Classification of a family of symmetric graphs with complete 2-arc transitive quotients

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Abstract

In this paper we give a classification of a family of symmetric graphs with complete 2-arc transitive quotients. Of particular interest are two subfamilies of graphs which admit an arc-transitive action of a projective linear group. The graphs in these subfamilies can be defined in terms of the cross ratio of certain 4-tuples of elements of a finite projective line, and thus may be called the second type 'cross ratio graphs', which are different from the 'cross ratio graphs' studied in [A. Gardiner, C. E. Praeger and S. Zhou, Cross-ratio graphs, *J. London Math. Soc.* (2) 64 (2001), 257-272]. We also give a combinatorial characterisation of such second type cross ratio graphs.

Keywords: Symmetric graph; arc-transitive graph; 2-arc transitive graph; quotient graph; 3-arc graph; cross ratio graph

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1 Introduction

Exploring the symmetry of various mathematical structures is a lasting endeavor of human beings. In particular, the study of symmetric graphs has long been one of the main streams in algebraic graph theory [2] since Tutte's seminal work [17]. Over decades researchers in this area have produced a large number of beautiful results on symmetric graphs. The reader is referred to [14, 15] for surveys on symmetric graphs and intertwined connections between them and finite groups.

The purpose of this paper is to give a classification of a family of symmetric graphs with complete 2-arc transitive quotients. This forms part of the broad program [9, 11, 12,

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19, 20, 21, 22] of studying symmetric graphs with 2-arc transitive quotients. Of particular interest arising from the classification are two subfamilies of symmetric graphs which admit an arc-transitive action of a projective linear group. Such graphs can be defined in terms of the cross ratio of certain 4-tuples of elements of a finite projective line, but they are different from the 'cross ratio graphs' studied in [6, 8]. The main result of the paper relies on the classification [4, 10] of finite 3-transitive groups, which in turn is a consequence of the classification of finite simple groups.

To explain our results let us introduce some notation and terminology, beginning with the following standard definitions [5] for permutation groups. Let G be a finite group and Ω a finite set. An *action* of G on Ω (with *degree* $|\Omega|$) is a mapping $\Omega \times G \to \Omega$, $(\alpha, g) \mapsto \alpha^g$ such that $\alpha^1 = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$ for any $\alpha \in \Omega$ and $g, h \in G$, where 1 is the identity of G; we also say that G *acts on* Ω. If $\alpha^g = \alpha$, then g *fixes* α. If 1 is the only element of G which fixes every $\alpha \in \Omega$, then G is *faithful* on Ω; otherwise G is *unfaithful* on Ω. Define $G_\alpha := \{g \in G : \alpha^g = \alpha\}$, the *stabiliser* of α in G, and similarly $G_{\alpha\beta} := \{g \in G : \alpha^g = \alpha, \beta^g = \beta\}$ for fixed $\alpha, \beta \in \Omega$. The *setwise stabiliser* of $X \subseteq \Omega$ in G is $G_X := \{g \in G : X^g = X\}$, where $X^g := \{\alpha^g : \alpha \in X\}$, and the *$G$-orbit* containing $\alpha \in \Omega$ is $\alpha^G := \{\alpha^g : g \in G\}$. If $\alpha^G = \Omega$ for some (and hence all) $\alpha \in \Omega$, then G is *transitive on* Ω. If G is transitive on the set of k-tuples of distinct members of Ω in its coordinate-wise induced action, then G is said to be *k-transitive* on Ω, where $k \geq 1$ is an integer. All groups in this paper can be found in [5, 13]. In particular, $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ are the well-known Mathieu groups, and $K.H$ denotes the semidirect product of K by H.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a (finite, undirected, simple) graph and $s \geq 1$ an integer. An *s-arc* of Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices of Γ such that α_i, α_{i+1} are adjacent for $i = 0, \ldots, s-1$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. In particular, a 1-arc is called an *arc*. Denote by $\mathtt{Arc}_s(\Gamma)$ the set of s-arcs and $\mathtt{Arc}(\Gamma) := \mathtt{Arc}_1(\Gamma)$ the set of arcs of Γ. The graph Γ is said to *admit* a finite group G as a group of automorphisms if G acts on $V(\Gamma)$ and, for any $\alpha, \beta \in V(\Gamma)$ and $g \in G$, α and β are adjacent in Γ if and only if α^g and β^g are adjacent in Γ. In the case where G is transitive on $V(\Gamma)$ and, under the induced action, transitive on $\mathtt{Arc}_s(\Gamma)$, Γ is said to be *(G, s)-arc transitive.* Thus, if G is transitive on $V(\Gamma)$, then Γ is $(G, 2)$-arc transitive if and only if for some $\alpha \in V(\Gamma)$, G_α is 2-transitive on $\Gamma(\alpha)$, the *neighbourhood* of α. A $(G, 1)$-arc transitive graph is usually called a *G-symmetric graph.* Such a graph Γ is said to be *imprimitive* if G is imprimitive on $V(\Gamma)$, that is, $V(\Gamma)$ admits a nontrivial partition $\mathcal{B}$ such that $B^g := \{\alpha^g : \alpha \in B\} \in \mathcal{B}$ for $B \in \mathcal{B}$ and $g \in G$; call $\mathcal{B}$ a *nontrivial G-invariant partition.* In this case the *quotient graph* of Γ with respect to $\mathcal{B}$, $\Gamma_{\mathcal{B}}$, is the graph with vertex set $\mathcal{B}$ such that $B, C \in \mathcal{B}$ are adjacent if and only if there exist $\alpha \in B$, $\beta \in C$ such that α, β are adjacent in Γ. This quotient graph is G-symmetric under the induced action (possibly unfaithful) of G on $\mathcal{B}$. *Throughout the paper we always use $b := |\Gamma_{\mathcal{B}}(B)|$ to denote the valency of $\Gamma_{\mathcal{B}}$, where $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of $B \in \mathcal{B}$ in $\Gamma_{\mathcal{B}}$.* Denote $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$.

Continuing the work in [12], in this paper we study imprimitive G-symmetric graphs $(\Gamma, \mathcal{B})$ such that the quotient $\Gamma_{\mathcal{B}}$ is a complete $(G, 2)$-arc transitive graph. The main result,

$P_3(K_{b+1})$	$\mathrm{CR}^2(b;x,s)$	$\mathrm{TCR}^2(b;x,s)$
Example 2.2	Example 2.8	Example 2.9
$\Gamma^{(=)}(d,2)$ and $\Gamma^{(\simeq)}(d,2)$	$\Gamma^{(1)}(\mathrm{M}_{11})$ and $\Gamma^{(2)}(\mathrm{M}_{11})$	$\Gamma^{(1)}(\mathrm{M}_{22})$ and $\Gamma^{(2)}(\mathrm{M}_{22})$
Examples 2.3 and 2.4	Example 2.5	Example 2.6

Table 1: Graphs in Theorem 1.1.

Theorem 1.1 below, is a classification of such graphs under certain additional conditions. Details of the graphs in the classification together with the corresponding groups will be given in the next section; see Table 1 for a summary. Denote by $n \cdot \Sigma$ the vertex-disjoint union of n copies of a given graph Σ, and K_n the complete graph with n vertices. Note that, for a prime power $b = p^e$ and $x \in \mathrm{GF}(b) \setminus \{0, 1\}$, the subfield of the Galois field $\mathrm{GF}(b)$ generated by x has the form $\mathrm{GF}(p^{s(x)})$, for some divisor $s(x)$ of e.

Theorem 1.1 *Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$, where $G \leq \mathrm{Aut}(\Gamma)$. Suppose that $\Gamma_{\mathcal{B}}$ is a complete $(G,2)$-arc transitive graph of valency b, each vertex of Γ has neighbours in exactly two blocks of $\mathcal{B}$, and for any three blocks B, C, D of $\mathcal{B}$ there exists exactly one vertex in B which has neighbours in both C and D. Then $b(b-1)/2 = |B| > |\Gamma(C) \cap B| = b - 1$, G is 3-transitive and faithful on $\mathcal{B}$ of degree $b + 1$, and either $\Gamma = ((b-1)b(b+1)/6) \cdot K_3$ with G an arbitrary 3-transitive group of degree $b+1$, or one of the following* (a)-(f) *holds.*

(a) $\Gamma = P_3(K_{b+1})$ *(path graph of K_{b+1}) and G is either S_{b+1} ($b \geq 3$), or A_{b+1} ($b \geq 5$), or M_{b+1} ($b = 10, 11, 22, 23$).*

(b) $\Gamma = \mathrm{CR}^2(b;x,s)$ *and it admits $\mathrm{PGL}(2,b).\langle\psi^s\rangle$ as an arc-transitive group of automorphisms, where $b = p^e$ with p a prime and $e \geq 1$, $x \in \mathrm{GF}(b) \setminus \{0,1\}$, and s is a divisor of $s(x)$.*

(c) $\Gamma = \mathrm{TCR}^2(b;x,s)$ *and it admits $\mathrm{M}(s/2,b)$ as an arc-transitive group of automorphisms, where $b = p^e$ with p an odd prime and $e \geq 2$ an even integer, $x \in \mathrm{GF}(b) \setminus \{0,1\}$ with $s(x)$ even and $x-1$ a square of $\mathrm{GF}(b)$, and s is an even divisor of $s(x)$.*

(d) $\Gamma = \Gamma^{(=)}(d,2)$ *or $\Gamma^{(\simeq)}(d,2)$, $b = 2^d - 1$, where $d \geq 2$, and either $G = \mathrm{AGL}(d,2)$, or $d = 4$ and $G = \mathbb{Z}_2^4.A_7$.*

(e) $(\Gamma, G) = (\Gamma^{(1)}(\mathrm{M}_{11}), \mathrm{M}_{11})$ *or $(\Gamma^{(2)}(\mathrm{M}_{11}), \mathrm{M}_{11})$, and $b = 11$.*

(f) $(\Gamma, G) = (\Gamma^{(1)}(\mathrm{M}_{22}), \mathrm{M}_{22})$ *or $(\Gamma^{(2)}(\mathrm{M}_{22}), \mathrm{M}_{22})$, and $b = 21$.*

The class of imprimitive G-symmetric graphs $(\Gamma, \mathcal{B})$ such that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, $|\{C \in \mathcal{B} : \Gamma(\alpha) \cap C \neq \emptyset\}| = 2$ for $\alpha \in V(\Gamma)$ and $|\Gamma(C) \cap \Gamma(D) \cap B| = 1$ for distinct $C, D \in \Gamma_{\mathcal{B}}(B)$ has been studied in [12, Section 4.1]. Theorem 1.1 classifies such graphs under the additional condition that $\Gamma_{\mathcal{B}}$ is a complete graph. Among the graphs above, $\mathrm{CR}^2(b;x,s)$ and $\mathrm{TCR}^2(b;x,s)$ in (b)-(c) of Theorem 1.1 are especially interesting to us. They can be defined in terms of the cross ratio of certain 4-tuples of elements of the projective line $\mathrm{PG}(1,b)$ over $\mathrm{GF}(b)$, where $b \geq 3$ is a prime power. (See Examples 2.8 and 2.9 for details.) Theorem 1.2 below gives a combinatorial characterisation of such 'cross ratio graphs', which is similar to the characterisation [8, Theorem 5.1] of the first type 'cross ratio graphs'. Let

$$V_2(b) := \{xyz : x, y, z \in \mathrm{PG}(1, b), x, y, z \text{ pairwise distinct}\} \tag{1}$$

with the understanding that $xyz = zyx$. (The sequence xyz will be interpreted as a path of length 2 in the complete graph with vertex set $\mathrm{PG}(1, b)$.)

Theorem 1.2 *Let $b = p^e \geq 3$ with p a prime and $e \geq 1$ an integer. Suppose that Γ is a G-symmetric graph with $V(\Gamma) = V_2(b)$ and $E(\Gamma) \subseteq \{\{wuy, w'u'y'\} : wuy, w'u'y' \in V_2(b), u \in \{w', y'\}, u' \in \{w, y\}\}$, where G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, b)$ with the induced natural action on $V_2(b)$. Then one of the following holds:*

(a) *$\Gamma = ((b-1)b(b+1)/6) \cdot K_3$ with connected components $\{\{wuy, uyw, ywu\}\}$, for distinct $w, u, y \in \mathrm{PG}(1, b)$;*

(b) *Γ is isomorphic to $\mathrm{CR}^2(b; x, s)$ or $\mathrm{TCR}^2(b; x, s)$, for some x, s as in (b) or (c) of Theorem 1.1.*

2 Proof of Theorem 1.1

Let $(\Gamma, \mathcal{B})$ be an imprimitive G-symmetric graph such that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Then the setwise stabiliser of $B \in \mathcal{B}$ in G, G_B, is 2-transitive on the neighbourhood $\Gamma_{\mathcal{B}}(B)$ of B in $\Gamma_{\mathcal{B}}$. Thus, for distinct $C, D \in \Gamma_{\mathcal{B}}(B)$, $|\Gamma(C) \cap \Gamma(D) \cap B|$ is a constant. That is, $\lambda := |\Gamma(C) \cap \Gamma(D) \cap B|$ is independent of the choice of the 2-arc (C, B, D) of $\Gamma_{\mathcal{B}}$. Moreover, if $\lambda \geq 1$, then the incidence structure with 'point set' $\Gamma_{\mathcal{B}}(B)$ and 'block set' B such that $C \in \Gamma_{\mathcal{B}}(B)$ is incident with $\alpha \in B$ if and only if $\alpha \in \Gamma(C)$ is a 2-(b, r, λ) design, where $r := |\{C \in \mathcal{B} : \Gamma(\alpha) \cap C \neq \emptyset\}| = |\{C \in \mathcal{B} : \alpha \in \Gamma(C)\}|$, which is independent of the choice of $\alpha \in V(\Gamma)$. (See [1] for terminology on block designs.)

A major tool for establishing Theorem 1.1 is a construction given in [12], which we explain now. A 2-*path* is a 2-arc identified with its reverse 2-arc. Let $\tau\sigma\tau'$ denote the 2-path with mid-vertex σ and end-vertices τ, τ'. Note that $\tau'\sigma\tau$ represents the same 2-path, and $\tau\sigma\tau' = \eta\varepsilon\eta'$ if and only if $\sigma = \varepsilon$ and $\{\tau, \tau'\} = \{\eta, \eta'\}$. For a regular graph Σ with valency at least 2,

$$\mathcal{B}_2(\Sigma) := \{B_2(\sigma) : \sigma \in V(\Sigma)\}$$

is a partition of the 2-paths of Σ, where $B_2(\sigma)$ is the set of 2-paths of Σ with mid-vertex σ. Let Δ be a *self-paired* subset of $\mathtt{Arc}_3(\Sigma)$, that is, $(\tau, \sigma, \varepsilon, \eta) \in \Delta$ implies $(\eta, \varepsilon, \sigma, \tau) \in \Delta$. Define [12] $\Gamma_2(\Sigma, \Delta)$ to be the graph with the set of 2-paths of Σ as vertex set such that two distinct 'vertices' $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are adjacent if and only if they have a common edge (that is, $\sigma \in \{\eta, \eta'\}$ and $\varepsilon \in \{\tau, \tau'\}$) and moreover the two 3-arcs (which are reverses of each other) formed by 'gluing' this common edge are in Δ. For example, if $\sigma = \eta', \varepsilon = \tau'$, then the 3-arcs thus formed are $(\tau, \sigma, \varepsilon, \eta)$ and $(\eta, \varepsilon, \sigma, \tau)$, which should be in Δ if $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are adjacent in $\Gamma_2(\Sigma, \Delta)$. The self-parity of Δ ensures that $\Gamma_2(\Sigma, \Delta)$ is a well-defined undirected graph. We will use the following result in the proof of Theorem 1.1.

Theorem 2.1 *([12, Theorem 4.3]) Let Σ be a $(G,2)$-arc transitive graph with valency $b \geq 3$ and Δ a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$. Then $\Gamma := \Gamma_2(\Sigma, \Delta)$ is a G-symmetric graph admitting $\mathcal{B} := \mathcal{B}_2(\Sigma)$ as a G-invariant partition such that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, $|B| > |\Gamma(C) \cap B|$ for adjacent $B, C \in \mathcal{B}$, and $(\lambda, r) = (1,2)$. Moreover, G is faithful on $V(\Gamma)$ if and only if it is faithful on $V(\Sigma)$.*

Conversely, any imprimitive G-symmetric graph $(\Gamma, \mathcal{B})$ such that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive of valency $b \geq 3$, $|B| > |\Gamma(C) \cap B|$ for adjacent $B, C \in \mathcal{B}$, and $(\lambda, r) = (1,2)$ is isomorphic to $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ for a self-paired G-orbit on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. Moreover, G is faithful on $V(\Gamma)$ if and only if it is faithful on $\mathcal{B}$.

Furthermore, in both parts of this theorem we have $|B| = b(b-1)/2$ and $|\Gamma(C) \cap B| = b-1$ for adjacent $B, C \in \mathcal{B}$. A 3-arc is called *proper* if it is not a 3-cycle.

Proof of Theorem 1.1 Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma_{\mathcal{B}} \cong K_{b+1}$ is a complete $(G,2)$-arc transitive graph of valency b, where $G \leq \mathrm{Aut}(\Gamma)$. Suppose that each vertex of Γ has neighbours in exactly two blocks of $\mathcal{B}$, and for any three blocks B, C, D of $\mathcal{B}$ exactly one vertex in B has neighbours in both C and D. Then $(\lambda, r) = (1,2)$, and hence $|B| = b(b-1)/2$ and $|\Gamma(C) \cap B| = b-1$ for adjacent $B, C \in \mathcal{B}$. Also $b \geq 3$ for otherwise $\mathcal{B}$ would be a trivial partition. Hence $|B| > |\Gamma(C) \cap B|$.

From Theorem 2.1, $\Gamma \cong \Gamma_2(K_{b+1}, \Delta)$ for a self-paired G-orbit Δ on $\mathtt{Arc}_3(K_{b+1})$, and G is faithful on $\mathcal{B}$ since $G \leq \mathrm{Aut}(\Gamma)$ is faithful on $V(\Gamma)$. Moreover, since $\Gamma_{\mathcal{B}}$ is a complete graph and is $(G,2)$-arc transitive, G is 3-transitive on $\mathcal{B}$. Thus, from the classification of highly transitive permutation groups (see e.g. [4, 10]), G is one of the following groups of degree $b+1$ with the natural 3-transitive permutation representation on $V(K_{b+1})$: (i) S_{b+1} ($b \geq 3$); (ii) A_{b+1} ($b \geq 4$); (iii) $\mathrm{AGL}(d,2)$ ($b = 2^d - 1 \geq 3$); (iv) $\mathbb{Z}_2^4.A_7$ ($b = 15$); (v) Mathieu groups M_{b+1} ($b = 10, 11, 21, 22, 23$) and M_{11} ($b = 11$); and (vi) 3-transitive groups G satisfying $\mathrm{PGL}(2,b) \leq G \leq \mathrm{P\Gamma L}(2,b)$ ($b \geq 3$ is a prime power, note that $\mathrm{PGL}(2,4) \cong A_5$).

In the case where Δ contains a 3-cycle, it must consist of all 3-cycles of K_{b+1}, and one can check that $\Gamma_2(K_{b+1}, \Delta) \cong ((b-1)b(b+1)/6) \cdot K_3$ and G can be any 3-transitive group of degree $b+1$ in the list above. Henceforth we may assume that Δ is a self-paired G-orbit on the set of proper 3-arcs of K_{b+1}. In the remaining part of this section we will determine, for each 3-transitive group G above, all such self-paired G-orbits Δ together with the corresponding graphs $\Gamma_2(K_{b+1}, \Delta)$, and thus complete the proof of Theorem 1.1. These graphs will be given in Examples 2.2, 2.3, 2.4, 2.5, 2.6, 2.8 and 2.9, and sorted in terms of the corresponding groups. $\square$

The self-paired G-orbits Δ above were used in the classification of another family of symmetric graphs with complete 2-arc transitive quotients; see [21, Theorem 3.19] and [7]. However, the graphs arisen therein are different from those in Theorem 1.1, and they are defined as 3-arc graphs. (For a regular graph Σ and a self-paired subset Δ of $\mathtt{Arc}_3(\Sigma)$, the 3-*arc graph* [11, 18] $\Xi(\Sigma, \Delta)$ of Σ with respect to Δ is defined to have vertex set $\mathtt{Arc}(\Sigma)$

such that $(\sigma, \tau), (\sigma', \tau')$ are adjacent if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$.) The information about Δ derived from [21] will be used in our subsequent discussions.

Example 2.2 (*path graph* $P_3(K_{b+1})$) If G is 4-transitive on $V(K_{b+1})$, then either $G = S_{b+1}$ ($b \geq 3$), or $G = A_{b+1}$ ($b \geq 5$), or $G = \mathrm{M}_{b+1}$ ($b = 10, 11, 22, 23$). In each case G is transitive on the set Δ of proper 3-arcs of K_{b+1}, and hence Δ is the unique self-paired G-orbit on such 3-arcs. Thus, we obtain a unique graph, namely, $\Gamma := \Gamma_2(K_{b+1}, \Delta)$. For each proper 3-arc $(\tau, \sigma, \varepsilon, \eta)$, $\{\tau\sigma\varepsilon, \eta\varepsilon\sigma\}$ is an edge of Γ, and moreover any edge of Γ is of this form. Hence Γ is exactly the path graph $P_3(K_{b+1})$ of K_{b+1}. (Given a graph Σ, the *path graph* $P_3(\Sigma)$ [3] is the graph with 2-paths of Σ as vertices such that two 'vertices' are adjacent if and only if the corresponding 2-paths have exactly one common edge. In the case when Σ is regular of valency ≥ 2 and Δ is the set of all proper 3-arcs of Σ, we have $\Gamma_2(\Sigma, \Delta) \cong P_3(\Sigma)$ [12, Remark 4.5(d)].)

For any distinct $\sigma, \varepsilon \in V(K_{b+1})$, the subgraph of Γ induced by edges between $B_2(\sigma)$ and $B_2(\varepsilon)$ is the complete bipartite graph $K_{b-1,b-1}$ with a perfect matching deleted. Thus, Γ is a connected graph with order $(b-1)b(b+1)/2$, valency $2(b-1)$ and diameter 3.

From now on we consider the case where G is 3-transitive but not 4-transitive on $V(K_{b+1})$.

Example 2.3 (*Affine graphs* $\Gamma^{(=)}(d, 2)$ *and* $\Gamma^{(\simeq)}(d, 2)$) The group $\mathrm{AGL}(d, 2)$ is 3-transitive on the point set $V(d, 2)$ of $\mathrm{AG}(d, 2)$, where $d \geq 2$. Let K_{b+1} be defined on $V(d, 2)$, where $b = 2^d - 1 \geq 3$. For $d \geq 3$, by [21, Lemma 3.9] and the discussion in [21, Remark 3.12], there are exactly two self-paired $\mathrm{AGL}(d, 2)$-orbits on the proper 3-arcs of K_{b+1}, namely,

$$\Delta^{=} := \{(\mathbf{w}, \mathbf{u}, \mathbf{y}, \mathbf{z}) : \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \in V(d, 2) \text{ distinct}, \mathbf{u} - \mathbf{w} = \mathbf{y} - \mathbf{z}\}$$

and

$$\Delta^{\simeq} := \{(\mathbf{w}, \mathbf{u}, \mathbf{y}, \mathbf{z}) : \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \in V(d, 2) \text{ independent}\}.$$

Therefore, $\Gamma^{(=)}(d, 2) := \Gamma_2(K_{b+1}, \Delta^{=})$ and $\Gamma^{(\simeq)}(d, 2) := \Gamma_2(K_{b+1}, \Delta^{\simeq})$ are the only graphs arising from $\mathrm{AGL}(d, 2)$ when $d \geq 3$. The vertices of these graphs are $\mathbf{wuy}$ ($= \mathbf{yuw}$) for distinct $\mathbf{w}, \mathbf{u}, \mathbf{y} \in V(d, 2)$, and the edges of them are $\{\mathbf{wuy}, \mathbf{uyz}\}$ for distinct $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \in V(d, 2)$, where $\mathbf{u} - \mathbf{w} = \mathbf{y} - \mathbf{z}$ for $\Gamma^{(=)}(d, 2)$, and $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z}$ are independent for $\Gamma^{(\simeq)}(d, 2)$. In other words, $\mathbf{wuy}$ and $\mathbf{uyz}$ are adjacent in $\Gamma^{(=)}(d, 2)$ precisely when in $\mathrm{AG}(d, 2)$ the unique line through $\mathbf{u}$, $\mathbf{w}$ and the unique line through $\mathbf{y}$, $\mathbf{z}$ are parallel, and they are adjacent in $\Gamma^{(\simeq)}(d, 2)$ precisely when these two lines are skew. In the case when $d = 2$, we have $b + 1 = 4$ and by [21, Lemma 3.9] $\Delta^{=}$ above is the only self-paired $\mathrm{AGL}(2, 2)$-orbit on the proper 3-arcs of K_4. Hence $\Gamma^{(=)}(2, 2) \cong 3 \cdot C_4$ is the unique graph arising from $\mathrm{AGL}(2, 2)$.

Example 2.4 (*Affine graphs* $\Gamma^{(=)}(4, 2)$ *and* $\Gamma^{(\simeq)}(4, 2)$) The group $\mathbf{Z}_2^4.A_7$ is a subgroup of $\mathrm{AGL}(4, 2)$, where $\mathbf{Z}_2^4$ acts on $V(K_{16}) := V(4, 2)$ by translations and $G_{\mathbf{0}} \cong A_7$ is a subgroup of $\mathrm{GL}(4, 2) \cong A_8$ acting 2-transitively on $V(4, 2) \setminus \{\mathbf{0}\}$ in its natural action. As shown in

[21, Example 3.16, Remark 3.12], $\Delta^{=}$ and $\Delta^{\simeq}$ (for $d=4$) defined in Example 2.3 are the only self-paired $(\mathbf{Z}_2^4.A_7)$-orbits on the proper 3-arcs of K_{16}. Thus, $\Gamma^{(=)}(4,2)$ and $\Gamma^{(\simeq)}(4,2)$ are the only graphs arising from $\mathbf{Z}_2^4.A_7$.

Example 2.5 (*Mathieu graphs* $\Gamma^{(1)}(\mathrm{M}_{11})$ *and* $\Gamma^{(2)}(\mathrm{M}_{11})$) The Mathieu group M_{11} with degree $b+1=12$ is the automorphism group of the unique 3-$(12,6,2)$ design $\mathcal{D}$. Let K_{12} be defined on the point set of $\mathcal{D}$, so that M_{11} is 3-transitive on $V(K_{12})$. For a 2-arc $(\tau,\sigma,\varepsilon)$ of K_{12}, there are exactly two blocks of $\mathcal{D}$ which contain τ,σ,ε simultaneously. Let $X(\tau,\sigma,\varepsilon)$ denote the union of these two blocks of $\mathcal{D}$. It was proved in [21, Example 3.17] that there are exactly two self-paired M_{11}-orbits on the proper 3-arcs of K_{12}, namely,

$$\Delta_1 := \{(\tau,\sigma,\varepsilon,\eta) : \eta \in V(K_{12}) \setminus X(\tau,\sigma,\varepsilon)\}$$

$$\Delta_2 := \{(\tau,\sigma,\varepsilon,\eta) : \eta \in X(\tau,\sigma,\varepsilon) \setminus \{\tau,\sigma,\varepsilon\}\}.$$

Thus, $\Gamma^{(1)}(M_{11}) := \Gamma_2(K_{12},\Delta_1)$ and $\Gamma^{(2)}(M_{11}) := \Gamma_2(K_{12},\Delta_2)$ are the only graphs arising from M_{11}. Since $\mathcal{B}_2(K_{12})$ has block size 55, these graphs have order $12\times 55=660$. For distinct $\sigma,\varepsilon\in V(K_{12})$, there are 10 vertices in $B_2(\sigma)$ which have neighbours in $B_2(\varepsilon)$ in each of these graphs. Since $|V(K_{12})\setminus X(\tau,\sigma,\varepsilon)|=3$ and $|X(\tau,\sigma,\varepsilon)\setminus\{\tau,\sigma,\varepsilon\}|=6$ ([21, Example 3.17]), it follows that $\Gamma^{(1)}(M_{11})$ and $\Gamma^{(2)}(M_{11})$ have valencies 6 and 12, respectively.

Example 2.6 (*Mathieu graphs* $\Gamma^{(1)}(\mathrm{M}_{22})$ *and* $\Gamma^{(2)}(\mathrm{M}_{22})$) The Mathieu group M_{22} of degree $b+1=22$ is the automorphism group of the 3-$(22,6,1)$ Steiner system $\mathcal{D}$. Let K_{22} be defined on the point set of $\mathcal{D}$, so that M_{22} is 3-transitive on $V(K_{22})$. For a 2-arc $(\tau,\sigma,\varepsilon)$ of K_{22}, let $Y(\tau,\sigma,\varepsilon)$ be the unique block of $\mathcal{D}$ containing τ,σ and ε. It was proved in [21, Example 3.18] that there are exactly two self-paired M_{22}-orbits on the proper 3-arcs of K_{22}, namely,

$$\Delta_1 := \{(\tau,\sigma,\varepsilon,\eta) : \eta \in V(K_{22}) \setminus Y(\tau,\sigma,\varepsilon)\}$$

$$\Delta_2 := \{(\tau,\sigma,\varepsilon,\eta) : \eta \in Y(\tau,\sigma,\varepsilon) \setminus \{\tau,\sigma,\varepsilon\}\}.$$

Hence $\Gamma^{(1)}(M_{22}) := \Gamma_2(K_{22},\Delta_1)$ and $\Gamma^{(2)}(M_{22}) := \Gamma_2(K_{22},\Delta_2)$ are the only graphs arising from M_{22}. Since $\mathcal{B}_2(K_{22})$ has block size 210, the order of these graphs is $22\times 210=4620$. For distinct $\sigma,\varepsilon\in V(K_{22})$ there are 20 vertices in $B_2(\sigma)$ which have neighbours in $B_2(\varepsilon)$ in each of these graphs. Since $|V(K_{22})\setminus Y(\tau,\sigma,\varepsilon)|=16$ and $|Y(\tau,\sigma,\varepsilon)\setminus\{\tau,\sigma,\varepsilon\}|=3$ ([21, Example 3.18]), $\Gamma^{(1)}(M_{22})$ and $\Gamma^{(2)}(M_{22})$ have valencies 32 and 6, respectively.

The remainder of this section is devoted to graphs arising from 3-transitive subgroups of $\mathrm{P\Gamma L}(2,b)$, where $b=p^e\geq 3$ with p a prime and $e\geq 1$. The projective line $\mathrm{PG}(1,b)$ over $\mathrm{GF}(b)$ can be identified with $\mathrm{GF}(b)\cup\{\infty\}$, where ∞ satisfies the usual arithmetic rules such as $1/\infty=0, \infty+y=\infty, \infty^p=\infty$, etc. The 2-dimensional projective group $\mathrm{PGL}(2,b)$ consists of all Möbius transformations

$$t_{\alpha,\beta,\gamma,\delta} : z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta} \quad (\alpha,\beta,\gamma,\delta\in \mathrm{GF}(b), \alpha\delta-\beta\gamma\neq 0)$$

of $\mathrm{PG}(1,b)$ (see e.g. [13, pp.20-21]). Note that $t_{\alpha,\beta,\gamma,\delta} = t_{\alpha',\beta',\gamma',\delta'}$ if and only if $(\alpha,\beta,\gamma,\delta)$ is a non-zero multiple of $(\alpha',\beta',\gamma',\delta')$. The group $\mathrm{PGL}(2,b)$ is sharply 3-transitive in this action on $\mathrm{PG}(1,b)$, that is, it is 3-transitive and only the identity element $t_{1,0,0,1}$ of $\mathrm{PGL}(2,b)$ fixes three elements of $\mathrm{PG}(1,b)$. The Frobenius automorphism $\psi : x \mapsto x^p$ of $\mathrm{GF}(b)$ induces an automorphism of $\mathrm{PGL}(2,b)$ by $\psi : t_{\alpha,\beta,\gamma,\delta} \mapsto t_{\alpha^p,\beta^p,\gamma^p,\delta^p}$, and the group generated by $\mathrm{PGL}(2,b)$ and ψ is the semidirect product $\mathrm{P\Gamma L}(2,b) := \mathrm{PGL}(2,b).\langle\psi\rangle$. This group is the automorphism group of $\mathrm{PGL}(2,b)$, and it acts on $\mathrm{PG}(1,b)$ (with $\psi : z \mapsto z^p$, where $\infty^p = \infty$) as a 3-transitive group. We will use the following result about 3-transitive subgroups of $\mathrm{P\Gamma L}(2,b)$.

Lemma 2.7 *([8, Theorem 2.1]) Let $b = p^e$ with p a prime and $e \geq 1$ an integer. A subgroup of $\mathrm{P\Gamma L}(2,b)$ is 3-transitive on $\mathrm{PG}(1,b)$ if and only if it is one of the following groups:*

(a) $\mathrm{PGL}(2,b).\langle\psi^s\rangle$, *where s is a divisor of e;*

(b) $\mathrm{M}(s/2,b) := \langle \mathrm{PSL}(2,b), \psi^{s/2}t_{\alpha,0,0,1}\rangle$, *where p is odd, e is even, s is an even divisor of e, and α is a primitive element of* $\mathrm{GF}(b)$.

For a 4-tuple (u,w,y,z) of distinct elements of $\mathrm{PG}(1,b)$, the *cross ratio* (see e.g. [13, pp. 59]) is defined as

$$c(u,w;y,z) := \frac{(u-y)(w-z)}{(u-z)(w-y)}.$$

The cross ratio can take all values in $\mathrm{GF}(b)$ except 0 and 1, and is invariant under the induced action of $\mathrm{PGL}(2,b)$ on 4-tuples of distinct elements of $\mathrm{PG}(1,b)$. Under the Frobenius mapping ψ, we have

$$c(u^\psi, w^\psi; y^\psi, z^\psi) = c(u,w;y,z)^\psi. \tag{2}$$

Recall that, for each $x \in \mathrm{GF}(b) \setminus \{0,1\}$, the subfield of $\mathrm{GF}(b)$ generated by x has the form $\mathrm{GF}(p^{s(x)})$, for some divisor $s(x)$ of e. Given x, for each divisor s of $s(x)$, let $B(x,s)$ denote the $\langle\psi^s\rangle$-orbit containing x, that is,

$$B(x,s) := \{x^{\psi^{si}} : 0 \leq i < s(x)/s\}.$$

In the remaining part of this paper K_{b+1} is taken as defined on $\mathrm{PG}(1,b)$, so that $V_2(b)$ defined in (1) is the set of 2-paths of K_{b+1}. Since there are two kinds of 3-transitive subgroups of $\mathrm{P\Gamma L}(2,b)$ (Lemma 2.7), there are two kinds of 'second type cross ratio graphs', $\mathrm{CR}^2(b;x,s)$ and $\mathrm{TCR}^2(b;x,s)$, which will be given in the next two examples. The name 'cross ratio' is used since the adjacency relations of these graphs can be defined in terms of cross ratio, just as the case for the (first type) 'cross ratio graphs' [8] $\mathrm{CR}(b;x,s)$ and $\mathrm{TCR}(b;x,s)$. The 'untwisted cross ratio graphs' $\mathrm{CR}^2(b;x,s)$ and $\mathrm{CR}(b;x,s)$ share a self-paired G-orbit Δ on the proper 3-arcs of K_{b+1}, that is, $\mathrm{CR}^2(b;x,s) = \Gamma_2(K_{b+1},\Delta)$ and $\mathrm{CR}(b;x,s) = \Xi(K_{b+1},\Delta)$ for the same Δ. Similar relation holds for the 'twisted cross ratio graphs' $\mathrm{TCR}^2(b;x,s)$ and $\mathrm{TCR}(b;x,s)$.

Example 2.8 (*Second type untwisted cross ratio graphs* $\mathrm{CR}^2(b;x,s)$) Let $x \in \mathrm{GF}(b) \setminus \{0,1\}$, and let s be a divisor of $s(x)$. Define $\mathrm{CR}^2(b;x,s)$ to have vertex set $V_2(b)$ (defined in (1)) and arc set

$$\{(wuy, uyz) : u,w,y,z \in \mathrm{PG}(1,b) \text{ distinct}, \ c(u,w;y,z) \in B(x,s)\}.$$

Since $(uyz, wuy) = (zyu, yuw)$ and $c(y,z;u,w) = c(u,w;y,z)$, (wuy, uyz) is an arc of $\mathrm{CR}^2(b;x,s)$ if and only if (uyz, wuy) is an arc of $\mathrm{CR}^2(b;x,s)$. Thus, $\mathrm{CR}^2(b;x,s)$ is well-defined as an undirected graph.

Since $c(\infty,0;1,x) = x \in B(x,s)$, by (2) we have

$$c(\infty^{\psi^{si}}, 0^{\psi^{si}}; 1^{\psi^{si}}, x^{\psi^{si}}) = c(\infty,0;1,x)^{\psi^{si}} = x^{\psi^{si}} \in B(x,s).$$

Let $G := \mathrm{PGL}(2,b).\langle\psi^s\rangle$. Then G is transitive on 4-tuples of distinct elements of $\mathrm{PG}(1,b)$ with a fixed cross ratio (see e.g. [13, pp. 59]). Thus, (wuy, uyz) is an arc of $\mathrm{CR}^2(b;x,s)$ $\Leftrightarrow c(u,w;y,z) \in B(x,s) \Leftrightarrow c(u,w;y,z) = x^{\psi^{si}}$ for some i, $0 \le i < s(x)/s \Leftrightarrow (w,u,y,z) \in \Delta := (0,\infty,1,x)^G$. It is readily seen that $t_{1,-x,1,-1} \in G$ reverses $(0,\infty,1,x)$. Thus, Δ is a self-paired G-orbit on $\mathtt{Arc}_3(K_{b+1})$, and $\mathrm{CR}^2(b;x,s)$ is precisely the graph $\Gamma_2(K_{b+1},\Delta)$. Consequently, by Theorem 2.1 it admits G as an arc-transitive group of automorphisms. Moreover, since G is 3-transitive on $\mathrm{PG}(1,b)$, every self-paired G-orbit on proper 3-arcs of K_{b+1} has the form $(0,\infty,1,x)^G$ for some $x \in \mathrm{GF}(b) \setminus \{0,1\}$, and hence no other graph arises from G.

For the same x,s,G,Δ as above, from the discussion in [21, Example 3.1] the 3-arc graph $\Xi(K_{b+1},\Delta)$ is exactly the (first type) untwisted cross ratio graph $\mathrm{CR}(b;x,s)$ [8]. That is, $\mathrm{CR}(b;x,s)$ has vertex set $\{(y,z) : y,z \in \mathrm{PG}(1,b), y \ne z\}$ such that $(u,w),(y,z)$ are adjacent if and only if $(w,u,y,z) \in \Delta$, that is, $c(u,w;y,z) \in B(x,s)$.

Example 2.9 (*Second type twisted cross ratio graphs* $\mathrm{TCR}^2(b;x,s)$) Let $b = p^e$ with p an odd prime and $e \ge 2$ an even integer. Let $x \in \mathrm{GF}(b) \setminus \{0,1\}$ such that $x-1$ is a square of $\mathrm{GF}(b)$ and $s(x)$ is even, and s an even divisor of $s(x)$. Let $G := \mathrm{M}(s/2,b)$ and $\Delta := (0,\infty,1,x)^G$. Define $\mathrm{TCR}^2(b;x,s)$ to be the graph with vertex set $V_2(b)$ and arc set

$$\{(wuy, uyz) : u,w,y,z \in \mathrm{PG}(1,b) \text{ distinct}, \ (w,u,y,z) \in \Delta\}.$$

Since $x-1$ is a square, we have $t_{1,-x,1,-1} \in \mathrm{PSL}(2,b) \le \mathrm{M}(s/2,b)$. Thus, since $t_{1,-x,1,-1}$ reverses $(0,\infty,1,x)$, Δ is a self-paired G-orbit on the proper 3-arcs of K_{b+1}. Hence $\mathrm{TCR}^2(b;x,s)$ is indeed well-defined as an undirected graph, and it is the graph $\Gamma_2(K_{b+1},\Delta)$ and so admits G as an arc-transitive group of automorphisms by Theorem 2.1. Moreover, the 3-transitivity of G on $\mathrm{PG}(1,b)$ implies that every self-paired G-orbit on the proper 3-arcs of K_{b+1} has the form $(0,\infty,1,x)^G$ for some x. Hence all graphs $\Gamma_2(K_{b+1},\Delta)$ arising from G have the form $\mathrm{TCR}^2(b;x,s)$.

For the same x,s,G,Δ as in Example 2.9, it follows from [21, Example 3.1] that the 3-arc graph $\Xi(K_{b+1},\Delta)$ is exactly the (first type) twisted cross ratio graph $\mathrm{TCR}(b;x,s)$ [8].

That is, $\mathrm{TCR}(b;x,s)$ has vertex set $\{(y,z) : y,z \in \mathrm{PG}(1,b), y \neq z\}$ such that $(u,w),(y,z)$ are adjacent if and only if $(w,u,y,z) \in \Delta$.

The proof of Theorem 1.1 is complete up to now. $\square$

3 Proof of Theorem 1.2

Let

$$\mathcal{B}_2(b) := \{B_2(u) : u \in \mathrm{PG}(1,b)\},$$

where

$$B_2(u) := \{wuy : w,y \in \mathrm{PG}(1,b) \setminus \{u\}, w \neq y\}.$$

Then $\mathcal{B}_2(b)$ $(= \mathcal{B}_2(K_{b+1}))$ is a $\mathrm{P\Gamma L}(2,b)$-invariant partition of $V_2(b)$. The following lemma will be used in the proof of Theorem 1.2.

Lemma 3.1 *Let $b = p^e$ with p a prime and $e \geq 1$. Let $x \in \mathrm{GF}(b) \setminus \{0,1\}$, and let s be a divisor of $s(x)$. Let $\Gamma = \mathrm{CR}^2(b;x,s)$ and $G = \mathrm{PGL}(2,b).\langle \psi^s \rangle$, or $\Gamma = \mathrm{TCR}^2(b;x,s)$ and $G = \mathrm{M}(s/2,b)$, where in the latter case $e, s(x), s$ are even and $x-1$ is a square of $\mathrm{GF}(b)$. Then the valency of Γ is $2s(x)/s$ and the neighbourhood of $0\infty 1$ in Γ is*

$$\Gamma(0\infty 1) = \{\infty 1 y : y \in B(x,s)\} \cup \{\infty 0 z : z \in -B(x,s)+1\}, \tag{3}$$

where $-B(x,s)+1 := \{-x^{\psi^{si}} + 1 : 0 \leq i < s(x)/s\}$. Moreover, the stabiliser in G of the vertex $0\infty 1$ is given by $H := (G_\infty)_{\{0,1\}} = \langle t_{-1,1,0,1} \rangle.\langle \psi^s \rangle$.

Proof Since $\Gamma = \Gamma_2(K_{b+1}, \Delta)$, where Δ is as in Example 2.8 or 2.9 respectively, by the definition of $\Gamma_2(K_{b+1}, \Delta)$ it follows that $\Gamma(0\infty 1) \cap B_2(u) \neq \emptyset$ if and only if $u = 1$ or 0.

Consider $\Gamma = \mathrm{CR}^2(b;x,s)$ and $G = \mathrm{PGL}(2,b).\langle \psi^s \rangle$ first. Since $c(\infty,0;1,y) = y$ for $y \in \mathrm{GF}(b) \setminus \{0,1\}$, $0\infty 1$ and $\infty 1 y \in B_2(1)$ are adjacent in Γ if and only if $y \in B(x,s)$. Since $|B(x,s)| = s(x)/s$, and since only $B_2(1)$ and $B_2(0)$ contain neighbours of $0\infty 1$, the valency of Γ is equal to $2s(x)/s$. Since $G_{\infty 01} = \langle \psi^s \rangle$ ([8, Corollary 2.2]) and $(\mathrm{PGL}(2,b)_\infty)_{\{0,1\}} = \langle t_{-1,1,0,1} \rangle$, we have $H = \langle t_{-1,1,0,1} \rangle.\langle \psi^s \rangle$. Note that $(1,\infty,0,z) = (0,\infty,1,y)^g$ for some $g \in G \Leftrightarrow g \in H$, $z = y^g$, and g swaps 0 and $1 \Leftrightarrow z \in -B(x,s)+1$. Therefore, $1\infty 0$ $(= 0\infty 1)$ is adjacent to $\infty 0 z$ if and only if $z \in -B(x,s)+1$, and hence (3) holds.

Next let us consider $\Gamma = \mathrm{TCR}^2(b;x,s)$ and $G = \mathrm{M}(s/2,b)$. Since e is even, -1 is a square of $\mathrm{GF}(b)$ and hence $(\mathrm{PSL}(2,b)_\infty)_{\{0,1\}} = \langle t_{-1,1,0,1} \rangle$. Since $G_{\infty 01} = \langle \psi^s \rangle$ ([8, Corollary 2.2]), it follows that $H = \langle t_{-1,1,0,1} \rangle.\langle \psi^s \rangle$. Using [8, Remark 3.5(b)], we have: $0\infty 1$ and $\infty 1 y$ are adjacent in $\Gamma \Leftrightarrow (0,\infty,1,y) \in (0,\infty,1,x)^G \Leftrightarrow (\infty,0)$ and $(1,y)$ are adjacent in $\mathrm{CR}(b;x,s) \Leftrightarrow y \in B(x,s)$. A similar argument as in the previous paragraph then establishes (3). $\square$

A transitive group G on a set Ω induces an action on $\Omega \times \Omega$, and the G-orbits on $\Omega \times \Omega$ are called G-orbitals on Ω. Such a G-orbital Δ is called *self-paired* if $(\alpha,\beta) \in \Delta$ implies $(\beta,\alpha) \in \Delta$. For a self-paired G-orbital Δ other than $\{(\alpha,\alpha) : \alpha \in \Omega\}$ (the trivial

G-orbital), the graph with vertex set Ω and arc set Δ is an undirected graph, called a *G-orbital graph* on Ω, which admits G as an arc-transitive group of automorphisms. Conversely, any G-symmetric graph with vertex set Ω is isomorphic to a G-orbital graph on Ω. (See [14] for a more detailed discussion.)

Proof of Theorem 1.2 Let Γ and G satisfy the conditions of Theorem 1.2. Then by Lemma 2.7, for some divisor s of e, $G = \mathrm{PGL}(2,b).\langle\psi^s\rangle$, or $G = \mathrm{M}(s/2,b)$ (where p is odd, and s and e are even). From the discussion above, the graph Γ must be an orbital graph for some non-trivial self-paired G-orbital on $V_2(b)$. Since G is 3-transitive on $\mathrm{PG}(1,b)$, without loss of generality we may assume that this orbital is $\Delta' := (0\infty1, xuz)^G$, where $x,u,z \in \mathrm{PG}(1,b)$ are pairwise distinct such that $(\infty,\{0,1\}) \neq (u,\{x,z\})$. The assumption on Γ implies that $u \in \{0,1\}$ and $\infty \in \{x,z\}$. If $u=1$, then $\Delta' := (0\infty1, x1\infty)^G$ and $x \in \mathrm{PG}(1,b)\setminus\{\infty,1\}$ by setting $z=\infty$ without loss of generality. Similarly, if $u=0$, then $\Delta' := (1\infty0, z0\infty)^G$ and $z \in \mathrm{PG}(1,b)\setminus\{\infty,0\}$ by letting $x=\infty$. From Lemma 3.1, we have $H := (G_\infty)_{\{0,1\}} = \langle t_{-1,1,0,1}\rangle.\langle\psi^s\rangle$. Given x, $t_{-1,1,0,1} \in H$ maps $(0\infty1, x1\infty)$ to $(1\infty0, z0\infty)$, where $z = -x+1$, and hence $(0\infty1, x1\infty)^G = (1\infty0, z0\infty)^G$. Similarly, for a given z, $t_{-1,1,0,1}$ maps $(1\infty0, z0\infty)$ to $(0\infty1, x1\infty)$, where $x=-z+1$, and hence $(1\infty0, z0\infty)^G = (0\infty1, x1\infty)^G$. Therefore, the two types of self-paired G-orbital Δ' above (arising from $u=1$ and $u=0$) produce the same family of G-symmetric graphs, and hence it suffices to consider only the first type.

Let $\Delta' := (0\infty1, x1\infty)^G$, where $x \in \mathrm{PG}(1,b)\setminus\{\infty,1\}$. It may happen that $x=0$, and in this case by the 3-transitivity of G on $\mathrm{PG}(1,b)$, Γ must consist of all 3-cycles $\{wuy, uyw, ywu\}$ for pairwise distinct $w,u,y \in \mathrm{PG}(1,b)$, and hence case (a) in Theorem 1.2 occurs. In the following we suppose $x \neq 0$. The self-parity of Δ' implies that $\Delta := (0,\infty,1,x)^G$ is a self-paired G-orbit on the set of proper 3-arcs of K_{b+1}. Since $0\infty1$ is adjacent to $x1\infty \in B_2(1)$ by the definition of Δ', we have $\Gamma(0\infty1)\cap B_2(1) \neq \emptyset$. Also, as shown above, $t_{-1,1,0,1} \in H$ maps $(0\infty1, x1\infty)$ to $(1\infty0, z0\infty)$, where $z=-x+1$. Thus, $0\infty1$ is adjacent to $z0\infty \in B_2(0)$ in Γ, and hence $\Gamma(0\infty1)\cap B_2(0) \neq \emptyset$. On the other hand, suppose $\Gamma(0\infty1)\cap B_2(u) \neq \emptyset$ for some u, say, $0\infty1$ is adjacent to $wuy \in B_2(u)$. Then there exists an element of G which maps $(0\infty1, x1\infty)$ to $(0\infty1, wuy)$. This element must be in H and maps 1 to u, and hence $u=1$ or 0. Therefore, $\Gamma(0\infty1)\cap B_2(u) \neq \emptyset$ if and only if $u=1$ or 0.

Since G is transitive on the arcs of Γ and $\mathcal{B}_2(b)$ is a G-invariant partition of $V_2(b)$, H must be transitive on the vertices of $\Gamma(0\infty1)\cap(B_2(1)\cup B_2(0))$. Thus this set consists of vertices $x'1\infty$, $z'0\infty$ for $x' \in E(x) := \{x^{\psi^{si}} : 0 \le i < e\}$, $z' \in E(z) := \{z^{\psi^{si}} : 0 \le i < e\}$, and vertices $x'0\infty$, $z'1\infty$ for $x' \in F(x) := \{-x^{\psi^{si}}+1 : 0 \le i < e\}$, $z' \in F(z) := \{-z^{\psi^{si}}+1 : 0 \le i < e\}$. Since $z=-x+1$, we have $F(x) = E(z) = -E(x)+1$, $F(z) = E(x)$, and consequently

$$\Gamma(0\infty1)\cap B_2(1) = \{\infty1x' : x' \in E(x)\} \tag{4}$$

$$\Gamma(0\infty1)\cap B_2(0) = \{\infty0z' : z' \in -E(x)+1\}. \tag{5}$$

Note that $E(x)$ is contained in the subfield $\mathrm{GF}(p^{s(x)})$ generated by x, and hence each element of $E(x)$ is left invariant by $\psi^{s(x)}$. Thus, $\Gamma(0\infty1)$ is left invariant by $\langle H, \psi^{s(x)}\rangle$

and consequently $\langle G, \psi^{s(x)} \rangle$ leaves the G-orbital of arcs of Γ invariant. In other words, $\langle G, \psi^{s(x)} \rangle$ is contained in $\mathrm{Aut}(\Gamma)$. Thus we may assume that $\psi^{s(x)} \in G$, and hence that s divides $s(x)$. This implies $E(x) = B(x, s)$. Comparing (3) with (4)-(5), we conclude that the neighbourhoods of $0\infty1$ in Γ and in $\mathrm{CR}^2(b; x, s)$ or $\mathrm{TCR}^2(b; x, s)$ are the same, and they admit the same arc-transitive group G of automorphisms. Therefore, $\Gamma \cong \mathrm{CR}^2(b; x, s)$ or $\mathrm{TCR}^2(b; x, s)$, depending on whether $G = \mathrm{PGL}(2, b).\langle \psi^s \rangle$ or $G = \mathrm{M}(s/2, b)$, and this completes the proof. $\square$

4 Remarks

Denote

$$B_2(u; w) := \{wuy : y \neq u, w\}$$

for each pair $u, w \in \mathrm{PG}(1, b)$ with $u \neq w$. The graph $\mathrm{CR}^2(b; x, s)$ can be thought as obtained from $\mathrm{CR}(b; x, s)$ by unfolding each vertex of the latter to $b - 1$ vertices of the former, namely, each vertex (u, w) of $\mathrm{CR}(b; x, s)$ is unfolded to the $b - 1$ vertices of $\mathrm{CR}^2(b; x, s)$ contained in $B_2(u; w)$. Note that each vertex wuy of $\mathrm{CR}^2(b; x, s)$ has precisely two pre-images (u, w), (u, y) under this operation. It is evident that $\{(u, w), (y, z)\} \mapsto \{wuy, uyz\}$ for $(w, u, y, z) \in \Delta$ defines a bijection between the edges of $\mathrm{CR}(b; x, s)$ and the edges of $\mathrm{CR}^2(b; x, s)$, and $\{wuy, uyz\}$ is the only edge of $\mathrm{CR}^2(b; x, s)$ between $B_2(u; w)$ and $B_2(y; z)$, where $\Delta := (0, \infty, 1, x)^G$ as in Example 2.8. Similarly, $\mathrm{TCR}^2(b; x, s)$ can be obtained from $\mathrm{TCR}(b; x, s)$ by 'unfolding' each vertex (u, w) to the $b - 1$ vertices in $B_2(u; w)$ in the same manner.

In a subsequent paper [16] we will have a detailed study of the second type cross ratio graphs and answer a few questions about them.

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