ELSEVIER

Discrete Mathematics 199 (1999) 291–297

DISCRETE MATHEMATICS

Note

A sequential coloring algorithm for finite sets

Sanming Zhou *,[1]

Department of Mathematics, The University of Western Australia, Nedlands, Perth, WA 6907, Australia

Received 29 May 1997; revised 14 September 1998; accepted 21 September 1998

Abstract

Let X be a finite set and P a hereditary property associated with the subsets of X. A partition of X into n subsets each with property P is said to be a P-n-coloring of X. The minimum n such that a P-n-coloring of X exists is defined to be the P-chromatic number of X. In this paper we give a sequential coloring algorithm for P-coloring X. From the algorithm we then get a few upper bounds for the P-chromatic number. In particular, we generalize the Welsh–Powell upper bound for ordinary chromatic number to the case of P-chromatic number of any finite set X.

Keywords: Finite set; Hereditary property; Sequential coloring; Conditional chromatic number; Algorithm

1. Introduction

A simple but useful way of coloring the vertices of a graph G is as follows. First order the vertices, say $x_1 \prec \cdots \prec x_p$, and then color them one by one: color x_1 with 1, then color x_2 with 1 if x_1 and x_2 are not adjacent and 2 otherwise, and so on; color each vertex with the smallest color it can have at that stage (see [18] for techniques based on this method). In this paper, we will show how the similar method works for conditional colorings of a finite set.

Let X be a finite set whose elements are called *vertices*. Let P be a property associated with the subsets of X. If $Y \subseteq X$ possesses P, we call Y a *P-set*; otherwise a *$\overline{P}$-set*. In the following we always suppose that P is *hereditary* in the sense that whenever Y is a P-set then each subset of Y is also a P-set. We also suppose without mentioning that each singleton $\{x\}$ for $x \in X$ is a P-set. A *P-n-coloring* of X is an assignment of n colors $1,2,\ldots,n$ to the vertices of X such that for each color i the

* E-mail: smzhou@maths.uwa.edu.au.

[1] Supported by an OPRS of the Australian Department of Education, Employment and Training and a UPA from The University of Western Australia.

0012-365X/99/$ – see front matter
PII: S0012-365X(98)00340-9

subset of all vertices colored with i is a P-set. Equivalently, a P-n-coloring of X is a partition of X into n P-sets. This concept of P-coloring of a finite set can date back to [7] when Cockayne, Miller and Prins proved the interpolation theorem for complete P-n-colorings. In [20], the *P-chromatic number* of X, denoted by $\chi_P(X)$, was defined to be the minimum n such that a P-n-coloring of X exists. Since each $\{x\}$ is a P-set, $\chi_P(X)$ is well-defined. If Y is a $\overline{P}$-set but $Y \setminus \{x\}$ is a P-set for each $x \in Y$, then we say that Y is a *minimal $\overline{P}$-set*. For each $x \in X$, we define the *P-degree* of x in X, denoted by $d_P(x,X)$, to be the largest number of members in a family $\mathscr{A}_x$ of minimal $\overline{P}$-sets of X with the property that $A \cap A' = \{x\}$ for any distinct $A, A' \in \mathscr{A}_x$. Let $\delta_P(X)$ and $\Delta_P(X)$ be, respectively, the *minimum P-degree* and the *maximum P-degree* of the vertices of X. Note that when restricted to a subset Y of X, P is also a hereditary property associated with the subsets of Y. Thus, $\chi_P(Y)$, $d_P(x,Y)$, $\delta_P(Y)$ and $\Delta_P(Y)$ are well-defined, where $x \in Y$. If $\chi_P(X \setminus \{x\}) < \chi_P(X)$ for each $x \in X$, then X is said to be *P-critical*. We use p to denote the cardinality of X.

The main purpose of this paper is to give an algorithm for sequentially P-coloring the finite set X. By using this algorithm we derive several upper bounds for $\chi_P(X)$. In particular, we generalize the well-known Welsh-Powell upper bound [3,17] for ordinary chromatic number to the case of P-chromatic number of any finite set. From this we can get upper bounds for some known graphical invariants such as the arboricity, vertex arboricity, and so on. Also, the upper bounds obtained for P-chromatic number imply a theorem of Tomescu [16] and an upper bound [2, Corollary 1, p. 117] for the chromatic number of a hypergraph.

2. The algorithm and its consequences

Algorithm SC(O)

(1) Suppose $O : x_1 \prec \cdots \prec x_p$ is a given order of the vertices of X. Color x_1 with color 1.

(2) Generally suppose $x_1, \ldots, x_{i-1}$ have been colored, let $X_i = \{x_1, \ldots, x_{i-1}, x_i\}$. If X_i is a P-set, then color x_i with any color that has been used already; otherwise X_i contains minimal $\overline{P}$-sets and we define

$\mathscr{A}_i = \{A \subseteq X_i : A$ is a minimal $\overline{P}$-set containing x_i and $A \setminus \{x_i\}$ is monochromatically colored$\}$,

$\mathscr{B}_i = \{B \subseteq X_i : B$ is a minimal $\overline{P}$-set containing x_i and $B \setminus \{x_i\}$ is not monochromatic$\}$, and

$\mathscr{C}_i = \{C \subseteq X_i : C$ is a minimal $\overline{P}$-set not containing $x_i\}$.

(3) If there exists a color used by a vertex in $\cup_{C \in \mathscr{C}_i} C$ but not used by any vertex in $\cup_{Y \in \mathscr{A}_i \cup \mathscr{B}_i} (Y \setminus \{x_i\})$, then assign this color to x_i.

(4) If all colors used in $\cup_{C \in \mathscr{C}_i} C$ are also used in $\cup_{Y \in \mathscr{A}_i \cup \mathscr{B}_i} (Y \setminus \{x_i\})$, then we distinguish two cases:

(4a) If there is a color used in $\cup_{B \in \mathscr{B}_i} B$ but not in $\cup_{A \in \mathscr{A}_i} A$, then assign this color to x_i;

(4b) otherwise, assign x_i a new color.

(5) If $i = p$, stop; otherwise set $i := i + 1$ and go to (2).

Inductively, one can show that the subset of the vertices in X_i with any one color contains no minimal $\overline{P}$-sets. Therefore, the algorithm SC(O) gives rise to a P-coloring of X for any given order O. The algorithm is justified also by the fact that it gives an optimal P-coloring (that is, a P-coloring using $\chi_P(X)$ colors) for some order O of X. In fact, let $n = \chi_P(X)$ and $\{A_1,\ldots,A_n\}$ be a P-n-coloring of X. Let us order the vertices of X in such a way that $x \prec y$ whenever $x \in A_i$, $y \in A_j$ with $i < j$. For such an order $O_M : x_1 \prec \ldots \prec x_p$ one can prove by induction on j that if $x_j \in X_i$, then SC(O_M) uses at most i colors for $\{x_1,\ldots,x_j\}$. Thus, SC(O_M) exploits at most n colors for coloring all the vertices in X. By the minimality of n, SC(O_M) gives a P-n-coloring of X.

Let O be the given order as in the algorithm. In applying SC(O), we denote $ds_1(x) = 0$ for all $x \in X$ and, if $x_1,\cdots,x_{i-1}$ have been colored, define $ds_i(x)$ to be the number of distinct colors that have been assigned to vertices in $\cup(A \setminus \{x\})$, where the union is taken over all minimal $\overline{P}$-sets $A \subseteq \{x_1,\ldots,x_{i-1},x\}$ such that $x \in A$ and $A \setminus \{x\}$ is monochromatic. Define

$$\mathrm{DS}(O) = \max_{1\leqslant i\leqslant p} \min\{i, ds_i(x_i)+1\} = \max_{1\leqslant i\leqslant p} ds_i(x_i)+1,$$

$$\mathrm{B}(O) = \max_{1\leqslant i\leqslant p} \min\{i, d_P(x_i,X_i)+1\} = \max_{1\leqslant i\leqslant p} d_P(x_i,X_i)+1,$$

$$\mathrm{WP}(O) = \max_{1\leqslant i\leqslant p} \min\{i, d_P(x_i,X)+1\}.$$

Note that if $A,A' \subseteq \{x_1,\ldots,x_{i-1},x\}$ are minimal $\overline{P}$-sets containing x such that $A \setminus \{x\}, A' \setminus \{x\}$ are monochromatic and A,A' contain a common vertex other than x, then the colors used for $A \setminus \{x\}, A' \setminus \{x\}$ are the same. Thus, $ds_i(x_i) \leqslant d_P(x_i,X_i)$. This, together with $d_P(x_i,X_i) \leqslant d_P(x_i,X)$, implies that $\mathrm{DS}(O) \leqslant \mathrm{B}(O) \leqslant \mathrm{WP}(O)$. From the algorithm SC($O$) we get the following upper bounds for P-chromatic number.

Theorem 1. *Let P be a hereditary property associated with the subsets of a finite set X. Then for any order O of the vertices of X, we have*

$$\chi_P(X) \leqslant \mathrm{DS}(O).$$

In particular, we have

$$\chi_P(X) \leqslant \mathrm{B}(O),$$

and

$$\chi_P(X) \leqslant \mathrm{WP}(O).$$

Proof. It suffices to prove that SC(O) uses at most DS(O) colors for coloring all the vertices in X. Let c_i be the number of colors used in coloring $X_i = \{x_1,\ldots,x_i\}$

by $SC(O)$. We prove $c_i \leqslant DS(O)$, $1 \leqslant i \leqslant p$, by induction on i. Clearly, $c_i \leqslant i$. If (3) or (4a) happens, then $c_i = c_{i-1} \leqslant DS(O)$ by the induction hypothesis. If (4b) occurs, then $c_i \leqslant |\mathscr{A}_i'|$, where $\mathscr{A}_i' \subseteq \mathscr{A}_i$ is such that for distinct $A, A' \in \mathscr{A}_i'$ the colors used for $A \setminus \{x_i\}, A' \setminus \{x_i\}$ are different and $\mathscr{A}_i'$ is maximal with respect to this property. Thus, $c_i \leqslant c_{i-1} + 1 \leqslant |\mathscr{A}_i'| + 1 \leqslant ds_i(x_i) + 1 \leqslant DS(O)$. This completes the proof. □

If $x_i \prec x_j$ in the order O and $d_P(x_i, X) < d_P(x_j, X)$, then we can exchange the positions of x_i and x_j and thus get a new order O'. It can be verified that $WP(O') \leqslant WP(O)$. So $WP(O)$ is minimized when X is ordered as $O_{WP} : x_1 \prec \cdots \prec x_p$ such that $d_P(x_1, X) \geqslant \cdots \geqslant d_P(x_p, X)$. From Theorem 1, we have

Corollary 1. *Suppose* $X = \{x_1, \ldots, x_p\}$ *with* $d_P(x_1, X) \geqslant \cdots \geqslant d_P(x_p, X)$. *Then*

$$\chi_P(X) \leqslant \max_{1 \leqslant i \leqslant p} \min\{i, d_P(x_i, X) + 1\}.$$

Note that if X is the vertex set of a graph G and P is the property of being a vertex independent set of G, then $\chi_P(X)$ is the ordinary chromatic number $\chi(G)$ of G and, for each $x \in X$, the P-degree $d_P(x, X)$ is just the degree $d(x)$ of x in G. So Corollary 1 is a generalization of the well-known Welsh–Powell bound [3,17]: If the vertices $x_1, x_2, \ldots, x_p$ of a graph G are indexed in such a way that $d(x_1) \geqslant \ldots \geqslant d(x_p)$, then

$$\chi(G) \leqslant \max_{1 \leqslant i \leqslant p} \min\{i, d(x_i) + 1\}.$$

An *n-coloring* of a hypergraph H [2] is a partition of the set of vertices of H into n subsets each contains no edges with cardinality greater than one. The *chromatic number* $\chi(H)$ of H [2] is the minimum n such that an n-coloring of H exists. The *β-degree* $d_H^{\beta}(x)$ of x in H [2] is the maximum number of edges different from $\{x\}$ whose pairwise intersections are $\{x\}$. Theorem 1 implies also the following theorem of Tomescu which is stronger than Corollary 1.

Corollary 2. (Tomescu [16]). *Let* $\{A_1, \ldots, A_n\}$ *be an n-coloring of a hypergraph H, and* $d_i = \max_{x \in A_i} d_H^{\beta}(x), 1 \leqslant i \leqslant n$. *Then we have*

$$\chi(H) \leqslant \max_{1 \leqslant i \leqslant n} \min\{i, d_i + 1\}.$$

Proof. Set X to be the vertex set of H and call $Y \subseteq X$ a P-set if it contains no edges of H with cardinality greater than one. Then $\chi(H) = \chi_P(X)$. Define an order of the vertices in X such that $x \prec y$ whenever $x \in A_i, y \in A_j, i < j$. For such an order $O : x_1 \prec \ldots \prec x_p$, it can be proved that if $x_j \in A_i$ then $ds_j(x_j) + 1 \leqslant \min\{i, d_i + 1\}$. Thus, $DS(O) \leqslant \max_{1 \leqslant i \leqslant n} \min\{i, d_i + 1\}$ and the result follows from $\chi_P(X) \leqslant DS(O)$. □

The order O in $SC(O)$ is arbitrarily set beforehand. To make the number of colors used reasonably small, an effective way is to use the so-called smallest-last technique for determining an order O_{SL}, as in the case of ordinary vertex colorings of graphs [14]. More precisely, we construct an order O_{SL} in the following way:

(1) Let x_p be a vertex of X with the minimum P-degree in $X_p = X$;

(2) for $i = p-1,\ldots,2,1$, let x_i be a vertex with the minimum P-degree in $X_i = X \setminus \{x_p,\ldots,x_{i+1}\}$ when $x_p,\ldots,x_{i+1}$ have been chosen.

Denote

$$\Lambda(X) = \max_{Y \subseteq X} \min_{x \in Y} d_P(x,Y) + 1 = \max_{Y \subseteq X} \delta_P(Y) + 1.$$

Since $d_P(x_i,X_i) = \min_{x \in X_i} d_P(x,X_i)$, we have $B(O_{SL}) \leqslant \Lambda(X)$. On the other hand, let Y_0 be a subset of X which attains the maximum in the definition of $\Lambda(X)$, and let x_i be the last vertex of Y_0 in the order O_{SL}. Then

$$d_P(x_i,X_i) + 1 \geqslant d_P(x_i,Y_0) + 1 \geqslant \min_{x \in Y_0} d_P(x,Y_0) + 1 = \Lambda(X).$$

Thus, $B(O_{SL}) \geqslant \Lambda(X)$ and hence $B(O_{SL}) = \Lambda(X)$. The use of this smallest-last technique leads to the upper bounds (i) and (ii) in the following corollary, where (ii) is a generalization of the known result $\chi(G) \leqslant 1 + \Delta(G)$ for the chromatic number $\chi(G)$ of a graph G to the case of P-chromatic number. The equivalent hypergraph forms of these two upper bounds can be found in [2, pp. 116–117].

Corollary 3. *Suppose X is a finite set and P is a hereditary property associated with the subsets of X. Then the following* (i)–(iii) *hold.*

(i) $\chi_P(X) \leqslant 1 + \max_{Y \subseteq X} \delta_P(Y)$.

(ii) $\chi_P(X) \leqslant 1 + \Delta_P(X)$.

(iii) *If X is P-critical, then* $\chi_P(X) \leqslant 1 + \delta_P(X)$.

Proof. We have proved that $B(O_{SL}) = \Lambda(X)$. So (i) follows from $\chi_P(X) \leqslant B(O_{SL})$ immediately. For each $x \in Y$, we have $d_P(x,Y) \leqslant d_P(x,X)$. Hence $\delta_P(Y) \leqslant \Delta_P(X)$ and (ii) follows from (i). For (iii), if there exists a vertex $x \in X$ with $d_P(x,X) < n-1$, where $n = \chi_P(X)$, then since X is P-critical there exists a P-$(n-1)$-coloring $\{A_1,\ldots,A_{n-1}\}$ of $X \setminus \{x\}$. Let O be the order of X such that the vertices in A_i precede those in A_j whenever $i < j$ and all vertices in $X \setminus \{x\}$ precede x. Clearly, $DS(O) \leqslant n-1$ and hence $\chi_P(X) \leqslant n-1$ by Theorem 1. This contradiction shows that $d_P(x,X) \geqslant \chi_P(X) - 1$ for each $x \in X$ and hence proves (iii). □

3. Concluding remarks

If P is a hereditary graphical property and X is the vertex set of a graph G, then $\chi_P(X)$ is the *P-chromatic number* [9] of G; if P is an edge hereditary graphical property [11] and X is the edge set of G, then $\chi_P(X)$ is just the *P-chromatic index*

[9] of G. A study on P-chromatic number of a graph was conducted in [4]. As shown in the literature (see e.g. [1,9,19,20]), a large number of known graphical invariants such as the ordinary chromatic number, edge chromatic number, thickness, arboricity, vertex arboricity [6], linear arboricity [10], vertex linear arboricity [13], unicyclicity [10], biparticity [10], n-th chromatic number [5], cochromatic number [12], chromatic partition number [15], subchromatic number [1], partite chromatic number [8] can be expressed as $\chi_P(X)$, where X is the vertex set or edge set of the graph and P is a specific graphical property. The algorithm in previous section can be applied to all these invariants and a Welsh-Powell-type upper bound can be obtained from Corollary 1 for each of them. As an example, we consider the vertex arboricity $a(G)$ which is precisely the P-chromatic number $\chi_P(X)$, where X is the vertex set of G and P is the property such that $Y \subseteq X$ is a P-set if and only if the subgraph $G[Y]$ induced by Y is a forest. Note that the minimal $\overline{P}$-sets are those $Y \subseteq V(G)$ such that $G[Y]$ is a chordless cycle. For each $x \in X$, let $c(x)$ be the maximum number of chordless cycles containing x such that any two of them have no common vertex other than x. Then $c(x) = d_P(x,X)$ and hence Corollary 1 gives $a(G) \leqslant \max_{1 \leqslant i \leqslant p} \min\{i, c(x_i)+1\}$, where $x_1,\ldots,x_p$ are the vertices of G and $c(x_1) \geqslant \ldots \geqslant c(x_p)$. This new upper bound for vertex arboricity is sharp in some cases since, for example, if G is the Petersen graph then $c(x) = 1$ for all vertices x and both sides of the inequality above are equal to 2.

The reader is referred to a subsequent paper [21] for an analysis on the structure of a P-critical set and for the relationship between $\chi_P(X)$ and the domination number of an associated graph.

Acknowledgements

The author would like to thank the referees for their valuable comments and suggestions. He also appreciates Professor Yu-Yuan Qin for his inspiring support in the preparation of a preliminary version of this paper.

References

[1] M.O. Albertson, R.E. Jamison, S.T. Hedetniemi, S.C. Locke, The subchromatic number of a graph, Discrete Math. 74 (1989) 33–49.
[2] C. Berge, Hypergraphs, North-Holland, Amsterdam, 1989.
[3] J.A. Bondy, Bounds for the chromatic number of a graph, J. Combin. Theory 7 (1969) 96–98.
[4] J.I. Brown, D.G. Corneil, On generalized graph colorings, J. Graph Theory 11 (1987) 87–99.
[5] G. Chartrand, D. Geller, S. Hedetniemi, A generalization of the chromatic number, Proc. Cambridge Phil. Soc. 64 (1968) 265–271.
[6] G. Chartrand, H. Kronk, C. Wall, The point arboricity of a graph, Israel J. Math. 6 (1968) 169–179.
[7] E.J. Cockayne, G.G. Miller, G. Prins, An interpolation theorem for partitions which are complete with respect to hereditary properties, J. Combin. Theory B 13 (1972) 290–297.
[8] G.S. Domke, R. Laskar, S.T. Hedetniemi, K. Peters, The partite-chromatic number of a graph, Congr. Numer. 53 (1986) 235–246.

[9] F. Harary, Conditional colorability in graphs, in: F. Harary, J.S. Maybee (Eds.), Graphs and Applications, Wiley, New York, 1985, 127–136.
[10] F. Harary, Covering and packing in graphs I, Ann. New York Acad. Sci. 175 (1970) 198–205.
[11] S.T. Hedetniemi, Hereditary properties of graphs, J. Combin. Theory, Ser. B 14 (1973) 94–99.
[12] F.L. Lesniak, J. Roberts, The cochromatic number of a graph, Ars Combin. 3 (1977) 39–45.
[13] M. Matsumoto, Bounds for the vertex linear arboricity, J. Graph Theory 14 (1990) 117–126.
[14] D.W. Matula, L.L. Beck, Smallest-last ordering and clustering and graph coloring algorithms, J. Association for Computing Machinery 30 (1983) 417–427.
[15] E. Sampathkumar, C.V. Venkatachalam, Chromatic partitions of a graph, Discrete Math. 74 (1989) 227–239.
[16] I. Tomescu, Sur le problème du coloriage des graphes généralisés, C.R. Acad. Sci. Paris 267 (1968) 250–252.
[17] D.J.A. Welsh, M.B. Powell, An upper bound for the chromatic number of a graph and its application to timetabling problems, Comput. J. 10 (1967) 85–86.
[18] D. de Werra, Heuristics for graph coloring, Computing (Suppl. 7) (1990) 191–208.
[19] D. Woodall, Improper colorings of graphs, in: R. Nelson, R.J. Wilson (Eds.), Graph Colorings, Pitman Research Notes in Mathematics, Ser. 218, Longman Scientific and Technical, Harlow, 1990, 45–63.
[20] S.M. Zhou, Interpolation theorems for graphs, hypergraphs and matroids, Discrete Math. 185 (1998) 221–229.
[21] S.M. Zhou, Two theorems for conditional colorings of finite sets, preprint.