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Harper-type lower bounds and the bandwidths of the compositions of graphs[1]

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Abstract

We give a general Harper-type lower bound for the bandwidth of a graph which is a common generalization of several known results. As applications we get a lower bound for the bandwidth of the composition of two graphs. By using this we determine the bandwidths of some composition graphs such as $(P_r \times P_s)[H], (P_r \times C_s)[H]$ $(2r \neq s), (C_r \times C_s)[H]$ $(6 \leqslant 2r \leqslant s)$, etc., for any graph H. Interestingly, the bandwidths of these graphs have nothing to do with the structure of H in general.

1. Introduction

Let $G=(V(G),E(G))$ be a simple graph with order $n=|V(G)|$. A bijection $f : V(G) \to \{1,2,\ldots,n\}$ is called a *labeling* of G, and $B(G,f)=\max_{uv\in E(G)} |f(u) - f(v)|$ is the *bandwidth* of labeling f. The *bandwidth* of G, denoted by $B(G)$, is defined to be the minimum bandwidth of labelings of G.

The bandwidth problem for graphs has attracted many graph theorists for its strongly practical background and theoretical interest. Because of the NP-completeness of the decision problem for finding the bandwidths of arbitrary graphs (even for trees of maximum degree 3, see [6,12]), people are interested in finding bandwidths for special graphs. In this direction, to make the lower bounds for the bandwidth as sharp as possible is of great significance. For example, Harper's lower bound [7] and various generalizations of it [8,10,11,15] have been used extensively in determining bandwidths for special graphs.

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The present paper consists of two parts. In the first part (Section 2) we will give a general Harper-type lower bound for the bandwidth from which several known lower bounds can be derived. In particular, we will give a generalization of Chvátal's density lower bound [4]. Such a bound is in fact Harper-type in a sense. In the second part (Section 3), we will first give a lower bound for the bandwidth of the composition of two graphs by using the results shown in Section 2. Then we will determine the bandwidths of some composition graphs. As we will see, the bandwidths of $(P_r \times P_s)[H]$ $(2\leqslant r\leqslant s)$, $(P_r \times C_s)[H]$ $(2r \neq s)$ and $(C_r \times C_s)[H]$ $(6\leqslant 2r\leqslant s)$ are independent of the structure of H in general. They rely only on the order of H.

2. Harper-type lower bounds

The terminology and notation in the paper follow those of [1]. Since the bandwidth of a graph G is the maximum bandwidth of the components of G [2], we will always suppose in this section that G is connected. For $S\subseteq V(G)$, let $\bar{S} = V(G)\backslash S$. For a positive integer r, let

$$\partial^{(r)}(S) = \{u \in S:\ \text{there exists } v \in \bar{S} \text{ such that } d(u,v)\leqslant r\}$$

and

$$\nabla^{(r)}(S) = \{v \in \bar{S}:\ \text{there exists } u \in S \text{ such that } d(u,v)\leqslant r\}.$$

Here $d(u,v)$ is the distance between u and v in G. In particular, $\partial(S) = \partial^{(1)}(S)$ and $\nabla(S) = \nabla^{(1)}(S)$ are the inner and outer boundaries of S, respectively. Define

$$\delta_-^{(r)}(S) = \min_{u\in\partial^{(r)}(S)} |\nabla^{(r)}(u) \cap \bar{S}|,$$

$$\delta_+^{(r)}(S) = \min_{v\in\nabla^{(r)}(S)} |\nabla^{(r)}(v) \cap S|.$$

For a labeling f of G, let $u_i = f^{-1}(i)$ be the vertex with label i, $1\leqslant i\leqslant n$, and $S_k = f^{-1}(\{1,2,\dots,k\}) = \{u_1,u_2,\dots,u_k\}$, $1\leqslant k\leqslant n$. Clearly, $\bar{S}_k = \{u_{k+1},u_{k+2},\dots,u_n\}$. Let

$$\alpha = \alpha^{(r)}(S_k) = \min\{i:\ u_i \in \partial^{(r)}(S_k)\},$$

$$\beta = \beta^{(r)}(S_k) = \max\{j:\ u_j \in \nabla^{(r)}(S_k)\},$$

$$D_-^{(r)}(S_k) = \{u \in S_k:\ f(u)\geqslant\alpha\} = \{u_\alpha, u_{\alpha+1},\dots,u_k\},$$

$$D_+^{(r)}(S_k) = \{v \in \bar{S}_k:\ f(v)\leqslant\beta\} = \{u_{k+1},u_{k+2},\dots,u_\beta\}.$$

Then $\partial^{(r)}(S_k)\subseteq D_-^{(r)}(S_k)\subseteq S_k$, $\nabla^{(r)}(S_k)\subseteq D_+^{(r)}(S_k)\subseteq \bar{S}_k$. Clearly,

$$\alpha = k + 1 - |D_-^{(r)}(S_k)| \tag{1}$$

and

$$\beta = k + |D_+^{(r)}(S_k)|. \tag{2}$$

Let $D_0^{(r)}(S_k) = X \cup Y$ be a subset of $V(G)$ with the largest cardinality such that (i) $X \subseteq \partial^{(r)}(S_k)$, $Y \subseteq \nabla^{(r)}(S_k)$, and (ii) $u \in X$, $v \in Y$ imply $d(u,v) \leqslant r$. With this notation we can prove the following:

Theorem 1. *Let f be a labeling of G. Then for any integers r and k with $1 \leqslant r$, $1 \leqslant k \leqslant n$,*

$$B(G,f) \geqslant \max\{\lceil A_1/r \rceil,\ \lceil A_2/r \rceil,\ \lceil A_3/r \rceil\},$$

where $A_1 = |D_-^{(r)}(S_k)| + \delta_-^{(r)}(S_k) - 1$, $A_2 = |D_+^{(r)}(S_k)| + \delta_+^{(r)}(S_k) - 1$ *and* $A_3 = |D_0^{(r)}(S_k)| - 1$.

Proof. It is easy to see that if $a < b$ and $d(u_a, u_b) \leqslant r$ then $B(G,f) \geqslant (b-a)/r$. Let $\alpha = \alpha^{(r)}(S_k)$ be as above. Since $u_\alpha \in \partial^{(r)}(S_k)$, there exists $u_b \in \nabla^{(r)}(S_k)$ such that $d(u_\alpha, u_b) \leqslant r$. Choosing b maximal gives

$$b \geqslant k + |\nabla^{(r)}(u_\alpha) \cap \bar{S}_k| \geqslant k + \delta_-^{(r)}(S_k). \tag{3}$$

Now (1) and (3) give $B(G,f) \geqslant (b-\alpha)/r \geqslant A_1/r$. In a similar way we find $B(G,f) \geqslant A_2/r$. Finally, let $\alpha' = \min\{i:\ u_i \in X\}, \beta' = \max\{j:\ u_j \in Y\}$. Then $\beta' - \alpha' \geqslant A_3$, and so $B(G,f) \geqslant A_3/r$. Theorem 1 follows. □

Theorem 1 implies several known results. First note that $D_-^{(1)}(S_k), D_+^{(1)}(S_k)$ and $D_0^{(1)}(S_k)$ are precisely the sets $D^-(S_k), D^+(S_k)$ and $D^0(S_k)$, respectively, defined in [8]. Theorem 1 gives

Corollary 1 (Li and Lin [8]). *For any labeling f of G,*

$$B(G,f) \geqslant \max_{1 \leqslant k \leqslant n} \max\{|D^-(S_k)|, |D^+(S_k)|, |D^0(S_k)| - 1\}.$$

For $S \subseteq V(G)$, let

$$B_1(S) = \lceil (|\partial^{(r)}(S)| + \delta_-^{(r)}(S) - 1)/r \rceil,$$

$$B_2(S) = \lceil (|\nabla^{(r)}(S)| + \delta_+^{(r)}(S) - 1)/r \rceil.$$

Since $\partial^{(r)}(S_k) \subseteq D_-^{(r)}(S_k)$ and $\nabla^{(r)}(S_k) \subseteq D_+^{(r)}(S_k)$, we have

Corollary 2. *For any labeling f of G and any integers r and k with $1 \leqslant r$ and $1 \leqslant k \leqslant n$,*

$$B(G,f) \geqslant \max\{B_1(S_k), B_2(S_k)\} \geqslant \max\{\lceil |\partial^{(r)}(S_k)|/r \rceil, \lceil |\nabla^{(r)}(S_k)|/r \rceil\}.$$

As in [10], an integer-valued function φ defined on subsets of $V(G)$ is called a *generalized weight function* if (i) $\varphi(\emptyset) = 0$ and (ii) $\varphi(S \cup \{x\}) = \varphi(S)$ or $\varphi(S) + 1$ for each $S \subseteq V(G)$ and $x \in \bar{S}$. For any such φ and integer k, let

$$M(\varphi,k) = \{S \subseteq V(G):\ \varphi(S) = k \text{ and } \varphi(S \backslash \{u\}) < k \text{ for at least one vertex } u \in S\}.$$

Theorem 2. *Let φ be a generalized weight function on subsets of $V(G)$. Then for any integers r and k with $1 \leqslant r$ and $1 \leqslant k \leqslant \varphi(V(G))$,*

$$B(G) \geqslant \min_{S \in M(\varphi,k)} \max\{B_1(S), B_2(S)\}.$$

Proof. For any labeling f, there exists ℓ such that $\varphi(S_\ell) = k$, and if ℓ is minimal then $\varphi(S_\ell \backslash \{u_\ell\}) < k$ and so $S_\ell \in M(\varphi,k)$. By Corollary 2,

$$B(G,f) \geqslant \max\{B_1(S_\ell), B_2(S_\ell)\} \geqslant \min_{S \in M(\varphi,k)} \max\{B_1(S), B_2(S)\}.$$

Since f was arbitrary, the result follows. □

Setting $r = 1$ in Theorem 2, we get

Corollary 3 (Yuan [15]). *For any generalized weight function φ,*

$$B(G) \geqslant \max_{1 \leqslant k \leqslant \varphi(V(G))} \min_{S \in M(\varphi,k)} \max\{|\partial(S)| + \delta_-(S) - 1, |\nabla(S)| + \delta_+(S) - 1\},$$

where $\delta_-(S) = \delta_-^{(1)}(S)$ and $\delta_+(S) = \delta_+^{(1)}(S)$.

Noting that $\delta_-(S) \geqslant 1, \delta_+(S) \geqslant 1$, we have

Corollary 4 (Lin [10]). *For any generalized weight function φ,*

$$B(G) \geqslant \max_{1 \leqslant k \leqslant \varphi(V(G))} \min_{S \in M(\varphi,k)} \max\{|\partial(S)|, |\nabla(S)|\}.$$

Setting $\varphi(S) = |S|$ in Corollary 4, we get Harper's classical result.

Corollary 5 (Harper [7]). $B(G) \geqslant \max_{1 \leqslant k \leqslant n} \min_{|S|=k} \max\{|\partial(S)|, |\nabla(S)|\}$.

For each $v \in V(G), \delta_-^{(r)}(v) = |\nabla^{(r)}(v)| = |\{u \in V(G) :\ 1 \leqslant d(u,v) \leqslant r\}|, \delta_+^{(r)}(v) = |\partial^{(r)}(v)| = 1$. So Theorem 2 implies

Corollary 6. $B(G) \geqslant \max_{1 \leqslant r \leqslant D(G)} \min_{v \in V(G)} \lceil |\nabla^{(r)}(v)|/r \rceil$. (4)

In particular, $B(G) \geqslant \lceil (|V(G)| - 1)/D(G) \rceil$ (Chvátal [4]), *where $D(G)$ is the diameter of G.*

This Corollary shows that Chvátal's density lower bound is Harper-type in a sense. The following example shows that the lower bound (4) is attainable and sometimes better than Chvátal's bound.

Example 1. Let $G = P_3 \times P_3$, the grid with 3 rows and 3 columns. Then by Chvátal's lower bound we get $B(G) \geqslant \lceil (9-1)/4 \rceil = 2$. But (4) gives $B(G) \geqslant \min_{v \in V(G)} \lceil |\nabla^{(2)}(v)|/2 \rceil = \lceil 5/2 \rceil = 3$. In fact, $B(G) = 3$.

3. Applications: the bandwidths of some composition graphs

Harper [7] found the exact value for the bandwidth of the n-cube Q_n by using Corollary 5. Among the graphs whose bandwidths are already known, several are in the form of graph products. The *(cartesian) product* of graphs G and H, written $G \times H$, is the graph whose vertex set is $V(G) \times V(H)$, two vertices (u,v) and (u',v') being adjacent if either $u = u'$ and v is adjacent to v' or vice versa.

Theorem 3. *Let P_n and C_n be the path and the cycle on n vertices, respectively. Then*
(i) *If* $\max\{r,s\} \geqslant 2$, *then* $B(P_r \times P_s) = \min\{r,s\}$ ([5]);
(ii) *If* $s \geqslant 3$, *then* $B(P_r \times C_s) = \min\{2r,s\}$ ([2]);
(iii) *If* $r,s \geqslant 3$, *then* $B(C_r \times C_s) = 2\min\{r,s\} - \delta_{r,s}$ ([9]), *where* $\delta_{r,s}$ *is the Kronecker delta.*

These results can be proved concisely by choosing the generalized weight function φ in Corollary 4 appropriately, as shown in [10]. Recently, Li and Lin ([8]) completely determined the bandwidth of the join of k arbitrary graphs by applying their boundary inequality (Corollary 1). Partial results on this problem can be found in [14] also.

Theorem 2 may be more convenient to use than Corollary 3, as shown in the proof of the following result.

Theorem 4 (Yuan [15]). *Let $G_{t,\ell}$ be any graph constructed from a cycle $C = v_1 v_2 \cdots v_t v_1$ and t pairwise disjoint graphs $G_1, \ldots, G_t$ of ℓ vertices each, by joining each vertex in G_i to v_i and $v_{i+1} (1 \leqslant i \leqslant t, v_{t+1} = v_1)$. Then $B(G) = h(\ell,t)$, where*

$$h(\ell,t) = 2\ell - \left\lfloor \frac{\ell + 1 - t}{\lceil t/2 \rceil} \right\rfloor .$$

Proof. We first obtain a lower bound for $B(G)$ from Theorem 2 by setting $r = \lceil t/2 \rceil$, $k = 1$ and $\varphi(S) = |S|$ for each $S \subseteq V(G)$. If t is odd, then

$$|\nabla^{(r)}(S)| = |V(G)| - 1 = t(\ell + 1) - 1 = 2\ell \lceil t/2 \rceil - (\ell + 1 - t)$$

for each S with $|S| = 1$, while if t is even then

$$|\nabla^{(r)}(S)| \geqslant |V(G)| - \ell - 1 = t(\ell + 1) - \ell - 1 = 2\ell \lceil t/2 \rceil - (\ell + 1 - t)$$

for each such S (with equality for some S). Noting that $\delta_+^{(r)}(S) \geqslant 1$, we find from Theorem 2 that

$$B(G) \geqslant \min_{|S|=1} \lceil |\nabla^{(r)}(S)|/r \rceil = h(\ell,t).$$

If each G_i in $G_{t,\ell}$ is complete, denote the latter by $G_{t,\ell}^*$. It was proved in [13] that $B(G_{t,\ell}^*) = h(\ell,t)$. But $B(G) \leqslant B(G_{t,\ell}^*)$ since $G \subseteq G_{t,\ell}^*$ and so it follows that $B(G) = h(\ell,t)$ as required.

For two graphs G and H, the *composition* $G[H]$ is the graph with vertex set $V(G)\times V(H)$ in which (u,v) is adjacent to (u',v') if u is adjacent to u' in G or $u=u'$ and v is adjacent to v' in H. The bandwidths of the compositions of certain graph powers were discussed in [3]. In what follows, we will use the lower bounds given in the previous section to determine the bandwidths of some composition graphs. For this purpose we need the following:

Theorem 5 (Chinn et al. [2]). $B(G[H])\leqslant(B(G)+1)|V(H)|-1$.

For each $u\in V(G)$, let $V_u=\{(u,v)\colon\ v\in V(H)\}\subseteq V(G[H])$. Then the subgraph H_u of $G[H]$ induced by V_u is isomorphic to H. Let

$$p\ :\ V(G[H])\rightarrow V(G)$$

be the projection defined by $p(x)=u$ for each $x=(u,v)$, and $p(S)=\{p(x)\colon\ x\in S\}$ for $S\subseteq V(G[H])$. For every generalized weight function φ on subsets of $V(G)$, define $\varphi^*(S)=\varphi(p(S))$ for $S\subseteq V(G[H])$. Then φ^* is a generalized weight function defined on subsets of $V(G[H])$. Also, for any $S\subseteq V(G[H])$,

$$|\nabla_{G[H]}(S)|\geqslant|\nabla_G(p(S))|\cdot|V(H)|+\delta(H), \tag{5}$$

where $\delta(H)$ is the minimum vertex degree in H.

Theorem 6. *Suppose both φ and μ are generalized weight functions defined on subsets of $V(G)$. Define*

$$\beta_G(A)=\max_{\mu(A)\leqslant j\leqslant\mu(V(G))}\ \min_{\substack{B\in M(\mu,j)\\ B\supseteq A}}|\nabla_G(B)|,$$

$$\eta_\varphi(G)=\max_{1\leqslant k\leqslant\varphi(V(G))}\ \min_{A\in M(\varphi,k)}|\nabla_G(A)|,$$

$$\eta_{\varphi,\mu}(G)=\max_{1\leqslant k\leqslant\varphi(V(G))}\ \min_{A\in M(\varphi,k)}\max\{|\nabla_G(A)|,\beta_G(A)\}.$$

Then

(i) $B(G[H])\geqslant\eta_\varphi(G)|V(H)|+\delta(H)$;

(ii) $B(G[H])\geqslant\eta_{\varphi,\mu}(G)|V(H)|+\delta(H)$.

Proof. Since (ii) implies (i), we prove only (ii). Choose k such that $1\leqslant k\leqslant\varphi(V(G))$ and let f be any labeling of $G[H]$. Since φ^* is a generalized weight function, there exists ℓ such that $\varphi^*(S_\ell)=k$, and if ℓ is minimal then $S_\ell\in M(\varphi^*,k)$. Thus Corollary 2 gives $B(G[H],f)\geqslant|\nabla_{G[H]}(S_\ell)|$ and (5) gives

$$B'(G,H,f)\geqslant|\nabla_G(p(S_\ell))|, \tag{6}$$

where $B'(G,H,f)$ is defined by $B(G[H],f)=B'(G,H,f)|V(H)|+\delta(H)$.

For each j with $\mu^*(S_\ell) \leqslant j \leqslant \mu(V(G))$, choose t minimal so that $\mu^*(S_t) = j$. Then $S_t \in M(\mu^*, j)$ and $S_t \supseteq S_\ell$, whence $p(S_t) \in M(\mu, j)$ and $p(S_t) \supseteq p(S_\ell)$. Corollary 2 gives $B(G[H], f) \geqslant \nabla_{G[H]}(S_t)|$, and so (5) gives

$$B'(G,H,f) \geqslant |\nabla_G(p(S_t))| \geqslant \min_{\substack{B \in M(\mu,j)\\ B \supseteq p(S_\ell)}} |\nabla_G(B)|. \tag{7}$$

Since j was arbitrary, it follows from this that $B'(G,H,f) \geqslant \beta_G(p(S_\ell))$, and together with (6) and the fact that $p(S_\ell) \in M(\varphi, k)$ this gives

$$B'(G,H,f) \geqslant \max\{|\nabla_G(p(S_\ell))|, \beta_G(p(S_\ell))\} \geqslant \min_{A \in M(\varphi,k)} \max\{|\nabla_G(A)|, \beta_G(A)\}.$$

The result now follows from the arbitrariness of k and f. □

Theorem 6 generalizes Theorem 2 of [3]. Combining Theorems 5 and 6, we get

Theorem 7. *If $B(G) = \eta_\varphi(G)$ or $\eta_{\varphi,\mu}(G)$ for generalized weight functions φ and μ, then*

$$B(G[K_m]) = (B(G)+1)m - 1.$$

This theorem can be applied to prove

Theorem 8. *Let G be the graph constructed from $(t+1)$ pairwise vertex-disjoint graphs $G_0, G_1, \ldots, G_t$ by joining each vertex of G_i to each vertex of $G_{i+1}, 0 \leqslant i \leqslant t-1$, where $|V(G_0)| = |V(G_t)| = 1$, $|V(G_1)| = \cdots = |V(G_{t-1})| = \ell$ and $t \geqslant 5$. Then $B(G[K_m]) = 2\ell m - 1$.*

Proof. It is easy to see that if $A \subseteq V(G)$ and $|A| = \ell + 2$, then $|\nabla_G(A)| \geqslant 2\ell - 1$; indeed, equality holds if and only if $A \subseteq G_0 \cup G_1 \cup G_2$ or $A \subseteq G_t \cup G_{t-1} \cup G_{t-2}$. Now Corollary 4, with $\varphi(A) = |A|$ for each A, implies that

$$B(G) \geqslant \eta_\varphi(G) \geqslant \min_{|A|=\ell+2} |\nabla_G(A)| \geqslant 2\ell - 1.$$

But it was proved in [15] that $B(G) = 2\ell - 1$, and so $B(G) = \eta_\varphi(G)$ and it follows from Theorem 7 that $B(G[K_m]) = 2\ell m - 1$. □

Theorem 7 also implies the following:

Theorem 9. (i) $B((P_r \times P_s)[K_m]) = m(\min\{r,s\}+1) - 1 \quad (\max\{r,s\} \geqslant 2)$;
(ii) $B((P_r \times C_s)[K_m]) = m(\min\{2r,s\}+1) - 1 \quad (s \geqslant 3)$;
(iii) $B((C_r \times C_s)[K_m]) = m(2\min\{r,s\} - \delta_{r,s} + 1) - 1 \quad (r,s \geqslant 3)$;
(iv) *If r,s are even and $r \geqslant s$, or r is odd and s is even, or both r and s are odd*

and $s \geqslant r$, *then*

$$B((K_r \times K_s)[K_m]) = \frac{s(r+1)}{2}m - 1;$$

$$\text{(v)}\ B(Q_n[K_m]) = m\left(1 + \sum_{i=0}^{n-1}\binom{i}{\left\lfloor \frac{i}{2} \right\rfloor}\right) - 1.$$

Proof. If G is one of $P_r \times P_s, P_r \times C_s, C_r \times C_s$, it was proved in [10] that $B(G) = \eta_\varphi(G)$ or $\eta_{\varphi,\mu}(G)$ for suitable generalized weight functions φ and μ. If $G = K_r \times K_s$, then $B(G) = \eta_{\varphi,\mu}(G)$ for some φ and μ (see [11]). If $G = Q_n$, then $B(G) = \eta_\varphi(G)$ for the function φ defined by $\varphi(S) = |S|$ [7]. Therefore, the result follows from Theorems 3 and 7 and the values of $B(G)$ given in Theorem 3 and [7,11]. □

Based on Theorem 9, we now endeavor to determine the bandwidths of $(P_r \times P_s)[H]$, $(P_r \times C_s)[H]$ and $(C_r \times C_s)[H]$ for arbitrary graph H. We take both $P_r \times P_s$ and $P_r \times C_s$ as defined on the set $V_{r,s} = \{(i,j){:}\ 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$ with the edge sets $E(P_r \times P_s) = \{(i,j)(i',j'){:}\ i = i' \text{ and } |j-j'| = 1, \text{ or } j = j' \text{ and } |i-i'| = 1\}$ and $E(P_r \times C_s) = E(P_r \times P_s) \cup \{(i,1)(i,s){:}\ 1 \leqslant i \leqslant r\}$. Denote $R_i = \{(i,j){:}\ 1 \leqslant j \leqslant s\}(1 \leqslant i \leqslant r)$ and $Q_j = \{(i,j){:}\ 1 \leqslant i \leqslant r\}(1 \leqslant j \leqslant s)$. For a subset A of the vertex set of $P_r \times P_r$, if

(i) $A \cap Q_1 = \{(i,1){:}\ 1 \leqslant i \leqslant r,\ i \neq r-1\}$, $A \cap R_r = \{(r,1)\}$, and

(ii) for $j = 1,2,\ldots,\{(i,j+1),(i+1,j+1),\ldots,(i+\alpha,j+1)\} \subseteq A$ implies $\{(\max\{1,i-1\},j),(\max\{1,i-1\}+1,j),\ldots,(i+\alpha,j),(i+\alpha+1,j)\} \subseteq A$, or if

(i′) $A \cap Q_1 = \{(i,1){:}\ 1 \leqslant i \leqslant r, i \neq 2\}, A \cap R_1 = \{(1,1)\}$, and

(ii′) for each $j = 1,2,\ldots,\{(i,j+1),(i+1,j+1),\ldots,(i+\alpha,j+1)\} \subseteq A$ implies $\{(i-1,j),(i,j),\ldots,(\min\{i+\alpha+1,r\},j)\} \subseteq A$,

then we say A is *descending* with respect to Q_1. Symetrically, we can define subsets A which are descending with respect to Q_r, R_1 or R_r. Note that if A is descending with respect to Q_1 or Q_r, then A intersects at most $r-2$ Q_j.

Lemma 1. *Suppose* $2 \leqslant r \leqslant s, A \subseteq V_{r,s}$, *and* $\nabla(A) = \nabla_{P_r \times P_s}(A)$.

(i) *If* $r < s$ *and there exists unique* Q_a *such that* $|A \cap Q_a| = r$, *then* $|\nabla(A)| \geqslant r$;

(ii) *If* $r = s$ *and there exists unique* Q_a *such that* $|A \cap Q_a| = r-1$, *then* $|\nabla(A)| \geqslant r$. *Furthermore, if* $A \cap Q_a$ *contains an isolated vertex of the subgraph* G_A *of* $P_r \times P_r$ *induced by* A, *then* $|\nabla(A)| = r$ *if and only if* A *is descending with respect to* Q_1 *or* Q_r.

Proof. We prove (ii) only since the proof of (i) is similar. Suppose $r = s$ and $|A \cap Q_a| = r-1$. If A intersects every Q_j, then clearly $|\nabla(A)| \geqslant r$. In the following suppose, say, $A \cap Q_{b+1} = \emptyset (a \leqslant b)$ and $A \cap Q_j \neq \emptyset$ for each $j, a \leqslant j \leqslant b$. We will prove $|\nabla(A) \cap (\bigcup_{j \geqslant a} Q_j)| \geqslant r$ by induction on b. The inequality is true if $b = a$. Suppose $b > a$ and $A' = A \backslash Q_b$. Let $Q_b' = \{(i,b) \in A \cap Q_b{:}\ (i,b-1) \in A\}, Q_b'' = (A \cap Q_b) \backslash Q_b'$.

Since $A \cap Q_{b+1} = \emptyset$, by the induction hypothesis we have

$$\begin{aligned}|\nabla(A) \cap (\bigcup_{j \geqslant a} Q_j)| &\geqslant (|\nabla(A') \cap (\bigcup_{j \geqslant a} Q_j)| - |Q'_b|) \\ &\quad + |(\nabla(Q_b'') \cap Q_{b-1}) \backslash \nabla(A \cap Q_{b-1})| \\ &\quad + |(\nabla(A \cap Q_b) \cap Q_b) \backslash \nabla(A \cap Q_{b-1})| + |A \cap Q_b| \\ &\geqslant (|\nabla(A') \cap (\bigcup_{j \geqslant a} Q_j)| - |Q'_b|) + |A \cap Q_b| \\ &\geqslant |\nabla(A') \cap (\bigcup_{j \geqslant a} Q_j)| \geqslant r. \end{aligned} \tag{8}$$

Thus, we have proved that $|\nabla(A) \cap (\bigcup_{j \geqslant a} Q_j)| \geqslant r$ and hence

$$|\nabla(A)| \geqslant |\nabla(A) \cap (\bigcup_{j \geqslant a} Q_j)| \geqslant r. \tag{9}$$

Note that if $|\nabla(A)| = r$, then from (8)–(9) we must have $a = 1$, $Q'_b = A \cap Q_b, \nabla(A \cap Q_b) \cap Q_b \subseteq \nabla(A \cap Q_{b-1}) \cap Q_b$, and $|\nabla(A')| = r$. Repeatedly using these facts we know if $|\nabla(A)| = r$ and $A \cap Q_1$ contains an isolated vertex of G_A, then A is descending with respect to Q_1. Similarly, if $A \cap Q_j = \emptyset$ for some $j < a$, and if $|\nabla(A)| = r$ and $A \cap Q_a$ contains an isolated vertex of G_A, then A is descending with respect to Q_r. Conversely, if A is descending with respect to Q_1 or Q_r, then $|\nabla(A)| = r$. This completes the proof. □

Lemma 2. *Suppose $A \subseteq V_{r,s}$ and $\nabla(A) = \nabla_{P_r \times C_s}(A)$.*
(i) *If $2r \geqslant s+1$ and there exists unique R_i with $|A \cap R_i| = s-1$, then $|\nabla(A)| \geqslant s$;*
(ii) *If $2r \leqslant s-1$ and there exists unique Q_j such that $|A \cap Q_j| = r$, then $|\nabla(A)| \geqslant 2r$.*

Proof. We prove (ii) only. Without loss of generality, we may suppose $|A \cap Q_1| = r$. If $A \cap Q_j \neq \emptyset$ for each j, then $|\nabla(A)| \geqslant s-1 \geqslant 2r$. In the following, suppose $A \cap Q_{b+1} = A \cap Q_{s-c} = \emptyset (1 \leqslant b \leqslant s-c-1)$, but $A \cap Q_j \neq \emptyset$ for each j with $1 \leqslant j \leqslant b$ or $s-c+1 \leqslant j \leqslant s$. If $b+c-1 = 0$, then $Q_2 \cup Q_s \subseteq \nabla(A)$ and hence $|\nabla(A)| \geqslant 2r$. If $b+c-1 = s-2$ (i.e., Q_{b+1} coincides with Q_{s-c}), then $|\nabla(A)| \geqslant (s-2)+1 \geqslant 2r$. If $1 \leqslant b+c-1 < s-2$, the induction on $b+c-1$ ensures that $|\nabla(A)| \geqslant 2r$. □

By using Lemmas 1 and 2 we can prove

Lemma 3. *Let $\overline{K}_m$ be the empty graph of order m.*
(i) *If $2 \leqslant r \leqslant s$, then $B((P_r \times P_s)[\overline{K}_m]) \geqslant m(r+1)-1$;*
(ii) *If $2r \neq s$, then $B((P_r \times C_s)[\overline{K}_m]) \geqslant m(\min\{2r,s\}+1)-1$.*

Proof. (i) Denote $G = P_r \times P_s$. Let f be a labeling of $G[\bar{K}_m]$ and $S_k = f^{-1}(\{1,2,\ldots,k\})$. Let p and V_u be defined as before.

Case 1: $r < s$. Let $a = \min\{k : \max_{1\leqslant j\leqslant s} |p(S_k) \cap Q_j| = r\}, A = p(S_a)$ and $u = p(f^{-1}(a))$. Then there is unique Q_j such that $|A \cap Q_j| = r$. By Lemma 1 (i) we have $|\nabla_G(A)| \geqslant r$. By the minimality of a, we have $V_u \cap S_a = \{f^{-1}(a)\}$. Note that u is adjacent to at least one vertex of A in G, so $V_u \backslash \{f^{-1}(a)\} \subseteq \nabla_{G[\bar{K}_m]}(S_a)$. Thus we get

$$B(G[\bar{K}_m], f) \geqslant |\nabla_{G[\bar{K}_m]}(S_a)| \geqslant m|\nabla_G(A)| + (m-1) \geqslant m(r+1) - 1.$$

Case 2: $r = s$. Let $a = \min\{k : \max_{1\leqslant j\leqslant r} |p(S_k) \cap Q_j| = r-1\}, A = p(S_a)$ and $u = p(f^{-1}(a))$. Then $|\nabla_G(A)| \geqslant r$ by Lemma 1 (ii), and $V_u \cap S_a = \{f^{-1}(a)\}$ by the minimality of a. If $|\nabla_G(A)| \geqslant r+1$ or u is adjacent to a vertex of A in G, then $B(G[\bar{K}_m], f) \geqslant m(r+1) - 1$, as we have just shown in Case 1. If $|\nabla_G(A)| = r$ and u is isolated in the subgraph G_A of G induced by A, then A is descending with respect to Q_1 or Q_r (Lemma 1(ii)). So A intersects at most $r-2$ Q_j. In such case we consider $b = \min\{k : \max_{1\leqslant i\leqslant r} |p(S_k) \cap R_i| = r-1\}$, $B = p(S_b)$ and $v = p(f^{-1}(b))$. Clearly, $a < b$ and $A \subset B$. So B intersects at least $r-1$ R_i. Thus, B is descending with respect to neither R_1 nor R_r. From Lemma 1(ii) we know either $|\nabla_G(B)| \geqslant r+1$ or v is not an isolated vertex of G_B. In both cases we get $B(G[\bar{K}_m], f) \geqslant |\nabla_{G[\bar{K}_m]}(S_b)| \geqslant m(r+1) - 1$. The result follows from the arbitrariness of f.

(ii) Let f be a labeling of $(P_r \times C_s)[\bar{K}_m]$. Let

$$a = \begin{cases} \min\{k : \max\limits_{1\leqslant i\leqslant r} |p(S_k) \cap R_i| = s-1\} & \text{if } 2r \geqslant s+1, \\ \min\{k : \max\limits_{1\leqslant j\leqslant s} |p(S_k) \cap Q_j| = r\} & \text{if } 2r \leqslant s-1. \end{cases}$$

Note that $p(f^{-1}(a))$ is adjacent to at least one vertex of $p(S_a)$ in $P_r \times C_s$. So we get from Lemma 2 that $B((P_r \times C_s)[\bar{K}_m], f) \geqslant m(\min\{2r, s\} + 1) - 1$. This completes the proof. □

Theorem 10. *Let H be any graph of order m. Then*

$$\text{(i) } B((P_r \times P_s)[H]) = \begin{cases} m + \max\left\{B(H), \left\lfloor \dfrac{m-1}{2} \right\rfloor\right\} & \textit{if } (r,s) = (1,2),(2,1), \\ m(\min\{r,s\}+1) - 1 & \textit{if } (r,s) \neq (1,1),(1,2),(2,1). \end{cases}$$

$$\text{(ii) } B((P_r \times C_s)[H]) = \begin{cases} \max\left\{B(H) + 2m, 3m - \left\lceil \dfrac{m+1}{2} \right\rceil\right\} & \textit{if } (r,s) = (1,3), \\ m(\min\{2r,s\}+1) - 1 & \textit{if } (r,s) \neq (1,3) \textit{ and } 2r \neq s. \end{cases}$$

Proof. (i) If $(r,s) = (1,2)$ or $(2,1)$, then the result follows from Proposition 3 of [8]. If $r = 1, s \geqslant 3$, or $r \geqslant 3, s = 1$, the result is exactly Corollary 5 of [3]. In general case, the result follows from Lemma 3(i), Theorem 9(i) and the fact that $(P_r \times P_s)[\bar{K}_m] \subseteq (P_r \times P_s)[H] \subseteq (P_r \times P_s)[K_m]$.

(ii) If $(r,s) = (1,3)$, then $(P_r \times C_s)[H]$ is the join of three copies of H and the result follows from Proposition 3 of [8]. If $r = 1$, $s \geqslant 4$, the result is just

Corollary 9 of [3]. If $r \geqslant 2$ and $2r \neq s$, then the result follows from Lemma 3(ii) and Theorem 9(ii). □

In a similar way we can prove $2rm - 1 \leqslant B((P_r \times C_{2r})[H]) \leqslant (2r+1)m - 1$ and $m(2\min\{r,s\} - \delta_{r,s}) - 1 \leqslant B((C_r \times C_s)[H]) \leqslant m(2\min\{r,s\} - \delta_{r,s} + 1) - 1$ for any graph H of order m. Unfortunately, these are insufficient to give exact values of the bandwidths of $(P_r \times C_{2r})[H]$ and $(C_r \times C_s)[H]$. Nevertheless, we have

Theorem 11. *Let H be any graph of m vertices.*

(i) *If $6 \leqslant 2r \leqslant s$, then* $B((C_r \times C_s)[H]) = m(2r+1) - 1;$ (10)

$$\text{(ii)}\ B((C_3 \times C_s)[H]) = \begin{cases} 7m - 1 & \text{if } s \geqslant 4, \\ 6m - 1 & \text{if } s = 3; \end{cases} \tag{11}$$

$$\text{(iii)}\ B((C_4 \times C_s)[H]) = \begin{cases} 9m - 1 & \text{if } s \geqslant 5, \\ 8m - 1 & \text{if } s = 4. \end{cases} \tag{12}$$

Proof. (i) The result follows from Theorems 9(iii) and 10(ii) and the fact that $(P_r \times C_s)[H] \subseteq (C_r \times C_s)[H] \subseteq (C_r \times C_s)[K_m]$.

(ii) Note that (11) has overlap with (10) when $s > 6$. Nevertheless, we will give an independent proof of (11) since it may shed light on determining $B((C_r \times C_s)[H])$ when $r \leqslant s \leqslant 2r$. From Theorem 9(iii) it suffices to show that

$$B((C_3 \times C_s)[\overline{K}_m]) \geqslant \begin{cases} 7m - 1 & \text{if } s \geqslant 4, \\ 6m - 1 & \text{if } s = 3. \end{cases}$$

Let $G = C_3 \times C_s$. For $S \subseteq V(G[\overline{K}_m])$, let $A = p(S)$ and A_0 be the set of isolated vertices of the subgraph of G induced by A. Then

$$|\nabla(S)| \geqslant m|\nabla(A)| \geqslant m|\nabla(A_0)|, \tag{13}$$

$$|\nabla(S)| \geqslant m(|\nabla(A)| + |A| - |A_0|) - (|S| - |A_0|), \tag{14}$$

where $\nabla(S) = \nabla_{G[\overline{K}_m]}(S), \nabla(A) = \nabla_G(A)$, etc. If $s \geqslant 4$, we consider S with $|S| = 3m+1$. Then $|A| \geqslant 4$ and by (13) and (14) one can check that $|\nabla(S)| \geqslant 7m - 1$. Hence $B(G[\overline{K}_m]) \geqslant \min_{|S|=3m+1} |\nabla(S)| \geqslant 7m - 1$ by Corollary 5. If $s = 3$, consider S with $|S| = m + 1$. A similar argument shows that $B(G[\overline{K}_m]) \geqslant 6m - 1$.

(iii) By an analogous discussion as above we can prove

$$B((C_4 \times C_s)[\overline{K}_m]) \geqslant \begin{cases} 9m - 1 & \text{if } s \geqslant 5, \\ 8m - 1 & \text{if } s = 4, \end{cases}$$

which implies (12). This completes the proof. □

A noticeable feature of Theorems 10 and 11 is that the bandwidths of the considered composition graphs have nothing to do with the structure of H. This is somewhat surprising. Theorem 10(i) is a generalization of Theorem 3(i), and Theorems 10(ii) and 11 are partial generalizations of Theorems 3(ii) and 3(iii), respectively. From the foregoing discussion we have a good reason to conjecture that $B((P_r \times C_{2r})[H]) = m(2r+1)-1$ and $B((C_r \times C_s)[H]) = m(2r-\delta_{r,s}+1)-1$ $(r \leqslant s \leqslant 2r)$ for any graph H on m vertices.

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