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Interpolation theorems for graphs, hypergraphs and matroids[1]

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Abstract

Let P and Q be, respectively, hereditary and cohereditary properties defined on the subsets of a finite set S. We first prove that several functions related to P and Q interpolate over some families of subsets of S. By using this we then derive a number of interpolation results for graphs, hypergraphs and matroids. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Let $\mathscr{F}$ be a family of objects under consideration and $\varphi : \mathscr{F} \to \mathbf{Z}$ an integer-valued function defined on $\mathscr{F}$. If for any $X, X' \in \mathscr{F}$ and integer k with $\varphi(X) < k < \varphi(X')$, there exists $X'' \in \mathscr{F}$ such that $\varphi(X'') = k$, then following [12] we say that φ *interpolates* over $\mathscr{F}$. Obviously, this is equivalent to saying that the image set $\varphi(\mathscr{F})$ consists of consecutive integers. The study on interpolation seems to be initiated by the homomorphism interpolation theorem for graphs [10]. In 1980, Chartrand [4] asked whether the number of pendant vertices interpolates over the family of spanning trees of a connected graph. With the affirmative answer to this question a lot of interpolation properties for some families of subgraphs of a given graph have been discovered in recent years. The reader can consult, for example [11,12,17,18,21,22]. The purpose of this paper is to extend the study on interpolation to some families of subsets of a finite set. This point of view enables us to generalize a number of interpolation results known for graphs to the cases of hypergraphs and matroids.

[1] Part of the work was done while the author was in The University of Hong Kong.

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2. An interpolation theorem for some functions related to hereditary and cohereditary properties

Let S be a finite set and P a property associated with the subsets of S. When a subset of S possesses the property P, call it a P-set, otherwise a $\overline{P}$-set. P is *hereditary* [6,13] if the subsets of any P-set are also P-sets. Dually, a property Q associated with the subsets of S is said to be *cohereditary* [6] if each superset of any Q-set is a Q-set. In this section we will always use P and Q to denote, respectively, a hereditary property and a cohereditary property. For $X \subseteq S$, a partition $\{X_1, \ldots, X_t\}$ of X is called a P-partition [6] (respectively Q-partition) of order t if each X_i is a P-set (respectively Q-set). It is clear that X has a P-partition if and only if $\{x\}$ is a P-set for each $x \in X$, or, equivalently, the P-sets contained in X cover X. In the following we always assume this is true. We also make the reasonable assumption that the empty set is a P-set. Thus any $X \subseteq S$ has a P-partition. We define the *P-chromatic number* of X, denoted by $\chi_P(X)$, to be the minimum order of a P-partition of X. As we will see later, this is a natural generalization of both conditional chromatic number and conditional chromatic index of a graph defined in [9]. Dually we call

$$\chi_Q(X) = \begin{cases} \max\{t\colon X \text{ has a } Q\text{-partition of order } t\}, & \text{if } X \text{ is a } Q\text{-set}, \\ 0, & \text{otherwise}, \end{cases}$$

the *Q-cochromatic number* of X.

Two other invariants related to hereditary property P were introduced in [13,3]. Denote by $\beta_P(X)$ the maximum cardinality of a P-set contained in X, and $\alpha_P(X)$ the minimum cardinality of a subset of X which has non-empty intersection with every $\overline{P}$-set contained in X. It was proved [3] that $\alpha_P(X) + \beta_P(X) = |X|$, which generalizes the classical equalities due to Gallai [7] and Hedetniemi [13].

Let φ be an integer-valued function defined on the subsets of S. If $\varphi(X) - 1 \leqslant \varphi(X - \{x\}) \leqslant \varphi(X)$ for any X and each $x \in X$, then we call φ a *positive function*, as used in [12] for graphical invariants. Similarly, if $\varphi(X) \leqslant \varphi(X - \{x\}) \leqslant \varphi(X) + 1$, we call φ a *negative function*. For two subsets X, X' of S, if $X' = X - \{x\}$ for an element $x \in X$ or $X = X' - \{x'\}$ for an element $x' \in X'$, then we call $X \to X'$ a *single element deletion or addition transformation* (EDA). If $X' = (X - \{x\}) \cup \{x'\}$ for some $x \in X - X'$ and $x' \in X' - X$, then we call $X \to X'$ a *single element transformation* (SET). A family $\mathscr{F}$ of subsets of S is said to be EDA-SET *connectable* if, for any $X, X' \in \mathscr{F}$, there exists a sequence $X = X_0, X_1, \ldots, X_n = X'$ with all terms in $\mathscr{F}$ and each $X_i \to X_{i+1}$ is either an EDA or a SET, $0 \leqslant i \leqslant n - 1$. The idea of the following simple but useful proposition was repeatedly used by many authors [11,12,17,18,21,22].

Proposition 1. *Suppose $\mathscr{F}$ is EDA-SET connectable and $N[X] = \{X\} \cup \{X' \in \mathscr{F}$: $X \to X'$ is an EDA or a SET$\}$. If φ interpolates over $N[X]$ for each $X \in \mathscr{F}$, then φ interpolates over $\mathscr{F}$. In particular, any positive or negative function interpolates over $\mathscr{F}$.*

In this section we will use this elementary observation to prove an interpolation theorem for the invariants defined above. First we have

Lemma 1. *The invariants χ_P, χ_Q, α_P and β_P are all positive.*

Proof. Suppose $x \in X \subseteq S, k = \chi_P(X)$ and $l = \chi_P(X - \{x\})$. If $\{X_1, \ldots, X_k\}$ is a P-partition of X and, say, $x \in X_1$, then $\{X_1 - \{x\}, X_2, \ldots, X_k\}$ is a P-partition of $X - \{x\}$ with order at most k. Thus, $l \leqslant k$. On the other hand, if $\{X_1, \ldots, X_l\}$ is a P-partition of $X - \{x\}$, then $\{\{x\}, X_1, \ldots, X_l\}$ is a P-partition of X since $\{x\}$ is a P-set. So we have $k \leqslant l + 1$ and hence χ_P is positive.

Now suppose $k = \chi_Q(X)$ and $l = \chi_Q(X - \{x\})$. If $k = 0$, then obviously $l = 0$. If $l = 0$, one can easily check that $k \leqslant 1$. In either case we get $k - 1 \leqslant l \leqslant k$. So we can suppose in the following that neither k nor l is zero. Let $\{X_1, \ldots, X_k\}$ be a Q-partition of X and $x \in X_1$. If $k = 1$, then evidently $l = 0$ or 1. If $k \geqslant 2$, then $\{(X_1 \cup X_2) - \{x\}, X_3, \ldots, X_k\}$ is a Q-partition of $X - \{x\}$ since Q is cohereditary. In both cases we get $k - 1 \leqslant l$. On the other hand, we have $l \leqslant k$ because any Q-partition of $X - \{x\}$ can be extended to a Q-partition of X by adding x to one block of it. So χ_Q is positive.

Let Y be a P-set contained in X with the maximum cardinality. If $x \notin Y$, then $Y \subseteq X - \{x\}$. If $x \in Y$, then $Y - \{x\}$ is a P-set contained in $X - \{x\}$. In either case we get $\beta_P(X) - 1 \leqslant \beta_P(X - \{x\})$. This, together with the obvious inequality $\beta_P(X - \{x\}) \leqslant \beta_P(X)$, ensures that β_P is positive. Combining this result with the equality $\alpha_P(Z) + \beta_P(Z) = |Z| (\forall Z \subseteq S)$ we know α_P is positive. This completes the proof. □

Let m, n be integers with $m \leqslant n$ and μ be a positive or negative function defined on the subsets of S. Let P' and Q' be hereditary and cohereditary properties associated with the subsets of S, respectively. Let

$$\mathscr{F}_1 = \{X \subseteq S\colon |X| = m\},$$

$$\mathscr{F}_2 = \{X \subseteq S\colon m \leqslant |X| \leqslant n\},$$

$$\mathscr{F}_3 = \{X \subseteq S\colon \mu(X) \leqslant n\},$$

$$\mathscr{F}_4 = \{X \subseteq S\colon \mu(X) \geqslant n\},$$

$$\mathscr{F}_5 = \{X \subseteq S\colon X \text{ is a } P'\text{-set}\},$$

$$\mathscr{F}_6 = \{X \subseteq S\colon X \text{ is a } Q'\text{-set}\}.$$

It can be proved that

Lemma 2. *Each $\mathscr{F}_i$ is EDA-SET connectable, $1 \leqslant i \leqslant 6$.*

Combining Proposition 1 with Lemmas 1 and 2 we get

Theorem 1. *Let $\mathscr{F}$ be a family of subsets of S.*
(1) *If $\mathscr{F}$ is EDA-SET connectable, then χ_P, χ_Q, α_P and β_P all interpolate over $\mathscr{F}$.*

(2) *Any positive or negative function interpolates over* $\mathscr{F}_i, 1 \leqslant i \leqslant 6$.
(3) *In particular,* χ_P, χ_Q, α_P *and* β_P *all interpolate over* $\mathscr{F}_i, 1 \leqslant i \leqslant 6$.

3. Applications: Interpolation theorems for graphs, hypergraphs and matroids

Indeed, Theorem 1 is based on a very simple idea. Nevertheless, it has wide applicability. Theoretically, it can be applied to any finite mathematical structure and interesting interpolation results can be derived by setting P and Q to be, respectively, hereditary and cohereditary properties associated with the substructures of the given structure. In this section we exemplify some results of such kind for graphs, hypergraphs and matroids.

3.1. Matroids and greedoids

First we have

Corollary 1. *Let* $M=(S, \mathscr{I})$ *be a matroid on a finite set* S *and* $\mathscr{B}$ *and* $\mathscr{B}^*$ *the sets of bases and co-bases of* M*, respectively. Then for any hereditary property* P *and cohereditary property* Q *associated with the subsets of* S, χ_P, χ_Q, α_P *and* β_P *all interpolate over* $\mathscr{B}$, $\mathscr{B}^*$ *and* $\mathscr{I}$.

Proof. The connectedness of the matroid base graph [14] implies the SET-connectability of $\mathscr{B}$ and $\mathscr{B}^*$. From this and the fact that the property of being independent set is hereditary we know that $\mathscr{I}$ is EDA-SET connectable. The result then follows from Theorem 1(1). □

The similar result is true for greedoids (see [16] for terminology and notation for greedoids). Korte and Lovász [15] proved that the base graph of every 2-connected greedoid is connected. So we get

Corollary 2. *Let* $(S, \mathscr{I})$ *be a greedoid with base set* $\mathscr{B}$ *and the set* $\mathscr{B}^*$ *of base complements. Let* P *and* Q *be hereditary and cohereditary properties, respectively, associated with the subsets of* S*. If* $(S, \mathscr{I})$ *is 2-connected, then* χ_P, χ_Q, α_P *and* β_P *all interpolate over* $\mathscr{B}$, $\mathscr{B}^*$ *and* $\mathscr{I}$.

It was proved [12,21] that the edge chromatic number χ', edge independence number β' and edge covering number α' interpolate over the family of spanning trees of a connected graph. Corollaries 1 and 2 generalize these results to the cases of matroids and greedoids.

3.2. Graphs

Theorem 1 and Corollary 1 are particularly applicable to graphs. For a graphical property P, if every edge-induced subgraph of any graph possessing P has P as well, then

P is said to be *edge-induced hereditary* [13]. Dually, we call a graphical property Q *edge-induced cohereditary* if whenever G is an edge-induced subgraph of G' and G has Q then G' has Q as well. For a given graph $G=(V(G),E(G))$ with order $p=|V(G)|$ and size $q=|E(G)|$, we set $S=E(G)$ and define $\chi'_P(G[X])=\chi_P(X), \chi'_Q(G[X])=\chi_Q(X)$, etc., for each $X\subseteq E(G)$. Then the concepts of positive and negative invariants coincide with that defined in [12] and we can view P and Q as hereditary and cohereditary properties defined on the subsets of $E(G)$. Note that $\chi'_P(G[X])$ is just the P-chromatic index [9] of $G[X]$ and $\alpha'_P(G[X])$ and $\beta'_P(G[X])$ are, respectively, the invariants $\alpha_1(P)$ and $\beta_1(P)$ defined in [13]. For each integer h with $p-\omega\leqslant h\leqslant q$ (where ω is the number of components of G), the family $C_h(G)$ of spanning h-edge subgraphs of G with exactly ω components is the set of bases of a matroid on $E(G)$. In fact, this matroid is just the elongation [19] of the cycle matroid of G to height h and the set of bases of its dual matroid is $C_h^*(G)=\{G-E(C)\colon C\in C_h(G)\}$. In particular, if G is connected then $C_{p-1}(G)$ and $C_{p-1}^*(G)$ are exactly the set of spanning trees and the set of cotrees of G, respectively. A graphical property P' is said to be *spanning hereditary* if a graph has P' implies all spanning subgraphs of it have P' as well. The *spanning cohereditary property* is understood in a similar way. From Theorem 1(2) and (3) and Corollary 1 we get

Corollary 3. *Let P' and Q' be spanning hereditary and spanning cohereditary properties, respectively. Then any positive or negative invariant interpolates over the following families of subgraphs of G:*

(a) *The family of spanning m-edge subgraphs.*
(b) *The family of spanning subgraphs having at least m and at most n edges.*
(c) *The family of spanning subgraphs having the property P'.*
(d) *The family of spanning subgraphs having the property Q'.*
(e) *The family $C_h(G)$ of spanning h-edge subgraphs with the same number of components as G.*
(f) *The family $C_h^*(G)$ of spanning subgraphs whose complements with respect to G belong to $C_h(G)$.*

In particular, for any edge-induced hereditary property P and edge-induced cohereditary property Q, χ'_P, χ'_Q, α'_P and β'_P all interpolate over the families (a)–(f).

Corollary 3 implies a number of interesting results of which some are known and the others are new. For given integer l and L, let P' (respectively Q') be the property of being graphs with maximum (resp. minimum) degree at most L (resp. at least l). Then (c) (resp. (d)) is just the family of spanning subgraphs of G with maximum (resp. minimum) degree $\leqslant L$ (resp. $\geqslant l$) discussed in [12]. As noticed in [12,21] there are a number of invariants, such as connectivity, edge-connectivity, independence number, edge-independence number, edge-covering number, domination number, etc., which are either positive or negative. So we get the results of Corollary 1a, Theorem 4 and Corollary 4a of [12] immediately from Corollary 3.

Table 1

P	χ'_P
Planarity	Thickness
Acyclicity	Arboricity
With at most one cycle	Unicyclicity [8]
Being linear forests	Linear arboricity [8]
Without paths of length n	n-edge chromatic number [22]
Without odd cycles	Biparticity [8]

Table 2

Q	χ'_Q
Nonplanarity	Coarseness
Non-acyclicity	Anarboricity [8]
Being graphs other than paths	Unpath number (apathy, [8])

As shown in [8,9,22], a number of invariants can be expressed as χ'_P or χ'_Q. We list some of them in Tables 1 and 2. From Corollary 3 we have

Corollary 4. *All the invariants listed in Tables* 1 *and* 2 *interpolate over the families* (a)–(f) *in Corollary* 3.

If we take $S = V(G)$, then the concepts of positive and negative function differ from that used in [12]. In such case we use the terminologies of *vertex positive invariant* and *vertex negative invariant* in the obvious way. A graphical property P is *induced hereditary* [13] if the vertex-induced subgraphs of any graph possessing P also possess P. We have the dual notation of *induced cohereditary property* Q. Call $X \subseteq V(G)$ a P-set (resp. Q-set) if $G[X]$ possesses P (resp. Q). Define $\chi_P(G[X]) = \chi_P(X), \chi_Q(G[X]) = \chi_Q(X)$, and so on. Then $\chi_P(G[X])$ is the P-chromatic number [9]. If P is the property of being edgeless graphs, then α_P and β_P are just the ordinary vertex covering number and vertex independence number. If we define the Q-sets to be dominating sets of a graph G, then Q is cohereditary and the domatic number [20] $d(G) = \chi_Q(V(G))$. Similarly, the total domatic number [20], f-domatic number [23] and total f-domatic number [23] can be expressed as χ_Q. If P is the property of being an irredundant set, then it is hereditary and the upper irredundance number [5] $\mathrm{IR}(G) = \beta_P(V(G))$. We assert from Theorem 1 that $\chi_P, \chi_Q, \alpha_P, \beta_P$ and particularly all the invariants mentioned above interpolate over, say, the family of m-vertex subgraphs of G.

3.3. Hypergraphs

We refer to [1] for the terminologies and results on hypergraphs, but repeat some basic definitions used in the following.

A *hypergraph* $H=(V,\mathscr{E})$ is a finite vertex set V together with a family $\mathscr{E}$ of subsets, called the edges, of V. Here we do not require that $\bigcup_{e\in\mathscr{E}} e=V$. If $\mathscr{C}\subseteq\mathscr{E}$, we call $(V,\mathscr{C})$ a *partial hypergraph* of H generated by $\mathscr{C}$ and we write $H-e=(V,\mathscr{E}-\{e\})$ for $e\in\mathscr{E}$. If all the edges of H have the same cardinality r, then H is *r-uniform*. $\mathscr{M}\subseteq\mathscr{E}$ is a *matching* if the edges in $\mathscr{M}$ are pairwise disjoint. The *matching number* $\nu(H)$ of H is the maximum number of edges in a matching of H. The *transversal number* $\tau(H)$ is the minimum cardinality of a set of vertices of H which intersects every edge of H. A set of vertices of H is a *stable set* of H if it contains no edge of cardinality great than one. The *stability number* $\beta(H)$ is the maximum cardinality of a stable set of H. The *chromatic number* $\chi(H)$ of H is the minimum order of a partition of its vertices into stable sets. The *degree* $d_H(x)$ of a vertex x is the maximum number of edges different from $\{x\}$ whose pairwise intersections are exactly $\{x\}$. Let $\delta(H)$ and $\Delta(H)$ be the minimum and the maximum degrees of the vertices of H, respectively. Let $r(H)=\max\{|e|: e\in\mathscr{E}\}$ be the rank of H and r an integer with $1\leqslant r\leqslant r(H)$. $X\subseteq V$ is called a *clique* of rank r if either $|X|<r$ or $|X|\geqslant r$ and each subset of X with cardinality r is contained in at least one edge of H. Denote by $\omega_r(H)$ the maximum cardinality of a clique of H with rank r.

Lemma 3. (1) *The invariants* ν, τ, χ, δ *and* Δ *are positive and* β *is negative (with respect to edges).*

(2) *If* H *is r-uniform, then* $\omega_r(H)-1\leqslant\omega_r(H-e)\leqslant\omega_r(H)$ *holds for each edge* e *of* H.

Proof. (1) As an example we prove the positiveness of δ and Δ. Obviously, we have $d_{H-e}(x)\leqslant d_H(x)$ for any vertex x and edge e of H. Let $e_1,\ldots,e_k$ be the edges of H with $k=d_H(x), e_i\neq\{x\}$ and $e_i\cap e_j=\{x\}$, $i\neq j$. If $e\neq e_i$ for each e_i, then we get $d_H(x)\leqslant d_{H-e}(x)$. Otherwise let, say, $e=e_1$. Then $e_2,\ldots,e_k$ are edges of H which intersect pairwise at $\{x\}$. Anyway we have $d_H(x)-1\leqslant d_{H-e}(x)\leqslant d_H(x)$, implying the positiveness of both δ and Δ.

(2) Let X be a maximum clique of rank r. If $|X|<r$, then X is also a clique of $H-e$ of rank r. If $|X|\geqslant r$ and $|X\cap e|<r$, then any subset of X with cardinality r cannot be contained in e and hence must be contained in an edge of $H-e$. In such case X is also a clique of $H-e$. If $|X\cap e|=r$, then $e\subseteq X$ and $X-\{x\}$ is a clique of $H-e$ for each $x\in e$. In all cases we get $\omega_r(H)-1\leqslant\omega_r(H-e)$. This, together with the evident inequality $\omega_r(H-e)\leqslant\omega_r(H)$, completes the proof of (2).

We observe that if P is the property associated with the subsets of $\mathscr{E}$ such that $\mathscr{C}$ is a P-set if and only if the edges in $\mathscr{C}$ are pairwise disjoint, then P is hereditary and $\nu(H)=\beta_P(H)$. If P is such that $X\subseteq V$ is a P-set if and only if X is a stable set of H, then $\beta(H)=\beta_P(H), \chi(H)=\chi_P(H)$. If P is the property such that the P-sets are cliques with rank r, then $\omega_r(H)=\beta_P(H)$. In general, for any hereditary property P and cohereditary property Q associated with the subsets of $\mathscr{E}$, we define $\chi'_P((V,\mathscr{C}))=\chi_P(\mathscr{C}), \chi'_Q((V,\mathscr{C}))=\chi_Q(\mathscr{C})$, and so on, for partial hypergraphs $(V,\mathscr{C})$ of H. We call a hypergraphical property P' *partially hereditary* if a hypergraph has P'

implies any partial hypergraph of it has P'. The *partial cohereditary property* is understood in the dual way. Note that these concepts coincide with that of spanning hereditary and spanning cohereditary properties in Section 3.2 when P' and Q' are graphical properties. Combining Theorem 1 and Lemma 3 we obtain

Corollary 5. *For any hereditary property P and cohereditary property Q associated with the subsets of $\mathscr{E}$, the invariants χ'_P, χ'_Q, α'_P, β'_P, ν, τ, χ, δ, Δ and β all interpolate over the following families of partial hypergraphs of $H=(V,\mathscr{E})$:*

(a) *The family of m-edge partial hypergraphs.*
(b) *The family of partial hypergraphs having at least m and at most n edges.*
(c) *The family of partial hypergraphs having a given partially hereditary property P'.*
(d) *The family of partial hypergraphs having a given partially cohereditary property Q'.*

Moreover, if H is r-uniform, then ω_r interpolates over all these families as well.

As examples we mention two familiar hereditary properties, namely the property P_1 such that $\mathscr{C}\subseteq\mathscr{E}$ is a P_1-set if and only if $\mathscr{C}$ has a system of distinct representatives (SDR) and the property P_2 such that $\mathscr{C}$ is a P_2-set if and only if $(V,\mathscr{C})$ is an l-Helly hypergraph. Here $(V,\mathscr{C})$ is an l-Helly hypergraph [2] if $\bigcap_{e\in\mathscr{C}'} e\neq\emptyset$ for any $\mathscr{C}'\subseteq\mathscr{C}$ satistying the condition that any l edges (not necessarily distinct) of $\mathscr{C}'$ have a nonempty intersection. Thus, a 2-Helly hypergraph is just a Helly hypergraph in the usual sense. Note that $\chi_{P_1}(\mathscr{C})$ is the minimum order of a partition of $\mathscr{C}$ into subsets each with a SDE and $\beta_{P_1}(\mathscr{C})$ is the maximum number of edges in a subset of $\mathscr{C}$ which has a SDR. The invariants χ_{P_2} and β_{P_2} can be accordingly interpreted. From Corollary 5, $\chi'_{P_i}, \alpha'_{P_i}, \beta'_{P_i}$, $i=1,2$, all interpolate over the families (a)–(d).

4. Concluding remarks

A P-partition $\{X_1,\ldots,X_t\}$ of $X\subseteq S$ is called *complete* [6] if $X_i\cup X_j$ is a $\overline{P}$-set for each pair $i,j, i\neq j$. The maximum order of a complete P-partition of X is the *P-achromatic number* [6], denoted by $\psi_P(X)$. As a generalization to the homomorphism interpolation theorem for graphs it was proved in [6] that for each k between $\chi_P(X)$ and $\psi_P(X)$ there exists a complete P-partition of X with order k. What is the counterpart of this result in the case of Q-partition?

Corollaries 1, 2 and 5 are not in their fullest version and one can derive more interpolation results from Theorem 1. For example, consider a matroid $M=(S,\mathscr{J})$ and a matroid invariant μ. Then the positiveness of μ means $\mu(M|X)-1\leqslant\mu(M|(X-\{x\}))\leqslant\mu(M|X)$ for any $X\subseteq S$ and $x\in X$, where $M|X$ is the restriction [19] of M to X, similarly for the negativeness of μ. (We mention that the rank function and in general any submodular function μ defined on the subsets of S satisfying $\mu(x)-\mu(\emptyset)\leqslant 1$, $x\in S$, is positive.). Let $\mathscr{F}(M,\mu,n^-)=\{X\subseteq S:\ \mu(M|X)\leqslant n\}$ and $\mathscr{F}(M,\mu,n^+)=\{X\subseteq S:\ \mu(M|X)\geqslant n\}$. Then we know from Theorem 1 that if φ and μ are positive or negative,

then φ interpolates over both $\mathscr{F}(M,\mu,n^-)$ and $\mathscr{F}(M,\mu,n^+)$. In particular, χ_P, χ_Q, α_P and β_P interpolate over these two families, where P and Q are as in Corollary 1.

Finally, although we discuss finite structures only the idea used can be applied to infinite sets and mathematical structures.

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