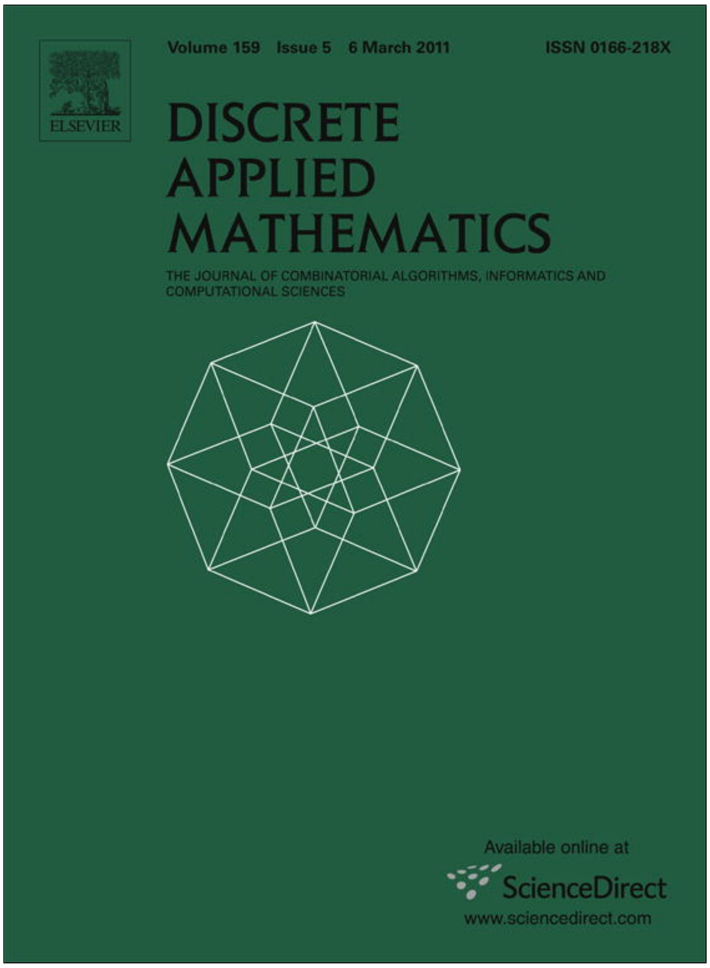
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A study of 3-arc graphs

Martin Knor [a,*], Guangjun Xu [b], Sanming Zhou [b]

[a] Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia
[b] Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

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ABSTRACT

An arc of a graph is an oriented edge and a 3-arc is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of length two. The 3-arc graph of a graph G is defined to have the arcs of G as vertices such that two arcs uv, xy are adjacent if and only if (v, u, x, y) is a 3-arc of G. In this paper, we study the independence, domination and chromatic numbers of 3-arc graphs and obtain sharp lower and upper bounds for them. We introduce a new notion of arc-coloring of a graph in studying vertex-colorings of 3-arc graphs.

1. Introduction

The 3-arc graph construction [12] has recently been proved to be useful in the classification or characterization of several families of arc-transitive graphs [6,9,12,13,18,19]. (A graph is arc-transitive if its automorphism group acts transitively on the set of oriented edges.) Although introduced initially in the context of graph symmetry, this construction is of interest for general graphs. It seems useful to investigate graph-theoretic properties of the 3-arc graph of any (not necessarily arc-transitive) connected graph. In [10] the diameter and connectivity of 3-arc graphs were studied and connections between 3-arc graphs and line and path graphs were explained. In the present paper, we study the independence, domination and chromatic numbers of 3-arc graphs.

An *arc* of a graph G is an ordered pair of adjacent vertices. For adjacent vertices u, v of G, we use uv to denote the arc from u to v, vu ($\neq uv$) the arc from v to u, and $\{u, v\}$ the edge between u and v. A 3-*arc* of G is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of G. It is allowed to have $v = y$ in a 3-arc (v, u, x, y).

Definition 1. Let G be a graph. The 3-*arc graph* of G, denoted by $X(G)$, is defined to have, for vertex set, the set of arcs of G. Two vertices corresponding to two arcs uv and xy are adjacent in $X(G)$ if and only if (v, u, x, y) is a 3-arc of G.

It follows that $X(G)$ is an undirected graph with $2\,|E(G)|$ vertices and $\sum_{\{u,v\}\in E(G)}(\deg_G(u) - 1)\,(\deg_G(v) - 1)$ edges, where $\deg_G(w)$ denotes the degree of w in G.

Let us illustrate the definition above by three simple examples. For the complete graph K_3 on three vertices, say, u, v and w, $X(K_3)$ consists of six vertices and three isolated edges joining uw to vw, uv to wv and vu to wu, respectively. For the complete bipartite graph $K_{2,3}$ with bipartition $\{\{u_1, u_2\}, \{v_1, v_2, v_3\}\}$, u_1v_1 is adjacent only to v_2u_2 and v_3u_2 in $X(K_{2,3})$, while v_1u_1 is adjacent only to u_2v_2 and u_2v_3 in $X(K_{2,3})$. By symmetry $X(G)$ consists of two 6-cycles, namely

* Corresponding author. Tel.: +421 2 59474409; fax: +421 2 52961477.
E-mail addresses: knor@math.sk (M. Knor), gx@ms.unimelb.edu.au (G. Xu), smzhou@ms.unimelb.edu.au (S. Zhou).

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$(u_1v_1, v_2u_2, u_1v_3, v_1u_2, u_1v_2, v_3u_2, u_1v_1)$ and $(u_2v_1, v_2u_1, u_2v_3, v_1u_1, u_2v_2, v_3u_1, u_2v_1)$. A necessary and sufficient condition for $X(G)$ to be connected was given in [10, Theorem 2]. From this condition, the smallest graph G such that $X(G)$ is connected is the complete graph on four vertices with one edge removed. Denote by v_1, v_2, v_3 and v_4 the vertices of this graph and assume the edge $\{v_3, v_4\}$ is removed. Then $X(G)$ consists of a 10-cycle $(v_1v_3, v_4v_2, v_1v_2, v_3v_2, v_1v_4, v_2v_4, v_3v_1, v_2v_1, v_4v_1, v_2v_3, v_1v_3)$ together with two chords $\{v_1v_3, v_2v_4\}$ and $\{v_1v_4, v_2v_3\}$.

From [10, Theorem 2], $X(G)$ is always connected if G is connected with *minimum degree* $\delta(G) \geq 3$. In [10, Theorem 3] it was proved further that, if the connectivity $\kappa(G) \geq 3$, then

$$\kappa(X(G)) \geq (\kappa(G) - 1)^2$$

and this bound is best possible. Regarding the diameter, it was proved in [10, Theorem 4] that, if G is connected with $\delta(G) \geq 3$, then

$$\mathrm{diam}(G) \leq \mathrm{diam}(X(G)) \leq \mathrm{diam}(G) + 2$$

and both bounds are attainable.

In this paper, we focus on independence, domination and vertex-coloring in 3-arc graphs. In the next section, we give a structural result (Theorem 2) on maximum independent sets of $X(G)$ when $\delta(G) \geq 3$. We also prove that the ratio of the independence number of $X(G)$ to that of G is between d and $d + 1$ for any connected d-regular graph G with $d \geq 2$ (Theorem 5), and that the independence number of $X(G)$ for any bipartite graph G with $\delta(G) \geq 2$ is equal to $|E(G)|$ (Theorem 6). In Section 3, for any graph G with $\delta(G) \geq 2$, we establish sharp lower and upper bounds for the domination number of $X(G)$ and we characterize the extremal graphs (Theorem 7). Further, we give an upper bound for the domination number of $X(G)$ in terms of the 2-domination number of G (Theorem 8). We also give a lower bound (Theorem 10) for the domination number of $X(G)$ in terms of the order and maximum degree of G and compare it with a well-known upper bound for domination number when G is regular (Corollary 11).

In Section 4, we study the chromatic number of 3-arc graphs. In doing so we introduce a new notion of arc-coloring of a graph which is different from the existing arc-coloring models [3,8,14,16]. In this new notion, we color the arcs of a graph G in such a way that two arcs uv and xy with $v \neq x$ and $y \neq u$, whose tails are joined by an edge in G, use distinct colors. The minimum number of colors required by such a coloring, $\chi'_3(G)$, is exactly the chromatic number of $X(G)$. We give sharp lower and upper bounds on $\chi'_3(G)$ in terms of $\chi(G)$ (Theorem 15), and prove that the problem of deciding whether $\chi'_3(G) \leq 3$ is NP-complete (Theorem 16). We finish the paper by a few remarks and open problems.

The reader is referred to [17] for notation and terminology undefined in the paper.

2. Independence in 3-arc graphs

An *independent set* of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The *independence number* of G, $\alpha(G)$, is the cardinality of a largest independent set of G.

If $\delta(G) = 1$, then the set of all arcs of G may form an independent set of $X(G)$, as exemplified by the star $K_{1,n}$. We thus consider graphs G with $\delta(G) \geq 2$. To facilitate presentation we introduce the following definition.

Definition 2. A set S of vertices of $X(G)$ is said to be *good* if there exists a partition of $V(G)$ into (not necessarily non-empty) subsets V_1, V_2, V_3 such that

(a) V_1 is an independent set of G, and $vu \in S$ for any $v \in V_1$ and $u \in N(v)$;
(b) V_2 is an independent set of G, any $v \in V_2$ has a unique neighbour u in V_1, and moreover u is the unique neighbour of v such that $vu \in S$, and
(c) $vu \notin S$ for any $v \in V_3$ and $u \in N(v)$.

In case of possible confusion, we use $\{V_1^S, V_2^S, V_3^S\}$ in place of $\{V_1, V_2, V_3\}$ to emphasize dependence of these subsets on S. Observe that a good set $S \subseteq V(X(G))$ is always an independent set of $X(G)$. A *good maximum independent set* is a maximum independent set which is good.

Lemma 1. *Let G be a graph with $\delta(G) \geq 2$. Then $X(G)$ has at least one good maximum independent set.*

Proof. Choose S to be a maximum independent set of $X(G)$ (i.e. $|S| = \alpha(X(G))$) such that $\{v \in V(G) : vu \in S \text{ for all } u \in N(v)\}$ has maximum cardinality.

We first prove that, for any $v \in V(G)$, if there are distinct $u_1, u_2 \in N(v)$ such that $vu_1, vu_2 \in S$, then $vu_3 \in S$ for any $u_3 \in N(v)$. Suppose otherwise. Then there exists $xy \in S$ such that $\{vu_3, xy\} \in E(X(G))$, so that $x \in N(v)$ and $y \in N(x) - \{v\}$. One of u_1 and u_2 is not identical to x. Without loss of generality, assume that $x \neq u_1$. Then $\{vu_1, xy\} \in E(X(G))$ (regardless of whether $x = u_2$ or not), which is a contradiction. Hence we have proved that, for any $v \in V(G)$, either $vu \in S$ for any $u \in N(v)$, or $vu \in S$ for a unique $u \in N(v)$, or $vu \notin S$ for any $u \in N(v)$. We denote the subsets of such vertices v by V_1, V_2, V_3, respectively. Then $\{V_1, V_2, V_3\}$ is a partition of $V(G)$.

Suppose that V_1 is not an independent set of G. Then there are $v_1, v_2 \in V_1$ such that $\{v_1, v_2\} \in E(G)$. Since $\delta(G) \geq 2$, there exist $x \in N(v_1) - \{v_2\}$ and $y \in N(v_2) - \{v_1\}$ such that $\{v_1, x\}, \{v_2, y\} \in E(G)$ and hence $v_1x, v_2y \in S$ by the definition of V_1. Hence $\{v_1x, v_2y\} \in E(X(G))$, which is a contradiction. Thus V_1 must be an independent set of G.

It remains to verify the first two statements in (b). Suppose $v \in V_2$ and let u be the unique neighbour of v such that $vu \in S$. Since $vu \in S$, for each $x \in N(v) - \{u\}$ and any $y \in N(x) - \{v\}$, we have $xy \notin S$. Thus, there exists $z \in N(u) - \{v\}$ such that $uz \in S$, for otherwise vx can be added to S to form a larger independent set, which violates the maximality of S. Now we show that $u \in V_1$. Suppose that uz is the unique arc starting at u and belonging to S. Set $S' = (S - \{uz\}) \cup \{vx : x \in N(v)\}$. Then S' is an independent set of $X(G)$ and $|S'| \geq |S|$. If $\deg(v) > 2$, then $|S'| > |S|$, which contradicts the maximality of S. Hence $\deg(v) = 2$ and $|S'| = |S|$. However, $|\{w : wy \in S' \text{ for every } y \in N(w)\}| > |\{w : wy \in S \text{ for every } y \in N(w)\}|$, which contradicts the choice of S. Thus there are at least two arcs starting from u which belong to S and so $u \in V_1$. So we have proved that the unique neighbour u of v such that $vu \in S$ must be in V_1. If there exists $x \in N(v) - \{u\}$ such that $x \in V_1$, then there exists $y \in N(x) - \{v\}$ as $\delta(G) \geq 2$. Since $x \in V_1$, we have $xy \in S$ and $\{xy, vu\} \in E(X(G))$, a contradiction. Therefore, u is the unique neighbour of v in V_1.

Finally, for distinct $v_1, v_2 \in V_2$, there is a unique $u_i \in N(v_i)$, $i = 1, 2$, such that $v_i u_i \in S$. Moreover, $u_1, u_2 \in V_1$ from the proof above. Thus, v_1 and v_2 cannot be adjacent in G, for otherwise $\{v_1u_1, v_2u_2\} \in E(X(G))$, a contradiction. Hence V_2 is an independent set of G. □

In the proof above the maximality of $|\{v \in V(G) : vu \in S \text{ for all } u \in N(v)\}|$ was used only when G contains a degree-two vertex. Thus, in the case when $\delta(G) \geq 3$, the proof of Lemma 1 gives the following result.

Theorem 2. *Let G be a graph with $\delta(G) \geq 3$. Then all maximum independent sets of $X(G)$ are good.*

The following lemma strengthens Lemma 1 and it will be used in subsequent discussion.

Lemma 3. *Let G be a graph with $\delta(G) \geq 2$. Then there exists a good maximum independent set S of $X(G)$ such that V_1^S is a maximal independent set of G.*

Proof. We start with a good maximum independent set S of $X(G)$ (whose existence is guaranteed by Lemma 1). Suppose that V_1^S is not a maximal independent set of G. Then there exists $w \in V(G)$ such that $V_1^S \cup \{w\}$ is an independent set of G. Since no neighbour of w is in V_1^S, we have $w \in V_3^S$. Moreover, all the neighbours of w are in $V_2^S \cup V_3^S$. Denote $T = (S - \{ux : u \in N(w), x \in N(u)\}) \cup \{wu : u \in N(w)\}$. Since $\delta(G) \geq 2$ and S is good, using (a)–(c) in Definition 2 one can see that T is an independent set of $X(G)$ such that $|T| \geq |S|$, and the equality occurs if and only if $N(w) \cap V_3^S = \emptyset$. Since S is a maximum independent set of $X(G)$, we have $|T| = |S|$ and hence $N(w) \cap V_3^S = \emptyset$, which implies $N(w) \subseteq V_2^S$. One can prove that T is a good maximum independent set of $X(G)$ with $V_1^T = V_1^S \cup \{w\}$, $V_2^T = V_2^S - N(w)$ and $V_3^T = (V_3^S - \{w\}) \cup N(w)$. If V_1^T is a maximal independent set of G, we are done; otherwise we repeat this procedure. Since G is finite, eventually we obtain a good maximum independent set R of $X(G)$ such that V_1^R is maximal. □

The word 'maximal' in Lemma 3 cannot be replaced by 'maximum' in general. For example, let $G = \bar{K}_3 + C_{2t}$ $(t \geq 4)$ be the join of three isolated vertices $\bar{K}_3$ and the cycle C_{2t} of length $2t$. Take S to be a good set of $X(G)$ such that V_1^S consists of the three vertices of $\bar{K}_3$. Then $V_2^S = \emptyset$, $|S| = 6t$ and S is an independent set of $X(G)$. However, V_1^S is not a maximum independent set of G since $\alpha(G) = t$. On the other hand, consider a good set T of $X(G)$ such that V_1^T is a maximum independent set of G. In such a case V_1^T consists of every second vertex of C_{2t} and V_2^T is empty, which gives $|T| = 5t$. Since $|S| > |T|$, there is not a good maximum independent set Q in $X(G)$ such that V_1^Q is a maximum independent set in G.

Since every good set is independent, the following formula is an immediate consequence of Lemma 3, where W plays the role of V_1^S and $\alpha(G_W) = |V_2^S|$.

Theorem 4. *Let G be a graph with $\delta(G) \geq 2$. Then*

$$\alpha(X(G)) = \max_{W} \left\{ \alpha(G_W) + \sum_{v \in W} \deg_G(v) \right\},$$

where the maximum is taken over all maximal independent sets W of G, and G_W is the subgraph of G induced by those vertices which have exactly one neighbour in W.

Theorem 4 can be used to find $\alpha(X(G))$ for some graphs G with $\delta(G) \geq 2$. Consider a cycle of length n, C_n, and let W be a maximal independent set of C_n. Then the graph induced by $V(C_n) - W$ consists of isolated vertices and edges. Therefore, if $|W| = k$, then G_W consists of $n - 2k$ isolated edges. Consequently, $\alpha(G_W) = n - 2k$ and $\alpha(X(C_n)) = (n - 2k) + 2k = n$. Another maximum independent set of $X(C_n)$ can be obtained by choosing all arcs of C_n in accordance with a fixed orientation of C_n. One can check that this maximum independent set is not good. This demonstrates that if $\delta(G) = 2$ then not every maximum independent set of $X(G)$ is good. In other words, the condition $\delta(G) \geq 3$ in Theorem 2 cannot be removed.

Next consider the wheel $G = W_n$ on $n + 1$ vertices. Let W be a maximal independent set of G. If W consists of the central vertex, then $\alpha(G_W) = \lfloor \frac{n}{2} \rfloor$ and so $\alpha(G_W) + \sum_{v \in W} \deg_G(v) = \lfloor \frac{3n}{2} \rfloor$. If the central vertex of G is not in W, then $k = |W| \leq \lfloor \frac{n}{2} \rfloor$ and $\alpha(G_W) = n - 2k$. Hence $\alpha(G_W) + \sum_{v \in W} \deg_G(v) = (n - 2k) + 3k \leq \lfloor \frac{3n}{2} \rfloor$. Therefore, $\alpha(X(W_n)) = \lfloor \frac{3n}{2} \rfloor$ by Theorem 4.

Theorem 4 implies the following bounds for regular graphs.

Theorem 5. *Let G be a connected d-regular graph with $d \geq 2$. Then*

$$d \leq \frac{\alpha(X(G))}{\alpha(G)} \leq d + 1. \tag{1}$$

Moreover, both bounds are attainable.

Proof. Choose a maximum independent set W of G. Then $\alpha(X(G)) \geq \sum_{v \in W} \deg_G(v) = d\,\alpha(G)$ by Theorem 4. On the other hand, by Theorem 4 there exists a maximal independent set W^* of G such that $\alpha(X(G)) = \alpha(G_{W^*}) + \sum_{v \in W^*} \deg_G(v)$. Since $\alpha(G_{W^*}) \leq \alpha(G)$, $|W^*| \leq \alpha(G)$ and G is d-regular, it follows that $\alpha(X(G)) \leq (d+1)\alpha(G)$.

Denote by $v_1, v_2, \ldots, v_n$ the vertices of a complete graph K_n. Then $S = \{v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_1\}$ is a good independent set of size n in $X(K_n)$, so that $\alpha(X(K_n)) \geq n$. Since K_n is $(n-1)$-regular and $\alpha(K_n) = 1$, we have $\alpha(X(K_n)) \leq n$ by (1). Thus $\alpha(X(K_n)) = n$, which achieves the upper bound in (1).

The lower bound in (1) is achieved by the complete bipartite graph $K_{n,n}$ because $\alpha(X(K_{n,n})) = |E(K_{n,n})| = n^2 = n\,\alpha(K_{n,n})$ by Theorem 6 below. $\square$

In the proof of Theorem 5, we demonstrated that the upper bound is achieved by complete graphs, which satisfy $\alpha(K_n) = 1$. However, this bound is achieved also by graphs G for which $\alpha(G) > 1$. Let $G_t = K_{2t} \square C_t$ be the Cartesian product of K_{2t} and C_t. That is, G_t consists of t vertex-disjoint copies of K_{2t} with vertices $\{v_0^i, v_1^i, \ldots, v_{2t-1}^i\}$ in the ith copy, $0 \leq i \leq t-1$, together with $2t^2$ edges $\{v_j^i, v_j^{i+1}\}$, $0 \leq j \leq 2t-1$, $0 \leq i \leq t-1$, where superscripts are taken modulo t. Obviously, G_t is a $(2t+1)$-regular graph. Since any independent set of G_t contains at most one vertex from each copy of K_{2t} and $V_1 = \{v_0^0, v_1^1, \ldots, v_{t-1}^{t-1}\}$ is an independent set of G_t, we have $\alpha(G_t) = t$. Now we take S to be the set of arcs of G_t starting from V_1 and those from $\{v_t^0, v_{t+1}^1, \ldots, v_{2t-1}^{t-1}\}$ to V_1. Then S is a good independent set of $X(G)$ with cardinality $|S| = t(2t+1) + t = (2t+2)t$, which is the upper bound in (1).

Using Lemma 3 we are able to determine $\alpha(X(G))$ when G is a bipartite graph.

Theorem 6. *Let G be a bipartite graph with $\delta(G) \geq 2$. Then*

$$\alpha(X(G)) = |E(G)|.$$

Proof. Let $\{U, W\}$ be the bipartition of G. Then $S = \{uv : u \in U \text{ and } v \in N(u)\}$ is a good independent set of $X(G)$ with size $|E(G)|$. It remains to prove that a maximum independent set of $X(G)$ has cardinality at most $|E(G)|$.

Let S be a good maximum independent set of $X(G)$ guaranteed by Lemma 3, and let $\{V_1^S, V_2^S, V_3^S\}$ be the corresponding partition of $V(G)$. Denote $U_i = U \cap V_i^S$ and $W_i = W \cap V_i^S$, $i = 1, 2, 3$. Since V_1^S is an independent set of G, there is no edge of G with one end-vertex in U_1 and the other end-vertex in W_1. Similarly, since V_2^S is an independent set, there is no edge between U_2 and W_2. By the definition of $\{V_1^S, V_2^S, V_3^S\}$, each vertex in U_2 is adjacent to a unique vertex in W_1, and each vertex in W_2 is adjacent to a unique vertex in U_1. Thus, since $\delta(G) \geq 2$, each vertex in U_2 (W_2, respectively) is adjacent to at least one vertex in W_3 (U_3, respectively). Hence $|(U_2, W_3)| \geq |(U_2, W_1)|$ and $|(U_3, W_2)| \geq |(U_1, W_2)|$, where (X, Y) is the set of edges of G between $X \subseteq U$ and $Y \subseteq W$. Therefore, $|S| = |(U_1, W)| + |(U, W_1)| + |(U_2, W_1)| + |(U_1, W_2)| \leq |(U_1, W)| + |(U, W_1)| + |(U_2, W_3)| + |(U_3, W_2)| \leq |E(G)|$. $\square$

The conclusion in Theorem 6 may not be true when $\delta(G) = 1$ as exemplified by $\alpha(X(K_{1,n})) = 2n$. On the other hand, if G is a bipartite graph with $\delta(G) \geq 3$, then from the proof of Theorem 6, for any good maximum independent set S of $X(G)$ we have $V_2^S = \emptyset$ and V_3^S is an independent set of G. Therefore, if in addition G is connected, then the bipartition of G must be $\{U_1, W_3\}$ or $\{U_3, W_1\}$.

3. Domination in 3-arc graphs

A *dominating set* of a graph G is a subset S of $V(G)$ such that $V(G) - S \subseteq \cup_{u \in S} N(u)$, where $N(u)$ is the neighbourhood of u in G. The *domination number* of G, $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Theorem 7. *Let G be a connected graph of order $n \geq 4$ and $\delta(G) \geq 2$. Then*

$$3 \leq \gamma(X(G)) \leq n \tag{2}$$

and both bounds are sharp. Moreover, $\gamma(X(G)) = n$ if and only if G is an n-cycle, and $\gamma(X(G)) = 3$ if and only if G contains a 3-cycle C_3 such that $|N(u) \cap V(C_3)| \geq 2$ for every $u \in V(G)$. Furthermore, for each integer k with $3 \leq k \leq n$, there exists a graph G with $\delta(G) \geq 2$ and order n such that $\gamma(X(G)) = k$.

Proof. Since uv does not dominate vu, $X(G)$ does not have any dominating set of cardinality one. Suppose that there exists a dominating set S of $X(G)$ with $|S| = 2$, say, $S = \{uv, wz\}$. Then $u \neq w$ for otherwise S does not dominate vu. Further, $\{u, w\} \in E(G)$ since otherwise S does not dominate uy for $y \in N(u) - \{v\}$. If $v \neq w$, then S does not dominate uw; if $z \neq u$ then S does not dominate wu, and if $v = w$ and $z = u$ then S does not dominate uy for $y \in N(u) - \{v\}$. Hence $\gamma(X(G)) \geq 3$.

Now we prove $\gamma(X(G)) \le n$. Since $\delta(G) \ge 2$, G contains at least one cycle. Let $C = (v_1, v_2, \dots, v_h, v_1)$ be a shortest induced cycle in G, where $h \ge 3$. Expand the path $v_1, v_2, \dots, v_h$ to a spanning tree T of G. Let U denote the unicyclic graph obtained by adding the edge $\{v_h, v_1\}$ to T. Let $S_1 = \{v_1v_2, v_2v_3, \dots, v_{h-1}v_h, v_hv_1\}$, and let S_2 consist of all arcs xy such that $\{x, y\} \in E(U) - E(C)$ and x is farther than y from C in U (that is, $d_U(x, v_i) > d_U(y, v_i)$ for one and hence all i with $1 \le i \le h$). Denote $S = S_1 \cup S_2$. Then $|S| = n$ since U is unicyclic. Moreover, by the definition of S_1 and S_2, the n arcs of S start at n different vertices of G. For each $u \in V(G)$, let $\bar{u}$ denote the neighbour of u such that $u\bar{u} \in S$. For any arc xy of $X(G)$, if $y = \bar{x}$, then $xy = x\bar{x} \in S$; if $y \ne \bar{x}$, then xy is dominated by $z\bar{z}$, where $z = \bar{x}$ (note that $\bar{z} \ne x$ by the choice of S). Hence S is a dominating set of $X(G)$ and $\gamma(X(G)) \le n$.

If G is a cycle, then since S_1 above is a dominating set of $X(G)$, we have $\gamma(X(G)) \le n$. However, each vertex of $X(G)$ dominates at most one vertex of S_1, so that $\gamma(X(G)) \ge |S_1| = n$. Thus $\gamma(X(G)) = n$ in this case.

Suppose that G is not a cycle, so that $V(U) - V(C) \ne \emptyset$. Let $w \in V(U) - V(C)$ such that w has degree one in U. Let $S' = S - \{w\bar{w}\}$. For any arc xy of G, we have $\bar{x} \ne w$ for otherwise both x and $\bar{w}$ are neighbours of w in U. Thus, if $xy \notin S$, then xy is dominated by $z\bar{z} \in S'$, where $z = \bar{x}$. Since $\delta(G) \ge 2$, there exists a neighbour u of w in G other than $\bar{w}$. Then $w\bar{w}$ is dominated by $u\bar{u}$. Therefore, S' is a dominating set of $X(G)$, which implies $\gamma(X(G)) \le |S'| < n$.

Next we characterize graphs attaining the lower bound.

Suppose first that G contains a 3-cycle $C_3 = (u_1, u_2, u_3, u_1)$ such that $|N(u) \cap V(C_3)| \ge 2$ for each $u \in V(G)$. Let $S = \{u_1u_2, u_2u_3, u_3u_1\}$. Consider any arc xy of G not in S. If $x \in V(C_3)$, say, $x = u_1$, then u_2u_3 dominates xy. If $x \notin V(C_3)$, without loss of generality, we may assume $u_1, u_2 \in N(x)$. If $y \ne u_1$, then xy is dominated by u_1u_2; if $y = u_1$, then xy is dominated by u_2u_3. Hence S is a dominating set of $X(G)$ and $\gamma(X(G)) = 3$.

Now suppose that $\gamma(X(G)) = 3$. Let $S = \{u_1v_1, u_2v_2, u_3v_3\}$ be a dominating set of $X(G)$. We first show that $|N(x) \cap \{u_1, u_2, u_3\}| \ge 2$ for any $x \in V(G) - \{u_1, u_2, u_3\}$. Since $\delta(G) \ge 2$, there is a neighbour y of x. Since S dominates xy, we have $|N(x) \cap \{u_1, u_2, u_3\}| \ge 1$. Without loss of generality, we may assume that $u_1 \in N(x)$. Since S dominates xu_1, we have either $\{u_2, x\} \in E(G)$ or $\{u_3, x\} \in E(G)$. Hence $|N(x) \cap \{u_1, u_2, u_3\}| \ge 2$. Consequently, $|\{u_1, u_2, u_3\}| \ge 2$. It remains to show that u_1, u_2 and u_3 form a 3-cycle in G.

We first prove that u_1, u_2 and u_3 are pairwise distinct. Suppose to the contrary that two of them are the same, say, $u_1 = u_2$. Then $u_3 \ne u_1$ and there exists a neighbour z_3 of u_3 such that $z_3 \ne v_3$. Since u_3z_3 is not dominated by u_3v_3, it must be dominated by u_1v_1 or u_2v_2 and hence $\{u_1, u_3\} \in E(G)$. We must have $u_1u_3 \in S$ for otherwise it is not dominated by any arc in S. Thus, u_1u_3 must be identical to one of u_1v_1 and u_2v_2. Without loss of generality, we may assume $u_1u_3 = u_1v_1$, so that $u_3 = v_1$. We must have $u_3u_1 \in S$ for otherwise none of the arcs in S dominates $u_3u_1 \in S$, a contradiction. Hence $u_3u_1 = u_3v_3$ and therefore $v_3 = u_1$. Since $n \ge 4$, there exists a vertex $x \in V(G) - \{u_1, u_3, v_2\}$. Since $u_1 = u_2$ and $|N(x) \cap \{u_1, u_2, u_3\}| \ge 2$ by the previous paragraph, x is adjacent to both u_1 and u_3. However, u_1x is not dominated by any arc in S, which is a contradiction. So we have proved that u_1, u_2 and u_3 are pairwise distinct.

Now we prove that u_1, u_2 and u_3 are pairwise adjacent. Suppose otherwise, say, $\{u_1, u_2\} \notin E(G)$. Since $\delta(G) \ge 2$, there is a neighbour z_1 of u_1 such that $z_1 \ne v_1$. Since $\{u_1, u_2\} \notin E(G)$, neither u_1v_1 nor u_2v_2 dominates u_1z_1. Hence u_1z_1 is dominated by u_3v_3 and so $\{u_1, u_3\} \in E(G)$. Similarly, there exists a neighbour z_2 of u_2 such that $z_2 \ne v_2$. An analogous argument shows that $\{u_2, u_3\} \in E(G)$. Note that neither u_2v_2 nor u_3v_3 dominates u_1u_3. Thus, if $u_1u_3 \ne u_1v_1$, then none of the arcs in S dominates u_1u_3, which is a contradiction. Hence $u_1v_1 = u_1u_3$ and so $v_1 = u_3$. Similarly, $v_2 = u_3$. Now if $v_3 \ne u_1$ then S does not dominate u_3u_1, while if $v_3 \ne u_2$ then S does not dominate u_3u_2. Hence S is not a dominating set of $X(G)$. This contradiction shows that u_1, u_2 and u_3 form a 3-cycle in G.

Example 1 below shows that every integer between 3 and n can be taken by $\gamma(X(G))$ for some graph G with $\delta(G) \ge 2$ and order n. □

Example 1. For any integers k and n with $3 \le k \le n$, there exists a graph with order n and $\delta(G) \ge 2$ such that $\gamma(X(G)) = k$. In fact, let $G_{n,k}$ be the graph with vertex set $\{u_1, u_2, \dots, u_{n-k}\} \cup \{v_0, v_1, \dots, v_{k-1}\}$ and edge set $\{\{u_i, v_0\}, \{u_i, v_1\} : 1 \le i \le n-k\} \cup \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-2}, v_{k-1}\}, \{v_{k-1}, v_0\}\}$. (Note that $G_{n,n}$ is exactly the n-cycle.) Let $S = \{v_0v_1, v_1v_2, \dots, v_{k-2}v_{k-1}, v_{k-1}v_0\}$. Observe that any arc of $G_{n,k}$ can dominate at most one arc of S. Hence $\gamma(X(G)) \ge |S| = k$. On the other hand, it is easy to see that S dominates $X(G)$. Therefore, $\gamma(X(G)) = k$.

A *k-dominating set* [5] of a graph H is a subset S of $V(H)$ such that $|N(u) \cap S| \ge k$ for every $u \in V(H) - S$. The *k-domination number* of H, denoted by $\gamma_k(H)$, is the minimum cardinality of a k-dominating set of H. Note that $\gamma_1(H) = \gamma(H)$. Our next result gives an upper bound for $\gamma(X(G))$ in terms of $\gamma_2(G)$. At the time of writing, we are unable to give a sharp upper bound for $\gamma(X(G))$ in terms of $\gamma(G)$.

Algorithm 1

Input: A graph G with $\delta(G) \ge 2$ and a minimum 2-dominating set T of G.
Output: A set S of arcs of G.

Set $S := \emptyset, i := 0, U := \emptyset$. {Comment: $U \subseteq V(G)$ and $i = |U|$}
While $T - N(U) \ne \emptyset$ **do**

$i := i + 1$;
choose $u_i \in V(G) - (T \cup U)$ with $N(u_i) \cap (T - N(U)) \ne \emptyset$;
let $N(u_i) \cap (T - N(U)) = \{v_1, v_2, \dots, v_k\}$;
if $k \ge 2$, **then**

if $N(u_i) \cap (V(G) - T) \neq \emptyset$, **then**

let $S_1 = \{v_1u_i, v_2u_i, \ldots, v_ku_i, u_iw_i, w_ix_i\}$,
where $w_i \in N(u_i) \cap (V(G) - T)$ and $x_i \in N(w_i) - \{u_i\}$;
set $S := S \cup S_1$;

else if $N(u_i) \cap N(U) \cap T \neq \emptyset$, **then**

let $S_2 = \{v_1u_i, v_2u_i, \ldots, v_ku_i, u_iw_i\}$,
where $w_i \in N(u_i) \cap N(U) \cap T$;
set $S := S \cup S_2$;

else

let $S_3 = \{v_1u_i, u_iv_1, v_2u_i, u_iv_2, \ldots, v_ku_i, u_iv_k\}$;
set $S := S \cup S_3$;

end;
if $k = 1$, **then**

let $S_4 = \{v_1u_i, u_iw_i\}$, where $w_i \in N(u_i) \cap N(U) \cap T$;
set $S := S \cup S_4$;

end;
$U := U \cup \{u_i\}$;

end.

Theorem 8. *Let G be a graph with $\delta(G) \geq 2$. Then*

$$\gamma(X(G)) \leq 2\gamma_2(G) \tag{3}$$

and this bound is sharp.

Proof. Let T be a minimum 2-dominating set of G. We apply Algorithm 1 to obtain a set S of arcs of G.

In the ith iteration of the **While** loop in Algorithm 1, we have $U = \{u_1, u_2, \ldots, u_{i-1}\}$. We choose $u_i \in V(G) - (T \cup U)$ so that $N(u_i) \cap (T - N(U)) \neq \emptyset$. In other words, u_i has a neighbour in T which is not a neighbour of any u_j, $j < i$. Since T is a minimum 2-dominating set of G, every vertex $v \in T$ has at least one neighbour not in T. Therefore, after several iterations we have $T - N(U) = \emptyset$ and the algorithm terminates.

In the ith iteration of the **While** loop in Algorithm 1, we have either $|N(u_i) \cap (T - N(U))| \geq 2$, for which we obtain one of S_1, S_2 and S_3, or $|N(u_i) \cap (T - N(U))| = 1$, for which we get S_4. In the last case, since T is a 2-dominating set of G, there must exist a neighbour w_i of u_i such that $w_i \in T \cap N(U)$. In all cases we add to S at most $2k$ arcs, where k is the number of neighbours of u_i in $T - N(U)$. Hence $|S| \leq 2|T|$. Now we prove that S dominates $X(G)$ and hence complete the proof of (3).

Let xy be an arbitrary arc of G not in S. We distinguish three cases.

Case 1. $x \in T$. Denote by u_i the first vertex in U such that $x \in N(u_i)$. If $y = u_i$, then $xy \in S$. If $y \neq u_i$, then u_iw_i dominates xy in cases S_1, S_2 and S_4, while in case S_3 either u_iv_1 or u_iv_2 dominates xy.

Case 2. $x \in U$. In this case $x = u_i$ for some i. Then w_ix_i dominates xy in case S_1. Observe that $w_i \in T \cap N(U)$ in cases S_2 and S_4. Hence there exists u_j such that $j < i$ and $w_iu_j \in S$. Since $xy \neq u_iw_i$, the arc w_iu_j dominates xy in cases S_2 and S_4. Since $xy \notin S$, case S_3 is impossible.

Case 3. $x \notin T \cup U$. Since T is a 2-dominating set in G, there are at least two neighbours of x in T. Let v be a neighbour of x in T such that $v \neq y$, and let u_i be the first vertex of U such that $v \in N(u_i)$. Then vu_i dominates xy.

So far we have completed the proof of (3). Observe that $\gamma(X(C_n)) = 2\gamma_2(C_n)$ if n is even. Hence the bound in (3) is sharp. □

Corollary 9. *Let G be a bipartite graph with bipartition $\{U, W\}$ and $\delta(G) \geq 2$. Then*

$$\gamma(X(G)) \leq 2\min\{|U|, |W|\}$$

and this bound is sharp.

Proof. Since $\delta(G) \geq 2$, each part of the bipartition of G is a 2-dominating set. By Theorem 8, we have $\gamma(X(G)) \leq 2\gamma_2(G) \leq 2\min\{|U|, |W|\}$. Similar to Theorem 8, the equality is attained by even cycles. □

In the next theorem, we give a lower bound for $\gamma(X(G))$ and compare it with an upper bound derived from the following known result [1,2,15] for any graph H of order n and minimum degree δ

$$\gamma(H) \leq \frac{n}{\delta + 1}(\ln(\delta + 1) + 1). \tag{4}$$

Theorem 10. *Let G be a graph with n vertices, m edges, maximum degree Δ and minimum degree $\delta \geq 2$. Then*

$$\left\lceil \frac{2n}{\Delta} \right\rceil \leq \gamma(X(G)) \leq \frac{2m}{\delta^2 - 2\delta + 2}(\ln(\delta^2 - 2\delta + 2) + 1). \tag{5}$$

Moreover, the lower bound is sharp.

Proof. $X(G)$ has $2m$ vertices and minimum degree $\delta(X(G)) \geq (\delta - 1)^2$. Applying (4) to $X(G)$ and noting that $\frac{\ln x+1}{x}$ is a decreasing function for $x \geq 1$, we obtain the upper bound in (5) immediately.

To prove the lower bound in (5), we partition the arcs of G into n disjoint parts A_u, $u \in V(G)$, where $A_u = \{uv : v \in N(u)\}$. Let S be a dominating set of $X(G)$ with minimum cardinality. Every arc in S dominates arcs in at most Δ different parts A_u. On the other hand, it requires at least two different arcs in S to dominate all arcs in a single A_u. Counting the number of ordered pairs (a, A_u), where a is an arc in S dominating some arcs in A_u, we obtain $\Delta|S| \geq 2n$. Hence $\gamma(X(G)) = |S| \geq \frac{2n}{\Delta}$. Cycles C_n demonstrate that the lower bound is sharp. □

Theorem 10 can be used to find $\gamma(X(G))$ for some graphs G with $\delta(G) \geq 2$.

Example 2. Let P_k be the prism on $2k$ vertices with $V(P_k) = \{u_0, u_1, \ldots, u_{k-1}, v_0, v_1, \ldots, v_{k-1}\}$ and $E(P_k) = \{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\}, \{u_i, v_i\} : 0 \leq i \leq k-1\}$, where the addition in subscripts is modulo k. Let $t = \lfloor \frac{k}{3} \rfloor$.

If $k \equiv 0 \bmod 3$, then $k = 3t$ and $\gamma(X(P_k)) \geq 4t$ by Theorem 10. On the other hand, $S = \{u_iu_{i+1}, u_{i+1}v_{i+1}, v_{i+1}v_i, v_iu_i : i = 0, 3, \ldots, 3(t-1)\}$ is a dominating set of $X(P_k)$ with cardinality $4t$. Hence $\gamma(X(P_k)) = 4t$.

If $k \equiv 1 \bmod 3$, then $k = 3t+1$ and $\gamma(X(P_k)) \geq 4t + \frac{4}{3}$ by Theorem 10. One can check that $S = \{u_iu_{i+1}, u_{i+1}v_{i+1}, v_{i+1}v_i, v_iu_i : i = 4, 7, \ldots, 3(t-1)+1\} \cup \{u_0u_1, u_1u_2, u_2v_2, v_2v_1, v_1v_0, v_0u_0\}$ is a dominating set of $X(P_k)$ with cardinality $4t + 2$. Hence $\gamma(X(P_k)) = 4t + 2$.

In the two cases above the lower bound in (5) is attained.

In the case where $k \equiv 2 \bmod 3$, by Theorem 10 and an analogous argument, we obtain $\lceil \frac{2n}{\Delta} \rceil = 4t + 3 \leq \gamma(X(P_k)) \leq 4t + 4$, where the lower bound is attained when, say, $t = 1$.

For a d-regular graph G with $d \geq 3$, the upper bound in (5) is strictly less than $\frac{n}{d-2}(\ln(d^2 - 2d + 2) + 1)$. Thus, by Theorem 10 and Example 2, we have the following corollary.

Corollary 11. *Let G be a d-regular graph of order n, where $d \geq 3$. Then*

$$\left\lceil \frac{2n}{d} \right\rceil \leq \gamma(X(G)) < \frac{n}{d-2}(\ln(d^2 - 2d + 2) + 1)$$

and the lower bound is sharp.

For sufficiently large d, this upper bound is better than the one in Theorem 7.

4. Coloring 3-arc graphs

We observe that a proper vertex-coloring of $X(G)$ is exactly a coloring of arcs of G, such that any two arcs uv and xy with $v \neq x$ and $y \neq u$, whose tails u and x are joined by an edge in G, receive different colors. (A vertex-coloring of a graph or directed graph is called *proper* if adjacent vertices receive different colors.) The latter, called a 3-*arc coloring* of G, is a new notion of arc-coloring for graphs that is different from the existing arc-coloring models [3,8,14,16]. Define $\chi_3'(G)$ to be the minimum number of colors needed by a 3-arc coloring of G and call it the 3-*arc chromatic index* of G. Equivalently, $\chi_3'(G)$ is defined as the chromatic number of $X(G)$.

The notion of 3-arc coloring can be extended to directed graphs in an obvious way by requiring that any two arcs uv and xy with $v \neq x$ and $y \neq u$, whose tails are joined by an arc (in either direction), receive distinct colors. So we can speak of the 3-arc chromatic index $\chi_3'(D)$ of a directed graph D, though we will mainly discuss the undirected case. Of course $\chi_3'(G)$ is equal to the 3-arc chromatic index of the directed graph obtained from G by replacing each edge by two arcs of opposite directions.

A *tournament* is a digraph T, such that for every $u, v \in V(T)$, $u \neq v$, we have either $uv \in E(T)$ or $vu \in E(T)$. The tournament is *transitive* if $uv, vw \in E(T)$ implies $uw \in E(T)$ for every triple $u, v, w \in V(T)$. A *Halin graph* is a planar graph $H = T \cup C$ whose edge set can be partitioned into a tree T with no vertex of degree two and a cycle C whose vertices are exactly the degree-one vertices of T.

Theorem 12. *The following hold:*

(a) *if T_n is a transitive tournament on n vertices, then $\chi_3'(T_n) = n - 1$;*
(b) $\chi_3'(K_n) = n - 1$;
(c) *for a connected graph G, $\chi_3'(G) = 1$ if and only if G is a star;*
(d) *for a connected graph G, $\chi_3'(G) = 2$ if and only if G is not a star and the subgraph of G induced by the vertices of degree at least three is bipartite;*
(e) *if H is a Halin graph, then $\chi_3'(H) = 2$ if H is bipartite and $\chi_3'(H) = 3$ otherwise.*

Proof. (a) Since T_n is acyclic, each vertex v_i of T_n can be assigned an integer n_i such that $n_i < n_j$ for each arc v_iv_j of T_n. Color each arc v_iv_j of T_n by n_i. Since no arc emanates from v_n, this is a 3-arc coloring of T_n using $n-1$ colors. Hence $\chi'_3(T_n) \le n-1$. On the other hand, we have $\chi'_3(T_n) \ge n-1$ because $v_1v_n, v_2v_n, \ldots, v_{n-1}v_n$ require pairwise distinct colors in any 3-arc coloring of T_n.

(b) Since T_n is an orientation of K_n, we have $\chi'_3(K_n) \ge \chi'_3(T_n) = n-1$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Color all arcs emanating from v_i by i, and color v_nv_i by i, for $i = 1, 2, \ldots, n-1$. In this way we get a 3-arc coloring of K_n, so that $\chi'_3(K_n) = n-1$.

(c) $\chi'_3(G) = 1$ if and only if $X(G)$ is a graph without edges. Since $X(G)$ is edgeless if and only if G has neither 3-cycles nor paths of length 3, $\chi'_3(G) = 1$ if and only if G is a star.

(d) Suppose that G is not a star. Denote by G_0 the subgraph of G induced by all vertices of degree at least three. Since $\chi'_3(G) = 2$ if and only if $X(G)$ is bipartite, it suffices to prove that $X(G)$ has an odd cycle if and only if G_0 has an odd cycle.

Suppose that G_0 has an odd cycle $(u_0, u_1, \ldots, u_{k-1}, u_0)$. Since the degree of u_i is at least three, $0 \le i < k$, there is a vertex v_i in G such that $v_i \ne u_{i-1}$ and $v_i \ne u_{i+1}$, the subscripts being modulo k. Then $(u_0v_0, u_1v_1, \ldots, u_{k-1}v_{k-1}, u_0v_0)$ is an odd cycle in $X(G)$.

Now suppose that $X(G)$ contains odd cycles. Let $C = (u_0v_0, u_1v_1, \ldots, u_{k-1}v_{k-1}, u_0v_0)$ be a shortest odd cycle in $X(G)$. We prove that $u_0, u_1, \ldots, u_{k-1} \in V(G_0)$. The vertex u_i is adjacent to u_{i-1}, u_{i+1} and v_i, $0 \le i < k$, indices being modulo k. Suppose that there is a subscript j such that $u_{j-1} = u_{j+1}$. If $u_{j-2} = u_j$ then $(u_0v_0, \ldots, u_{j-2}v_{j-2}, u_{j+1}v_{j+1}, \ldots, u_0v_0)$ is an odd cycle of length $k-2$ in $X(G)$, which contradicts the choice of C. Thus, $u_{j-2} \ne u_j$ and analogously we get $u_{j+2} \ne u_j$. But then $(u_0v_0, \ldots, u_{j-2}v_{j-2}, u_{j-1}u_j, u_{j+2}v_{j+2}, \ldots, u_0v_0)$ is a cycle of length $k-2$ in $X(G)$, which contradicts the choice of C. Therefore $u_{i-1} \ne u_{i+1}$ for all i, $0 \le i < k$. As v_i is distinct from both u_{i-1} and u_{i+1}, the degree of u_i is at least 3 in G. Therefore $(u_0, u_1, \ldots, u_{k-1}, u_0)$ is a closed walk of odd length in G_0, so that G_0 has an odd cycle.

(e) Let $H = T \cup C$. Denote $C = (v_1, v_2, \ldots, v_t, v_1)$ and let f be a proper vertex-coloring of T using colors 1 and 2. Define $f' : V(H) \to \{1, 2, 3\}$ such that $f'(x) = 3$ if $x = v_j$ for an odd j and $f'(x) = f(x)$ otherwise. If t is even, then color each arc xy of H by $f'(x)$. One can check that this is a 3-arc coloring of H. Assume that t is odd. Since H is a Halin graph, there is a unique neighbour w of v_t not on C. Color each arc xy with $x \ne v_t$ by $f'(x)$ and color v_tv_1 by 3. If $f'(v_{t-1}) = f'(w) = 1$, then color v_tv_{t-1} and v_tw by 2; if $f'(v_{t-1}) = f'(w) = 2$, then color v_tv_{t-1} and v_tw by 1, and if $f'(v_{t-1}) \ne f'(w)$, then color v_tv_{t-1} by $f'(v_{t-1})$ and v_tw by $f'(w)$. It can be verified that this is a 3-arc coloring of H. Hence $\chi'_3(H) \le 3$. Since each vertex of H has degree at least three, by (d), $\chi'_3(H) = 2$ if H is bipartite and $\chi'_3(H) = 3$ otherwise. □

A major result in this section is Theorem 15 below, which gives sharp lower and upper bounds on $\chi'_3(G)$ in terms of the chromatic number of G. To prove Theorem 15 we first discuss directed graphs with 3-arc chromatic index one.

Lemma 13. *Let D_n be an orientation of a cycle of length $n \ge 3$. Then $\chi'_3(D_n) = 1$ if and only if either D_n is a directed cycle, or n is even and any two consecutive arcs of D_n have opposite directions.*

Proof. The sufficiency is easy to see, so we prove the necessity only.

Hence, suppose that $\chi'_3(D_n) = 1$. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of D_n in a cyclic order. Suppose that there are two consecutive arcs of D_n having the same direction. Without loss of generality, we may assume that v_0v_1 and v_1v_2 are arcs of D_n. Since $\chi'_3(D_n) = 1$, $v_{n-1}v_0$ is an arc of D_n. Similarly, one can show successively that $v_{n-2}v_{n-1}, v_{n-3}v_{n-2}, \ldots, v_2v_3$ are arcs of D_n. Thus, D_n is a directed cycle.

Now suppose that no two consecutive arcs of D_n have the same direction. Then n is even and any two consecutive arcs of D_n have opposite direction. □

A *semi-cycle* (*semi-path*, respectively) in a directed graph is a directed subgraph whose underlying graph is a cycle (path, respectively). A semi-cycle is *odd* if its length is odd. A directed graph is *weakly connected* if its underlying graph is connected.

Lemma 14. *Let D be a weakly connected directed graph with $\chi'_3(D) = 1$. Then D contains at most one odd semi-cycle, and moreover such a semi-cycle must be a directed cycle.*

Proof. Since $\chi'_3(D) = 1$, every odd semi-cycle in D should have 3-arc chromatic index equal to one and hence is a directed cycle by Lemma 13.

Suppose that D contains two distinct odd semi-cycles, say, D_1 and D_2. Then D_1 and D_2 are directed cycles in D as shown above. If D_1 and D_2 have common vertices, then there exists an arc uv of D_1 such that u is the initial vertex of a directed path u, x, y of length two in D_2. So any 3-arc coloring of D assigns different colors to uv and xy, which contradicts the assumption $\chi'_3(D) = 1$. Hence D_1 and D_2 are vertex-disjoint. Since D is weakly connected, there exists a semi-path $v_1, v_2, \ldots, v_k$ in D such that $v_1 \in V(D_1)$ and $v_k \in V(D_2)$. Since D_1 is a directed cycle and v_1 is on D_1, the arc between v_1 and v_2 must be v_2v_1 and the arc between v_2 and v_3 must be v_3v_2. Based on these and the assumption $\chi'_3(D) = 1$, one can show that $v_{j+1}v_j$, $j = 1, \ldots, k-1$, are arcs of D. Since D_2 is also a directed cycle and v_k is on D_2, any 3-arc coloring of D assigns different colors to v_kv_{k-1} and an arc of D_2. This contradiction proves the result. □

Theorem 15. *Let G be a connected graph. Then*

$$\left\lceil \frac{\chi(G)+1}{3} \right\rceil \le \chi'_3(G) \le \chi(G) \tag{6}$$

and moreover both bounds are attainable.

Proof. Let G be a connected graph. For a proper vertex-coloring of G using $\chi(G)$ colors, the coloring of arcs of G such that each arc is assigned the color of its tail is a 3-arc coloring of G. Hence $\chi'_3(G) \leq \chi(G)$. The equality holds when, for example, G is an even cycle, or the graph obtained from any complete graph K_n by adding a new vertex v' for each $v \in V(K_n)$ and joining v and v' by an edge.

Now we prove the lower bound in (6). By Theorem 12(c), if $\chi'_3(G) = 1$ then G must be a star, which implies $\chi(G) = 2$ and hence $(\chi(G) + 1)/3 = \chi'_3(G)$. In the following, we assume $k = \chi'_3(G) \geq 2$.

Denote by D the directed graph obtained from G by replacing every edge by a pair of arcs of opposite directions. Let f be a 3-arc coloring of G (and hence of D) using colors $1, 2, \ldots, k$. Denote by D_i the directed subgraph of D induced by those arcs of D which are colored by i under f. Then $\chi'_3(D_i) = 1$, and so by Lemma 14, each component of D_i has at most one odd semi-cycle. Hence the vertices of D_i can be colored properly by three colors. Based on this we give a proper vertex-coloring of G as follows.

First, we color properly the vertices of D_1 other than sinks by three colors. (A *sink* in a directed graph is a vertex which is not the tail of any arc of the directed graph.) We show that in this way we obtain a proper partial coloring of G. Assume that two vertices of D_1, say u and x, receive the same color. Then there are $v, y \in V(D_1)$ such that uv and xy are arcs of D_1. Since $\chi'_3(D_1) = 1$, the arcs uv and xy are not adjacent in $X(G)$. Moreover, as u and x are not adjacent in D_1, we have $v \neq x$ and $y \neq u$. Consequently, in G we cannot have the edge $\{u, x\}$, so that the described partial coloring of G is proper.

Now we use three new colors to color properly those vertices of D_2, which are not sinks. However, we color only those non-sink vertices of D_2, which did not receive any color in the previous step. Analogous to the argument above, one can show that in such a way we obtain a proper partial coloring of G, in which we color all the vertices of $D_1 \cup D_2$, which are not sinks in $D_1 \cup D_2$. (That is, we color those vertices of $D_1 \cup D_2$ at which there starts at least one arc either in D_1 or in D_2.)

Repeating this process for $D_3, D_4, \ldots, D_{k-1}$ we obtain a proper partial coloring of G using at most $3(k-1)$ colors. Now consider a component of D_k. If this component has no odd semi-cycle, then its non-sink vertices can be colored properly by two colors. If this component contains a (unique) odd semi-cycle, then by Lemma 14 this odd semi-cycle is a directed cycle. Hence all its vertices are already colored and the remaining non-sink vertices can be colored properly by two colors. In any case, to color properly non-colored non-sink vertices of any component of D_k it suffices to use two colors. Thus, non-sink vertices of $\cup_{i=1}^{k} D_i = D$ can be colored properly by at most $3k - 1$ colors. Since every vertex of D is a non-sink vertex of some D_i, we have $\chi(G) \leq 3\chi'_3(G) - 1$ and the lower bound in (6) is established.

Let G_n be a graph consisting of $n \geq 1$ edge-disjoint triangles with a common vertex u. Then $\chi(G_n) = 3$ and $\chi'_3(G_n) \geq 2$. For each triangle of G_n, color the two arcs starting from u by color 1, and color one of the two arcs entering into u by 1 and the other one by 2. Color the remaining two arcs in each triangle by 2. One can verify that this is a 3-arc coloring of G. Hence $\chi'_3(G_n) = 2$ and the lower bound in (6) is attained by G_n. □

It is easily seen that the problem of deciding whether $\chi'_3(G) \leq k$ can be solved in polynomial time when $k = 1$ or 2; see Theorem 12. The following result shows that this problem is NP-complete when $k = 3$ even when restricted to planar graphs.

Theorem 16. *The problem of deciding whether $\chi'_3(G) \leq 3$ for a planar graph G is NP-complete.*

Proof. Given a planar graph G, we construct a graph G^* from G by adding a new vertex v' for each $v \in V(G)$ and joining v and v' by an edge. Obviously G^* is planar and G^* can be constructed in polynomial time. We show that $\chi'_3(G^*) \leq 3$ if and only if G is 3-colorable. Suppose that G^* has a 3-arc coloring f using three colors. Color the vertices of G such that $v \in V(G)$ receives color $f(vv')$. It can be easily seen that this is a proper vertex-coloring of G by three colors. Thus G is 3-colorable. Conversely, suppose G is 3-colorable and g is a proper vertex-coloring of G by three colors. For each $v \in V(G)$, assign color $g(v)$ to all arcs of G^* with tail v, and assign any of the three colors of g to $v'v$. It can be verified that this is a 3-arc coloring of G^* and hence $\chi'_3(G^*) \leq 3$. Thus we have proved that $\chi'_3(G^*) \leq 3$ if and only if G is 3-colorable. Since the problem of deciding whether a planar graph is 3-colorable is NP-complete [7], the problem of deciding whether $\chi'_3(G) \leq 3$ is NP-complete, too. □

5. Problems

It is known that line graphs and 2-path graphs (that is, 3-path graphs as used in [11]) have forbidden subgraph characterizations; see [17, 7.1] and [11], respectively. In contrast, as far as we are aware, there is no known characterization of 3-arc graphs.

Problem 1. Characterize 3-arc graphs of connected graphs.

Other interesting problems include the following two.

Problem 2. Let G be a connected graph with $\delta(G) \geq 3$. Under what conditions is $X(G)$ Hamiltonian?

Problem 3. Give a sharp upper bound on $\gamma(X(G))$ in terms of $\gamma(G)$ for any connected graph G with $\delta(G) \geq 2$.

There is a wide space for improving results of this paper. For instance, the gap between the upper and lower bounds in (2) is big (though both bounds are sharp in general) and it may be improved for some special families of graphs. Also, the lower bound in (6) seems to be far from optimal for $\chi'_3(G) > 2$.

There is an extensive literature on line graphs with hundreds of publications, and also dozens of papers on path graphs have been published (e.g. [4,11]). As these two graph operators are related [10] to the 3-arc graph operator, we expect that techniques used previously for line graphs and path graphs may be utilized to derive properties of 3-arc graphs.

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