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Imprimitive symmetric graphs, 3-arc graphs and 1-designs

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Abstract

Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$. For $B \in \mathscr{B}$, let $\mathscr{D}(B) = (B, \Gamma_{\mathscr{B}}(B), \mathrm{I})$ be the 1-design in which $\alpha \mathrm{I} C$ for $\alpha \in B$ and $C \in \Gamma_{\mathscr{B}}(B)$ if and only if α is adjacent to at least one vertex of C, where $\Gamma_{\mathscr{B}}(B)$ is the neighbourhood of B in the quotient graph $\Gamma_{\mathscr{B}}$ of Γ relative to $\mathscr{B}$. In a natural way the setwise stabilizer G_B of B in G induces a group of automorphisms of $\mathscr{D}(B)$. In this paper, we study those graphs Γ such that the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent, that is, there exists a bijection $\rho : B \to \Gamma_{\mathscr{B}}(B)$ such that $\rho(\alpha^x) = (\rho(\alpha))^x$ for $\alpha \in B$ and $x \in G_B$. In this case the vertices of Γ can be labelled naturally by the arcs of $\mathscr{B}$. By using this labelling technique we analyse $\Gamma_{\mathscr{B}}$, $\mathscr{D}(B)$ and the bipartite subgraph $\Gamma[B, C]$ induced by adjacent blocks B, C of $\mathscr{B}$, and study the influence of them on the structure of Γ. We prove that the class of such graphs Γ is precisely the class of those graphs obtained from G-symmetric graphs Σ and self-paired G-orbits on 3-arcs of Σ by a construction introduced in a recent paper of Li, Praeger and the author, and that Γ can be reconstructed from $\Gamma_{\mathscr{B}}$ via this construction.

Keywords: Symmetric graph; 3-arc graph; Design

1. Introduction

Let Γ be a finite graph and $s \geqslant 1$ an integer. An *s-arc* of Γ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices of Γ such that α_i, α_{i+1} are adjacent in Γ and $\alpha_{i-1} \neq \alpha_{i+1}$ for each i. If Γ admits a group G of automorphisms such that G is transitive on the vertex set $V(\Gamma)$ of Γ and, in its induced action, is transitive on the set $A_s(\Gamma)$ of s-arcs of Γ, then Γ is said to be *(G, s)-arc transitive*. In the literature a 1-arc is usually called an *arc* and a $(G, 1)$-arc transitive graph is called a *G-symmetric graph*. Clearly, a G-symmetric graph Γ is *regular*, that is, all the vertices of Γ have the same valency, which we call the valency of Γ and denote by $\mathrm{val}(\Gamma)$. Instead of $A_1(\Gamma)$ we

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will use $A(\Gamma)$ to denote the arc set of the graph Γ. For a group G acting on two finite sets Ω_1 and Ω_2, respectively, if there exists a bijection $\rho: \Omega_1 \to \Omega_2$ such that $\rho(\alpha^x) = (\rho(\alpha))^x$ for any $\alpha \in \Omega_1$ and $x \in G$, then the actions of G on Ω_1 and Ω_2 are said to be *permutationally equivalent* with respect to ρ.

The study of symmetric graphs and highly arc-transitive graphs has been one of the mainstreams in algebraic combinatorics over many years. The reader is referred to [9–11] for survey information about recent research in this area. Roughly speaking, in most cases the vertex set $V(\Gamma)$ of a G-symmetric graph Γ admits a *nontrivial* G*-invariant partition*, that is, a partition $\mathscr{B}$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in \mathscr{B}$ for any $B \in \mathscr{B}$ and $g \in G$, where $B^g := \{\alpha^g \colon \alpha \in B\}$. If this occurs then Γ is said to be an *imprimitive* G*-symmetric graph*. In the opposite case, G is primitive on $V(\Gamma)$ and the well-known O'Nan–Scott theorem has been proved to be very useful (see e.g. [10]). On the contrary, it seems that no such a powerful mathematical tool is available for imprimitive G-symmetric graphs. In an ambitious scheme, Gardiner and Praeger [3] introduced a geometric approach to studying such a graph Γ, which involves an analysis of the following three configurations (see Section 2, paragraph 2 for definitions) associated with $(\Gamma, \mathscr{B})$:

(i) the quotient graph $\Gamma_{\mathscr{B}}$ with respect to $\mathscr{B}$;
(ii) the bipartite subgraph $\Gamma[B, C]$ of Γ induced by adjacent blocks B, C of $\mathscr{B}$; and
(iii) the 1-design $\mathscr{D}(B) = (B, \Gamma_{\mathscr{B}}(B), \mathrm{I})$ induced on a block $B \in \mathscr{B}$ such that $\alpha \in B$ is *incident* with $C \in \Gamma_{\mathscr{B}}(B)$ (that is, $\alpha \mathrm{I} C$) if and only if α is adjacent in Γ to at least one vertex of C, where $\Gamma_{\mathscr{B}}(B)$ is the set of blocks of $\mathscr{B}$ adjacent to B in $\Gamma_{\mathscr{B}}$.

Note that $\mathscr{D}(B)$ may contain *repeated blocks*, that is, blocks of $\mathscr{D}(B)$ incident with the same subset of vertices of B. It was proved in [6, Theorem 1] that, if the *block size* $k := |\{\alpha \in B \colon \alpha \mathrm{I} C\}|$ (for $C \in \Gamma_{\mathscr{B}}(B)$) of $\mathscr{D}(B)$ satisfies $k = |B| - 1 \geqslant 2$, then $\mathscr{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive, and in this case (see [6, Theorem 5(b)]) the induced actions of G_B on the "points" B and the "blocks" $\Gamma_{\mathscr{B}}(B)$ of $\mathscr{D}(B)$ are permutationally equivalent with respect to the bijection defined by $\alpha \mapsto C$, for $\alpha \in B$, where C is the unique block in $\Gamma_{\mathscr{B}}(B)$ not incident with α. Moreover, in this case Γ can be reconstructed in a natural way from the $(G, 2)$-arc transitive quotient $\Gamma_{\mathscr{B}}$ and the action of G on $\mathscr{B}$ ([6, Theorem 1]), namely Γ is isomorphic to a 3-arc graph of $\Gamma_{\mathscr{B}}$ relative to a self-paired G-orbit on $A_3(\Gamma_{\mathscr{B}})$. For a regular graph Σ, a subset Δ of $A_s(\Sigma)$ is said to be *self-paired* if $(\alpha_0, \alpha_1, \ldots, \alpha_s) \in \Delta$ implies $(\alpha_s, \ldots, \alpha_1, \alpha_0) \in \Delta$. For a self-paired subset Δ of $A_3(\Sigma)$, the 3-*arc graph* $\mathrm{Arc}_\Delta(\Sigma)$ of Σ relative to Δ was defined in [6, Section 6] to have vertex set $A(\Sigma)$ in which $(\sigma, \tau), (\sigma', \tau')$ are adjacent if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. The self-parity of Δ ensures that adjacency in this graph is well-defined. For further development of the geometric approach above, the reader is referred to [4–6,12,13].

In a natural way, the setwise stabilizer G_B of B in G induces a group of automorphisms of the 'cross-sectional' geometry $\mathscr{D}(B)$, and G_B is transitive on the points, the blocks and the flags of $\mathscr{D}(B)$ (see Lemma 2.1). So the permutation equivalence between the actions of G_B on the "points" B and the "blocks" $\Gamma_{\mathscr{B}}(B)$ of $\mathscr{D}(B)$ is a geometric property bridging naturally the two parts of $\mathscr{D}(B)$. We notice that, besides the case mentioned above, this property is also possessed by some other imprimitive G-symmetric graphs (see Example 2.2). This motivated us to study such graphs Γ without necessarily assuming that $k = |B| - 1$ or $\mathscr{D}(B)$ contains no repeated blocks. That is, we will study in this paper G-symmetric graphs Γ admitting a nontrivial G-invariant partition $\mathscr{B}$ such that the following (PE) holds for some block $B \in \mathscr{B}$.

Assumption (*PE*)**.** The actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent with respect to some bijection $\rho : B \to \Gamma_{\mathscr{B}}(B)$.

To avoid triviality, we will assume without mentioning explicitly that $\mathrm{val}(\Gamma) > 1$ and that $\Gamma_{\mathscr{B}}$ *has at least one edge.* This latter assumption on $\Gamma_{\mathscr{B}}$ implies that each block of $\mathscr{B}$ is an independent set of Γ (see e.g. [3,8]). As we will see later, any G-symmetric graph Γ satisfying (PE) can be reconstructed from the G-symmetric quotient $\Gamma_{\mathscr{B}}$ via the 3-arc graph construction. In fact, we will show that the class of such graphs Γ is precisely the class of all 3-arc graphs $\mathrm{Arc}_{\Delta}(\Sigma)$ with Σ a G-symmetric graph and Δ a self-paired G-orbit on $A_3(\Sigma)$, and hence it is a quite large class of symmetric graphs.

The structure and main results of this paper are as follows. After introducing terminology and giving an example in Section 2, we develop in Section 3 a labelling technique for studying G-symmetric graphs Γ satisfying (PE). More precisely, we will show that each vertex of Γ can be labelled uniquely by an ordered pair "BC" of adjacent blocks B, C of $\Gamma_{\mathscr{B}}$, and we will prove some basic results relating to this labelling (Theorem 3.2). In particular, we will show that each vertex "BC" of Γ has a unique mate (Theorem 3.2(b)), labelled by "CB", and that $V(\Gamma)$ admits two other G-invariant partitions (Theorem 3.2(b) and (c)), namely $\mathscr{B}^* = \{B^*: B \in \mathscr{B}\}$ with B^* the set of mates of the vertices of B, and $\mathscr{P}$ with each block consisting of two mated vertices. Moreover, there is no edge of Γ between B and B^* (Theorem 3.2(d)). In the case where adjacent vertices of Γ have the same second coordinate in their labels, the $\mathrm{girth}(\Gamma_{\mathscr{B}})$ of $\Gamma_{\mathscr{B}}$ is equal to 3 and Γ is a disconnected graph with each connected component contained in some block of $\mathscr{B}^*$ (Theorem 3.3(b)). In the case where $\mathrm{girth}(\Gamma_{\mathscr{B}}) \geqslant 5$, we give some structural information about the bipartite graph $\Gamma[P, Q]$, where P, Q are adjacent blocks of $\mathscr{P}$ or adjacent blocks of $\mathscr{B}^*$, and show that the involution interchanging each pair of mated vertices induces a graph monomorphism from Γ to its complement $\bar{\Gamma}$, and induces a graph monomorphism from $\Gamma_{\mathscr{B}}$ to $\overline{\Gamma_{\mathscr{B}^*}}$ (Theorem 3.4). Based on this we obtain upper bounds for the valencies of Γ and $\Gamma_{\mathscr{B}}$ in this case (Corollary 3.5).

Under the assumption (PE), one may expect a more active role played by $\mathscr{D}(B)$ in influencing Γ, $\Gamma_{\mathscr{B}}$ and $\Gamma[B, C]$, and this will be studied in Section 4. In particular, we characterize the case where $k = v - 1 \geqslant 2$ and $\mathscr{D}(B)$ contains no repeated blocks as the only case such that $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive (Theorem 4.1(c)), and characterize the case where $\Gamma[B, C] \cong K_{v-1, v-1}$ as the only case such that $\Gamma_{\mathscr{B}}$ is $(G, 3)$-arc transitive (Theorem 4.1(d)).

In Section 5, we continue our study under the additional assumption that the mapping ρ in (PE) preserves the incidence relation of $\mathcal{D}(B)$ in the sense that, for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$, $\alpha \mathrm{I} C$ if and only if $\rho^{-1}(C)\mathrm{I}\rho(\alpha)$. In this case, $\mathcal{D}(B)$ is a self-dual 1-design and ρ induces a polarity of $\mathcal{D}(B)$ (Proposition 5.1). We will prove in particular that, if adjacent vertices of Γ have labels involving four distinct blocks and if $\Gamma[B,C] \cong kK_2$ is a matching of k edges, then there exists a G-orbit $\mathcal{O}$ on n-cycles of $\Gamma_{\mathcal{B}}$, for a certain even integer $n \geqslant 4$, such that the adjacency of Γ and the incidence relation of $\mathcal{D}(B)$ are determined completely by $\mathcal{O}$ (Theorem 5.3).

In Section 6, we will prove (Theorem 6.2) that any G-symmetric graph Γ satisfying (PE) can be reconstructed from $\Gamma_{\mathcal{B}}$ and the action of G on $\mathcal{B}$, namely Γ is isomorphic to a 3-arc graph of $\Gamma_{\mathcal{B}}$ relative to a certain self-paired G-orbit on $A_3(\Gamma_{\mathcal{B}})$. Conversely, we will show that, for any G-symmetric graph Σ and any self-paired G-orbit Δ on $A_3(\Sigma)$, the 3-arc graph $\mathrm{Arc}_{\Delta}(\Sigma)$ is a G-symmetric graph which admits a G-invariant partition $\mathcal{B}$ such that (PE) is satisfied for all $B \in \mathcal{B}$.

2. Definitions, notation and example

We refer to [1] for terminology and notation on incidence structures and designs, and to [2] for that on permutation groups. For a graph Γ and an integer $n \geqslant 1$, we use $n\Gamma$ to denote the graph consisting of n vertex-disjoint copies of Γ. So in particular nK_2 is a matching of n edges. We denote by $\Gamma(\alpha)$ the *neighbourhood* in Γ of a vertex $\alpha \in V(\Gamma)$, that is, the set of vertices of Γ adjacent to α in Γ. For two graphs Γ and Σ, a mapping $\varphi: V(\Gamma) \to V(\Sigma)$ is called a *graph homomorphism* if φ maps adjacent vertices of Γ to adjacent vertices of Σ; if in addition φ is one-to-one, then it is called a *graph monomorphism*.

Let Γ be a G-symmetric graph and $\mathcal{B}$ a nontrivial G-invariant partition of $V(\Gamma)$. The *quotient graph* of Γ with respect to $\mathcal{B}$, denoted by $\Gamma_{\mathcal{B}}$, is defined to be the graph with vertex set $\mathcal{B}$ in which two blocks $B, C \in \mathcal{B}$ are *adjacent* if and only if there is an edge of Γ joining a vertex of B and a vertex of C. For $\alpha \in V(\Gamma)$, we use $B(\alpha)$ to denote the block of $\mathcal{B}$ containing α. For $B \in \mathcal{B}$, denote by $\Gamma_{\mathcal{B}}(B)$ the neighbourhood of B in $\Gamma_{\mathcal{B}}$, and set $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$. For two adjacent blocks B, C of $\mathcal{B}$, denote by $\Gamma[B,C]$ the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Define $\mathcal{D}(B) := (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ to be the incidence structure such that, for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$, $\alpha \mathrm{I} C$ if and only if $\alpha \in \Gamma(C)$. Clearly, the set of points of $\mathcal{D}(B)$ incident with C is $\Gamma(C) \cap B$. Since Γ is G-symmetric and $\mathcal{B}$ is G-invariant, one can see that the following (i)–(iv) hold [3,6]:

(i) $B(\alpha^x) = (B(\alpha))^x$ for $\alpha \in V(\Gamma)$ and $x \in G$;
(ii) $\Gamma_{\mathcal{B}}$ is G-symmetric under the induced action (possibly unfaithful) of G on $\mathcal{B}$;
(iii) $\Gamma[B,C]$ and $\mathcal{D}(B)$ are, up to isomorphism, independent of the choice of adjacent blocks B, C and the block B, respectively; and
(iv) $\mathcal{D}(B)$ is a 1-(v,k,r) design, where $v := |B|$, $k := |\Gamma(B) \cap C|$ for $C \in \Gamma_{\mathcal{B}}(B)$, and $r := |\Gamma_{\mathcal{B}}(\alpha)|$ with $\Gamma_{\mathcal{B}}(\alpha) := \{C \in \Gamma_{\mathcal{B}}(B)\colon \alpha \mathrm{I} C\}$, for $\alpha \in B$.

By (i) above the number of times a block C of $\mathscr{D}(B)$ is repeated is independent of the choice of B, C. We call this number the *multiplicity* of $\mathscr{D}(B)$ and denote it by m. Clearly, m divides the valency $b := |\Gamma_{\mathscr{B}}(B)|$ of $\Gamma_{\mathscr{B}}$. Denote by G_B and $G_{(B)}$, respectively, the setwise and pointwise stabilizers of B in G, and by $G_{[B]}$ the subgroup of G_B fixing each $C \in \Gamma_{\mathscr{B}}(B)$ setwise. For $B, C, D \in \mathscr{B}$, we set $G_{B,C} := (G_B)_C = (G_C)_B$ and $G_{B,C,D} := (G_{B,C})_D$. For $\alpha \in V(\Gamma)$, denote by G_α the stabilizer of α in G, and set $G_{\alpha,B} := (G_\alpha)_B$. Clearly, G_B induces nature actions on B and on $\Gamma_{\mathscr{B}}(B)$. The following lemma shows that G_B induces an automorphism group of $\mathscr{D}(B)$. (This was observed in [3] in the case where G_α is primitive on $\Gamma(\alpha)$.)

Lemma 2.1 (Gardiner and Praeger [3, Section 3]). *Suppose that Γ is a finite G-symmetric graph and $\mathscr{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$. Let B be a block of $\mathscr{B}$. Then G_B induces a group of automorphisms of $\mathscr{D}(B)$ which is transitive on the points, the blocks and the flags of $\mathscr{D}(B)$.*

We now give the following example of G-symmetric graph Γ which satisfies (PE) for a G-invariant partition of $V(\Gamma)$. Note that in this example we have $k < v - 1$ and $\mathscr{D}(B)$ contains repeated blocks.

Example 2.2. Let PG$(2,2)$ be the Fano plane whose points $1, 2, \ldots, 7$ are as shown in Fig. 1. Let V be the set of ordered pairs of distinct points of PG$(2,2)$. Then $G := \mathrm{PGL}(3,2)$ is transitive on V (see e.g. [2, Section 2.8]). Define Γ to be the graph with vertex set V such that two vertices $\alpha\beta, \gamma\delta \in V$ are adjacent if and only if (i) $\alpha, \beta, \gamma, \delta$ are distinct, and (ii) β, δ and the unique point collinear with α, γ are distinct and are collinear in PG$(2,2)$. For example, $17, 26$ are adjacent in Γ since the unique point collinear with $1, 2$ is 3 and since $7, 6, 3$ are collinear in PG$(2,2)$. Similarly, we have $\Gamma(17) = \{26, 62, 35, 53\}$. Note that the pointwise stabilizer G_{17} of $1, 7$ in G contains an element which exchanges 2 and 6 and exchanges 3 and 5; also

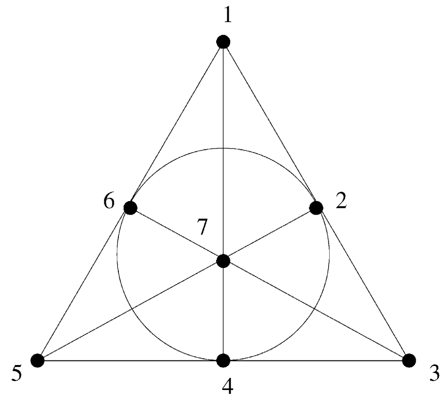

Fig. 1. Fano plane.

G_{17} contains an element which exchanges 2 and 3 and exchanges 6 and 5. So G_{17} is transitive on $\Gamma(17)$, and hence Γ is G-symmetric. One can see that $\Gamma \cong 7K_{2,2,2}$ and $\mathscr{B}:=\{B(\alpha)$: α is a point of PG(2,2)$\}$ is a G-invariant partition of V, where $B(\alpha):=\{\alpha\beta$: β is a point of PG(2,2) with $\beta\neq\alpha\}$. We have $\Gamma_{\mathscr{B}} \cong K_7$, $\Gamma[B(\alpha),B(\beta)] \cong 4K_2$ for $\alpha\neq\beta$, $\mathscr{D}(B(1))$ is a 1-(6,4,4) design, and the sets $\Gamma_{\mathscr{B}}(1\beta)$ for $1\beta\in B(1)$ are $\{12,13,14,17\}$, $\{14,15,16,17\}$, $\{12,13,16,15\}$ with each repeated twice. Thus, the block size of $\mathscr{D}(B(1))$ is less than $|B(1)|-1$. Clearly, the induced actions of $G_{B(\alpha)}$ on $B(\alpha)$ and $\Gamma_{\mathscr{B}}(B(\alpha))$ are permutationally equivalent with respect to the bijection $\rho:\alpha\beta\mapsto B(\beta)$. Note that $\alpha\beta$ is adjacent to a vertex in a block $B(\gamma)$ if and only if $\alpha\gamma$ is adjacent to a vertex in the block $B(\beta)$.

3. The labelling technique

As a fundamental fact, we now show that (PE) holds if and only if the vertices of Γ can be labelled in a natural way by the arcs of $\Gamma_{\mathscr{B}}$. For convenience we call a mapping $\mu: V(\Gamma)\to A(\Gamma_{\mathscr{B}})$ *compatible* with $\mathscr{B}$ if, for any $\alpha\in V(\Gamma)$, the arc $\mu(\alpha)$ of $\Gamma_{\mathscr{B}}$ is initiated at the block $B(\alpha)$.

Lemma 3.1. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$. Then (PE) holds for some $B\in\mathscr{B}$ if and only if the actions of G on $V(\Gamma)$ and $A(\Gamma_{\mathscr{B}})$ are permutationally equivalent with respect to a bijection $\mu: V(\Gamma)\to A(\Gamma_{\mathscr{B}})$ compatible with $\mathscr{B}$. Moreover, in this case we have $b=v\geqslant 2$, $G_{[B]}=G_{(B)}$, G is faithful on $\mathscr{B}$ if G is faithful on $V(\Gamma)$, and (PE) holds for all blocks B of $\mathscr{B}$.*

Proof. Suppose first that (PE) holds for some block $B\in\mathscr{B}$ and a bijection $\rho: B\to\Gamma_{\mathscr{B}}(B)$, and let α be a fixed vertex of B. Then, since Γ is G-vertex-transitive, each vertex of Γ has the form α^x for some $x\in G$. We will show that $\mu:\alpha^x\mapsto(B^x,(\rho(\alpha))^x)$, $x\in G$, defines a bijection from $V(\Gamma)$ to $A(\Gamma_{\mathscr{B}})$ which is compatible with $\mathscr{B}$. In fact, if $\alpha^x=\alpha^y$ for some $x,y\in G$, then $xy^{-1}\in G_\alpha$ ($\leqslant G_B$), and hence $B^{xy^{-1}}=B$ and $(\rho(\alpha))^{xy^{-1}}=\rho(\alpha^{xy^{-1}})=\rho(\alpha)$. Therefore, we have $\mu(\alpha^x)=\mu(\alpha^y)$ and thus μ is well-defined. Secondly, if $\mu(\alpha^x)=\mu(\alpha^y)$ for two vertices α^x,α^y, then $xy^{-1}\in G_B$ since $B^x=B^y$. This, together with $(\rho(\alpha))^x=(\rho(\alpha))^y$, implies that $\rho(\alpha)=(\rho(\alpha))^{xy^{-1}}=\rho(\alpha^{xy^{-1}})$. Note that $xy^{-1}\in G_B$ implies $\alpha^{xy^{-1}}\in B$, and that ρ is a bijection from B to $\Gamma_{\mathscr{B}}(B)$. So we have $\alpha^{xy^{-1}}=\alpha$, implying $\alpha^x=\alpha^y$ and hence μ is injective. Since G is transitive on arcs of $\Gamma_{\mathscr{B}}$, μ is in fact a bijection from $V(\Gamma)$ to $A(\Gamma_{\mathscr{B}})$. Since B and $\rho(\alpha)$ are adjacent blocks and $B^x=(B(\alpha))^x=B(\alpha^x)$, B^x and $(\rho(\alpha))^x$ are adjacent blocks and hence μ is compatible with $\mathscr{B}$. It follows from the definition that the actions of G on $V(\Gamma)$ and $A(\Gamma_{\mathscr{B}})$ are permutationally equivalent with respect to μ. Moreover, the definition of μ does not depend on the choice of $\alpha\in B$. In fact, for another vertex $\beta\in B$ and any vertex of Γ, say $\gamma=\alpha^x=\beta^y$ for some $x,y\in G$, we have $B^x=B(\alpha^x)=B(\beta^y)=B^y$ and hence $xy^{-1}\in G_B$. So $(\rho(\alpha))^{xy^{-1}}=\rho(\alpha^{xy^{-1}})=\rho(\beta)$, implying $(B,\rho(\alpha))^x=(B,\rho(\beta))^y$ and indeed the definition of μ is independent of the choice of $\alpha \in B$. One can see that, for each block $D \in \mathscr{B}$, the actions of G_D on D and $\Gamma_{\mathscr{B}}(D)$ are permutationally equivalent with respect to the bijection $\rho_D : D \to \Gamma_{\mathscr{B}}(D)$ defined by $(D, \rho_D(\gamma)) = \mu(\gamma)$, for $\gamma \in D$.

Now suppose conversely that the actions of G on $V(\Gamma)$ and $A(\Gamma_{\mathscr{B}})$ are permutationally equivalent with respect to a bijection $\mu : V(\Gamma) \to A(\Gamma_{\mathscr{B}})$ which is compatible with $\mathscr{B}$. Then $(B, \rho(\alpha)) = \mu(\alpha)$, for $\alpha \in B$, defines a bijection $\rho : B \to \Gamma_{\mathscr{B}}(B)$. It is easily checked that the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent with respect to ρ.

Finally, if (PE) holds, then $b = |\Gamma_{\mathscr{B}}(B)| = |B| = v \geqslant 2$. Also, in this case $G_{[B]} = G_{(B)}$ for each $B \in \mathscr{B}$. So if G is faithful on $V(\Gamma)$, then G is faithful on $\mathscr{B}$ as well. $\square$

Lemma 3.1 implies that, under the assumption (PE), each vertex α of Γ can be uniquely labelled by an ordered pair "BC" of adjacent blocks of $\Gamma_{\mathscr{B}}$, where $(B, C) = \mu(\alpha)$. In the following, we will identify α with the *label* "BC", so we have $G_{\text{“}BC\text{”}} = G_{B,C}$. Since $(\mu(\alpha))^x = \mu(\alpha^x)$, it follows that

$$\text{“}BC\text{”}^x = \text{“}B^x C^x\text{”} \tag{1}$$

for $x \in G$ and "BC" $\in V(\Gamma)$. One can see that the block B is precisely the set of those vertices of Γ whose labels have the first coordinate B, that is, $B = \{\text{“}BC\text{”}: (B, C) \in A(\Gamma_{\mathscr{B}})\}$. Note that each vertex $\alpha =$ "BC" of Γ has a unique *mate* $\alpha' :=$ "CB", and that z: $\alpha \mapsto \alpha'$ defines an involution on $V(\Gamma)$. Also, z centralises G since $\text{“}BC\text{”}^{zx} = \text{“}CB\text{”}^x = \text{“}C^x B^x\text{”} = \text{“}B^x C^x\text{”}^z = \text{“}BC\text{”}^{xz}$ for any $x \in G$. Since G preserves $\mathscr{B}$ invariant whilst it is easy to see that $B^z = \{\alpha' : \alpha \in B\} \notin \mathscr{B}$, we have $z \notin G$. Clearly, $\{\{\alpha, \alpha'\}: \alpha \in V(\Gamma)\}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$, and the graph Γ' with vertex set $V(\Gamma)$ and arc set $\{(\alpha, \alpha'): \alpha \in V(\Gamma)\}$ is G-symmetric. We record these basic results in the following theorem, which will be used repeatedly in our later discussion. For $B \in \mathscr{B}$, we set $B^* := B^z$.

Theorem 3.2. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent, for some $B \in \mathscr{B}$. Let $\mu : V(\Gamma) \to A(\Gamma_{\mathscr{B}})$ be the bijection guaranteed by Lemma* 3.1. *Then the following* (a)–(d) *hold*:

(a) *Each vertex α of Γ can be labelled uniquely by an ordered pair "BC" of adjacent blocks of $\Gamma_{\mathscr{B}}$, where $(B, C) = \mu(\alpha)$. Moreover, we have $G_{\text{“}BC\text{”}} = G_{B,C}$ and $\text{“}BC\text{”}^x = \text{“}B^x C^x\text{”}$ for "BC" $\in V(\Gamma)$ and $x \in G$.*
(b) *Each vertex $\alpha =$ "BC" has a unique mate $\alpha' :=$ "CB", the mapping $z : \alpha \mapsto \alpha'$ defines an involution such that $z \notin G$ and z centralises G, $\mathscr{P} := \{\{\alpha, \alpha'\}: \alpha \in V(\Gamma)\}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$, and the graph Γ' with vertex set $V(\Gamma)$ and arc set $\{(\alpha, \alpha'): \alpha \in V(\Gamma)\}$ is G-symmetric.*
(c) *$\mathscr{B}^* := \{B^* : B \in \mathscr{B}\}$ is a G-invariant partition of $V(\Gamma)$, $G_B = G_{B^*}$, and the actions of G_B on B and B^* are transitive and permutationally equivalent with respect to the restriction of z on B.*

(d) *There is no edge of Γ joining vertices of B and B^*. In particular, for each arc* ("BC", "DE") *of Γ, (C, B, D, E) is a 3-arc of $\Gamma_{\mathscr{B}}$.*

Proof. The truth of (a) and (b) has been shown above, and from this we get (c) by a routine argument. To prove (d), we assume that B, C are two adjacent blocks of $\Gamma_{\mathscr{B}}$. If "CB" is adjacent to "BC", then, since $\mathrm{val}(\Gamma) > 1$, "CB" is adjacent to a vertex "B_1C_1" distinct from "BC". By the G-symmetry of Γ, there exists $x \in G$ such that ("CB", "BC")^x = ("CB", "B_1C_1"). From (1) this implies that $C = C^x = C_1$, $B = B^x = B_1$, a contradiction. Hence each vertex "CB" of $V(\Gamma)$ is not adjacent to its mate "BC". Similarly, if "CB" is adjacent to a vertex "BD" $\in B \setminus \{$"BC"$\}$, then we can take a vertex "B_1D_1" which is distinct from "BD" and is adjacent to "CB", and hence ("CB", "BD")^x = ("CB", "B_1D_1") for some $x \in G$, implying $B = B^x = B_1$. On the other hand, there exists $y \in G$ such that ("CB", "BD")^y = ("B_1D_1", "CB"). This implies $C = B^y = D_1$, and hence "B_1D_1" = "BC". Again, this is a contradiction and hence there is no edge of Γ between B and B^*. In particular, if ("BC", "DE") is an arc of Γ, then $C \neq D$, $B \neq E$ and hence (C, B, D, E) is a 3-arc of $\Gamma_{\mathscr{B}}$. $\square$

As shown in the following theorem, the G-symmetric graphs satisfying (PE) fall into two categories according to the nature of labels of adjacent vertices of Γ.

Theorem 3.3. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent, for some $B \in \mathscr{B}$. Then one, and only one, of the following* (a) *and* (b) *occurs.*

(a) *Any two adjacent vertices have labels involving four distinct blocks. In this case, each block of $\mathscr{B}^*$ is an independent set of Γ.*
(b) *Any two adjacent vertices of Γ share the same second coordinate. In this case, Γ is disconnected with each block of $\mathscr{B}^*$ consisting of connected components of Γ. Moreover, we have* $\mathrm{girth}(\Gamma_{\mathscr{B}}) = 3$, $\Gamma[B, C] \cong kK_2$ *and* $\mathrm{val}(\Gamma) = |D^{G_{B,C}}|$, *where $B, C, D \in \mathscr{B}$ such that* "CB", "DB" *are adjacent in Γ. In particular, $\Gamma[B^*] \cong K_v$ if and only if $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive, and in this case we have $\Gamma \cong n(v+1)K_v$, $\Gamma[B, C] \cong (v-1)K_2$ and $\Gamma_{\mathscr{B}} \cong nK_{v+1}$ for an integer n, and the group induced on the vertex set of a connected component of $\Gamma_{\mathscr{B}}$ is* 3-*transitive.*

Proof. It is easy to see that either (a) or (b) occurs, and that (a) occurs if and only if each block of $\mathscr{B}^*$ is an independent set of Γ. In the following, we suppose (b) occurs, and let "CB", "DB" be adjacent vertices. Then $\mathrm{girth}(\Gamma_{\mathscr{B}}) = 3$ since (B, C, D, B) is a triangle of $\Gamma_{\mathscr{B}}$. Clearly, any two adjacent vertices of Γ lie in the same block of $\mathscr{B}^*$, and hence the subgraph $\Gamma[E^*]$ induced by each $E^* \in \mathscr{B}^*$ consists of connected components of Γ. By our assumption, "CB" is the unique vertex in C adjacent to "DB". So we have $\Gamma[C, D] \cong kK_2$. Moreover, a vertex "D_1B" $\in B^*$ is adjacent to "CB" in Γ $\Leftrightarrow$ there exists $g \in G$ such that ("CB", "DB")^g = ("CB", "D_1B") $\Leftrightarrow$ there exists

$g \in G_{B,C}$ such that $D^g = D_1$. Thus, we have $\mathrm{val}(\Gamma) = |D^{G_{B,C}}|$. In particular, $\Gamma[B^*] \cong K_v \Leftrightarrow G_{B,C}$ is transitive on $\Gamma_{\mathscr{B}}(B) \setminus \{C\} \Leftrightarrow G_B$ is 2-transitive on $\Gamma_{\mathscr{B}}(B) \Leftrightarrow \Gamma_{\mathscr{B}}$ is $(G,2)$-arc transitive. In this case, the argument above shows that (i) $\Gamma \cong |\mathscr{B}^*| K_v$, (ii) $\{B\} \cup \Gamma_{\mathscr{B}}(B)$ induces the complete graph K_{v+1} which is a connected component of $\Gamma_{\mathscr{B}}$ (note that $b = v$ by Lemma 3.1), and (iii) G induces a 3-transitive group on $\{B\} \cup \Gamma_{\mathscr{B}}(B)$. Therefore, we have $\Gamma_{\mathscr{B}} \cong nK_{v+1}$ and $\Gamma \cong n(v+1)K_v$ for an integer n. Counting the number of edges of Γ in two ways, we get $(n(v+1)v/2)k = n(v+1)(v(v-1)/2)$, which implies $k = v-1$ and hence $\Gamma[C,D] \cong (v-1)K_2$. $\square$

Note that case (a) in Theorem 3.3 occurs when girth $(\Gamma_{\mathscr{B}}) \geqslant 4$. If $\mathrm{girth}(\Gamma_{\mathscr{B}}) \geqslant 5$, then we get the following generalizations of [6, Theorem 9 and Corollary 1]—the proofs are much similar and hence omitted.

Theorem 3.4. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent, for some $B \in \mathscr{B}$. Suppose further that* $\mathrm{girth}(\Gamma_{\mathscr{B}}) \geqslant 5$. *Then*

(a) *$\Gamma[\{\alpha,\alpha'\},\{\beta,\beta'\}] \cong K_2$ for adjacent blocks $\{\alpha,\alpha'\}$ and $\{\beta,\beta'\}$ of $\mathscr{P}$.*
(b) *$\Gamma[B^*,C^*]$ is a matching for adjacent blocks B^*, C^* of $\mathscr{B}^*$; in particular $\Gamma[B^*,C^*] \cong K_2$ if* $\mathrm{girth}(\Gamma_{\mathscr{B}}) \geqslant 7$.
(c) *The involution $z : \alpha \mapsto \alpha'$ for $\alpha \in V(\Gamma)$ defines a graph monomorphism from Γ to the complement $\bar{\Gamma}$. Moreover, z induces graph monomorphisms from $\Gamma_{\mathscr{B}}$ to $\overline{\Gamma_{\mathscr{B}^*}}$, and from $\Gamma_{\mathscr{B}^*}$ to $\overline{\Gamma_{\mathscr{B}}}$, defined by $B \mapsto B^*$, and $B^* \mapsto B$, respectively.*

Corollary 3.5. *With the same assumptions as in Theorem* 3.4, *we have* $\mathrm{val}(\Gamma) \leqslant (|V(\Gamma)| - 2)/4$ *and* $\mathrm{val}(\Gamma_{\mathscr{B}^*}) \leqslant (|V(\Gamma)|/v) - v - 1$. *If in addition* $\mathrm{girth}(\Gamma_{\mathscr{B}}) \geqslant 7$, *then* $\mathrm{val}(\Gamma) \leqslant (|V(\Gamma)|/v^2) - (1/v) - 1$.

Remark 3.6. Let k^* denote the block size of the 1-design $\mathscr{D}(B^*)$. If $k^* = 1$, then $\mathrm{val}(\Gamma_{\mathscr{B}^*}) = v \cdot \mathrm{val}(\Gamma) > v = |B^*|$, and hence the actions of G_{B^*} on B^* and $\Gamma_{\mathscr{B}^*}(B^*)$ cannot be permutationally equivalent. From Theorem 3.4(b), this is the case in particular when $\mathrm{girth}(\Gamma_{\mathscr{B}}) \geqslant 7$. Thus the G-invariant partition $\mathscr{B}^*$ may not satisfy (PE). Moreover if $k^* = 1$, then the construction given in [13, Section 4] applies, and so Γ can be constructed from a certain G-point- and G-block-transitive 1-design with point set $\mathscr{B}^*$.

4. The 1-design $\mathscr{D}(B)$

Part (d) of Theorem 3.2 is equivalent to saying that, if ("BC", D) is a flag of $\mathscr{D}(B)$, then $C \neq D$ and hence (C,B,D) is a 2-arc of $\Gamma_{\mathscr{B}}$. Denote by $\bar{A}_2(\Gamma_{\mathscr{B}})$ the set of all such 2-arcs of $\Gamma_{\mathscr{B}}$, that is,

$$\bar{A}_2(\Gamma_{\mathscr{B}}) := \{(C,B,D) : \text{“}BC\text{”} \mathrm{I} D\}.$$

Denote by $\mathscr{D}^*(B)=(\Gamma_{\mathscr{B}}(B),B,\mathrm{I}^*)$ the dual 1-design of $\mathscr{D}(B)$, so $C\mathrm{I}^*\alpha$ for $C\in\Gamma_{\mathscr{B}}(B)$ and $\alpha\in B$ if and only if $\alpha\mathrm{I}C$. The main result of this section is the following theorem, which gives more information about the 1-design $\mathscr{D}(B)$.

Theorem 4.1. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent, for some $B\in\mathscr{B}$. Then the following* (a)–(d) *hold*:

(a) *Both $\mathscr{D}(B)$ and $\mathscr{D}^*(B)$ are 1-(v,k,k) designs.*
(b) *$\bar{A}_2(\Gamma_{\mathscr{B}})$ is a G-orbit on $A_2(\Gamma_{\mathscr{B}})$, $k=|C^{G_{B,D}}|$, and $k+m\leqslant v$, where $(C,B,D)\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and m is the multiplicity of $\mathscr{D}(B)$.*
(c) *The following conditions* (i)–(iv) *are equivalent*:
  (i) *$\Gamma_{\mathscr{B}}$ is $(G,2)$-arc transitive*;
  (ii) *$\bar{A}_2(\Gamma_{\mathscr{B}})=A_2(\Gamma_{\mathscr{B}})$*;
  (iii) *$k=v-1$*;
  (iv) *$k=v-1$ and $\mathscr{D}(B)$ contains no repeated blocks.*
(d) *$\Gamma[B,C]\cong K_{k,k}$ if and only if $G_{B,C,D}$ is transitive on $\Gamma(B)\cap D$, for $(C,B,D)\in\bar{A}_2(\Gamma_{\mathscr{B}})$. In particular, $\Gamma[B,C]\cong K_{v-1,v-1}$ if and only if $\Gamma_{\mathscr{B}}$ is $(G,3)$-arc transitive.*

Proof. (a) That $\mathscr{D}(B)$ is a 1-design implies $vr=bk$. Since $b=v$ (Lemma 3.1), we have $r=k$ and hence both $\mathscr{D}(B)$ and $\mathscr{D}^*(B)$ are 1-(v,k,k) designs.

(b) Let $(C,B,D),(C_1,B_1,D_1)\in\bar{A}_2(\Gamma_{\mathscr{B}})$. Then "$BC$" is adjacent to a vertex $\beta\in D$ and "B_1C_1" is adjacent to a vertex $\beta_1\in D_1$. So "B^xC^x" is adjacent to $\beta^x\in D^x$ for any $x\in G$. Thus $(C^x,B^x,D^x)\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and hence $\bar{A}_2(\Gamma_{\mathscr{B}})$ is G-invariant. On the other hand, since Γ is G-symmetric, there exists $y\in G$ such that ("BC", $\beta)^y=$ ("B_1C_1", β_1). This implies $(C,B,D)^x=(C_1,B_1,D_1)$ and hence G is transitive on $\bar{A}_2(\Gamma_{\mathscr{B}})$. Therefore, $\bar{A}_2(\Gamma_{\mathscr{B}})$ is a G-orbit on $A_2(\Gamma_{\mathscr{B}})$. From this we have: "$BE$" $\in B$ is adjacent to a vertex in $D\Leftrightarrow(E,B,D)\in\bar{A}_2(\Gamma_{\mathscr{B}})\Leftrightarrow$ there exists $x\in G$ such that $(C,B,D)^x=(E,B,D)\Leftrightarrow$ there exists $x\in G_{B,D}$ such that $C^x=E$. So we have $k=|C^{G_{B,D}}|$. Now suppose $D_1,\dots,D_m\in\Gamma_{\mathscr{B}}(B)$ are repeated blocks of $\mathscr{D}(B)$ (that is, $\Gamma(D_1)\cap B=\cdots=\Gamma(D_m)\cap B$). Then by Theorem 3.2(d), none of the m distinct vertices "BD_1",…,"BD_m" of B is in $\Gamma(D_1)\cap B$, and hence $k+m\leqslant v$.

(c) Clearly, (i) and (ii) are equivalent since $\bar{A}_2(\Gamma_{\mathscr{B}})$ is a G-orbit on $A_2(\Gamma_{\mathscr{B}})$. Note that $k=v-1$ implies $k=v-1\geqslant 2$ for otherwise we would have $\mathrm{val}(\Gamma)=1$, contradicting our assumption on the valency of Γ. From the argument in the proof of (b), we have: $k=v-1\Leftrightarrow k=v-1\geqslant 2\Leftrightarrow G_{B,D}$ is transitive on $\Gamma_{\mathscr{B}}(B)\setminus\{D\}\Leftrightarrow G_B$ is 2-transitive on $\Gamma_{\mathscr{B}}(B)\Leftrightarrow\Gamma_{\mathscr{B}}$ is $(G,2)$-arc transitive. So (i) and (iii) are equivalent. Clearly, (iv) implies (iii). Conversely, since $k+m\leqslant v$ as we have shown above, $k=v-1$ implies $m=1$ and hence $\mathscr{D}(B)$ has no repeated blocks. The equivalence of (i)−(iv) is then established.

(d) Let $(C,B,D)\in\bar{A}_2(\Gamma_{\mathscr{B}})$. Then, by the G-symmetry of Γ, $G_{\text{“}BC\text{”},D}=G_{B,C,D}$ is transitive on $\Gamma(\text{“}BC\text{”})\cap D\neq\emptyset$. Clearly, we have: $\Gamma[B,D]\cong K_{k,k}\Leftrightarrow\Gamma(\text{“}BC\text{”})\cap$

$D=\Gamma(B)\cap D \Leftrightarrow G_{\text{“}BC\text{”},D}$ is transitive on $\Gamma(B)\cap D \Leftrightarrow G_{B,C,D}$ is transitive on $\Gamma(B)\cap D$. In particular, from (c) above and Theorem 3.2(d) we have: $\Gamma[B,D] \cong K_{v-1,v-1} \Leftrightarrow k=v-1$ and $G_{B,C,D}$ is transitive on $D \setminus \{\text{“}DB\text{”}\} \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive and $G_{B,C,D}$ is transitive on $\Gamma_{\mathcal{B}}(D)\setminus\{B\} \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G,3)$-arc transitive. □

Remark 4.2. As mentioned in the introduction, if $k=v-1\geqslant 2$ and $\mathcal{D}(B)$ contains no repeated blocks, then the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent. So part (c) of Theorem 4.1 implies the result [6, Theorem 8] that in this case $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. Furthermore, it shows that, under the assumption (PE), this is the only case where $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. Part (d) of Theorem 4.1 implies that in such a case $\Gamma_{\mathcal{B}}$ is $(G,3)$-arc transitive if and only if $\Gamma[B,C]\cong K_{v-1,v-1}$ ([6, Theorem 2]), and that this is the only case where $\Gamma_{\mathcal{B}}$ is $(G,3)$-arc transitive.

5. The case where ρ is incidence-preserving

In this section, we study the case where the bijection ρ in (PE) is *incidence-preserving* in the sense that it satisfies

$$\alpha \mathrm{I} D \iff \rho^{-1}(D)\mathrm{I}\rho(\alpha) \tag{2}$$

for $\alpha\in B$ and $D\in\Gamma_{\mathcal{B}}(B)$. Using labels for vertices of Γ, this condition can be restated as

$$\text{“}BC\text{”}\mathrm{I}D \iff \text{“}BD\text{”}\mathrm{I}C \tag{3}$$

for distinct $C,D\in\Gamma_{\mathcal{B}}(B)$, which in turn is equivalent to saying that

$$(C,B,D)\in\bar{A}_2(\Gamma_{\mathcal{B}}) \iff (D,B,C)\in\bar{A}_2(\Gamma_{\mathcal{B}}). \tag{4}$$

Thus, in view of Theorem 4.1(b), one of the above holds if and only if $\bar{A}_2(\Gamma_{\mathcal{B}})$ is a self-paired G-orbit on $A_2(\Gamma_{\mathcal{B}})$. By Theorem 4.1(c), this is the case in particular when $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. However, there are other cases in which (3) is satisfied. This happens for the graph Γ in Example 2.2, where (3) is satisfied (see the last sentence in that example) but $\Gamma_{\mathcal{B}}$ is not $(G,2)$-arc transitive by Theorem 4.1(c) and the fact that $4=k<v-1=5$.

The additional requirement above implies immediately that $\mathcal{D}(B)$ is a self-dual 1-design, as stated below.

Proposition 5.1. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that, for some $B\in\mathcal{B}$, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent with respect to an incidence-preserving bijection ρ. Then $\mathcal{D}(B)$ is a self-dual 1-(v,k,k) design and ρ induces a polarity of $\mathcal{D}(B)$.*

Proof. Let ϕ be the bijection from $B \cup \Gamma_{\mathscr{B}}(B)$ to $\Gamma_{\mathscr{B}}(B) \cup B$ defined by $\phi(\alpha)=\rho(\alpha)$, $\phi(C)=\rho^{-1}(C)$ for $\alpha \in B$ and $C \in \Gamma_{\mathscr{B}}(B)$. Then $\phi(B)=\Gamma_{\mathscr{B}}(B)$, $\phi(\Gamma_{\mathscr{B}}(B))=B$, and (2) implies that $\alpha \mathrm{I} C \Leftrightarrow \phi(C)\mathrm{I}\phi(\alpha) \Leftrightarrow \phi(\alpha)\mathrm{I}^*\phi(C)$. Thus, ϕ is an isomorphism from $\mathscr{D}(B)$ to $\mathscr{D}^*(B)$ and hence $\mathscr{D}(B)$ is self-dual. Clearly, we have $\phi^2=1$ and hence ϕ is a polarity of $\mathscr{D}(B)$. □

For brevity we call a chordless 6-cycle in a given graph a *hexagon*, where a chord of a cycle is an edge joining two non-consecutive vertices of the cycle. Recall that in Section 3, we defined Γ' to be the graph with vertex set $V(\Gamma)$ and edge set $\{\{\alpha,\alpha'\}\colon \alpha \in V(\Gamma)\}$. In the case where (b) in Theorem 3.3 occurs, we have the following result which is interesting from a combinatorial point of view.

Theorem 5.2. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that, for some $B \in \mathscr{B}$, the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent with respect to an incidence-preserving bijection. Suppose further that adjacent vertices of Γ have the same second coordinate. Then there exists a G-invariant set $\mathscr{H}$ of hexagons of the graph $\Gamma \cup \Gamma'$ such that*

(a) *the edges of each hexagon of $\mathscr{H}$ lie in Γ and Γ' alternatively;*
(b) *each edge of Γ belongs to a unique hexagon of $\mathscr{H}$, and each edge of Γ' belongs to exactly k hexagons of $\mathscr{H}$; and*
(c) *any two hexagons of $\mathscr{H}$ have at most one common edge.*

Proof. Let {"BC", "DC"} be an edge of Γ. Then "BC"ID and "DC"IB. From (3) and our assumption on labels of adjacent vertices, it follows that "BD" is adjacent to "CD" and "DB" is adjacent to "CB". It is easy to see that h{"BC","DC"}:= ("BC","DC","CD","BD","DB","CB","BC") is a hexagon of $\Gamma \cup \Gamma'$ whose edges belong to Γ and Γ' alternatively. (See Fig. 2, where the dashed lines represent edges of Γ'.) Set

$\mathscr{H}$:={h{"BC","DC"}: ("BC","DC") $\in A(\Gamma)$}.

Since both Γ and Γ' are G-symmetric, $\mathscr{H}$ is G-invariant. One can see that h{"BC","DC"} $=h${"CD","BD"} $=h${"DB","CB"}, and this is the unique hexagon in $\mathscr{H}$ containing the edge {"BC","DC"} of Γ. By Theorem 3.3(b), we have $\Gamma[B,D] \cong kK_2$. When {"BC","DC"} runs over all the edges of $\Gamma[B,D]$, we get k hexagons h{"BC","DC"}, and these are the only members of $\mathscr{H}$ containing the edge {"BD","DB"} of Γ'. By the definition of the hexagons of $\mathscr{H}$, the validity of (c) is clear. □

The case where two adjacent vertices of Γ have labels involving four distinct blocks seems to be much more complicated, even under our additional assumption that ρ is incidence-preserving. So we concentrate on the extreme case where $\Gamma[B,C] \cong kK_2$ is a matching. In this case, the following theorem shows that there exists a G-orbit

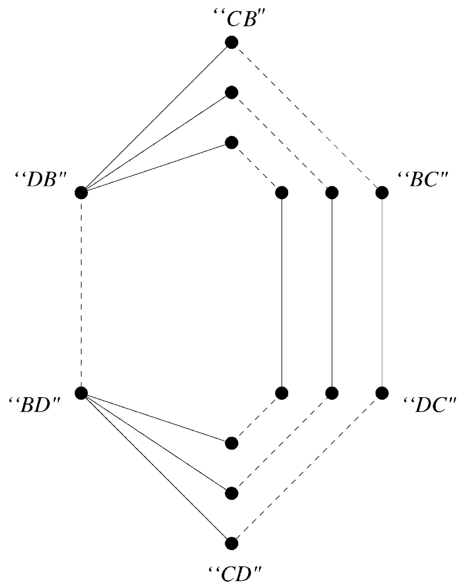

Fig. 2. Hexagons in $\Gamma \cup \Gamma'$.

on n-cycles of $\Gamma_{\mathscr{B}}$, for some even integer $n \geqslant 4$, which determines completely the adjacency of Γ.

Theorem 5.3. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that, for some $B \in \mathscr{B}$, the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent with respect to an incidence-preserving bijection. Suppose further that adjacent vertices of Γ have labels involving four distinct blocks and that $\Gamma[B,C] \cong kK_2$ for adjacent blocks B, C of $\mathscr{B}$. Then there exist an even integer $n \geqslant 4$ and a G-orbit $\mathscr{O}$ on n-cycles of $\Gamma_{\mathscr{B}}$ such that*

(a) *no two n-cycles of $\mathscr{O}$ have a 2-arc in common*;
(b) $(C,B,D) \in \bar{A}_2(\Gamma_{\mathscr{B}})$ *if and only if* (C,B,D) *is contained in a* (*unique*) *n-cycle of $\mathscr{O}$*; *and*
(c) *two vertices "BC","DE" of Γ are adjacent if and only if (C,B,D,E) is a 3-arc of $\Gamma_{\mathscr{B}}$ contained in an n-cycle of $\mathscr{O}$.*

Proof. Let $(B_0,B_1,B_2) \in \bar{A}_2(\Gamma_{\mathscr{B}})$, that is, "$B_1B_0$"I$B_2$. Then, since $\Gamma[B_1,B_2] \cong kK_2$ by our assumption, there exists a unique block $B_3 \in \Gamma_{\mathscr{B}}(B_2)$ such that "B_2B_3" is the unique vertex in B_2 adjacent to "B_1B_0". This implies $(B_3,B_2,B_1) \in \bar{A}_2(\Gamma_{\mathscr{B}})$ and hence $(B_1,B_2,B_3) \in \bar{A}_2(\Gamma_{\mathscr{B}})$ by (4). Thus "B_2B_1"IB_3 and hence there exists a unique block $B_4 \in \Gamma_{\mathscr{B}}(B_3)$ such that "B_3B_4" is the unique vertex in B_3 adjacent to "B_2B_1". This in turn implies that $(B_4,B_3,B_2) \in \bar{A}_2(\Gamma_{\mathscr{B}})$ and hence $(B_2,B_3,B_4) \in \bar{A}_2(\Gamma_{\mathscr{B}})$. Inductively, suppose that $B_0,B_1,B_2,\ldots,B_i$ have been determined for some $i \geqslant 3$ such that

$(B_{j-1},B_j,B_{j+1}),(B_{j+1},B_j,B_{j-1})\in\bar{A}_2(\Gamma_{\mathscr{B}})$ for $j=1,2,\ldots,i-1$, and that "B_jB_{j-1}" is adjacent to "$B_{j+1}B_{j+2}$" for $j=1,2,\ldots,i-2$. Then in particular "$B_{i-1}B_{i-2}$"$\mathrm{I}B_i$ and hence there exists a unique block $B_{i+1}\in\Gamma_{\mathscr{B}}(B_i)$ such that "B_iB_{i+1}" is the unique vertex in B_i adjacent to "$B_{i-1}B_{i-2}$". Thus we have $(B_{i+1},B_i,B_{i-1})\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and hence $(B_{i-1},B_i,B_{i+1})\in\bar{A}_2(\Gamma_{\mathscr{B}})$. Continuing this process, we see that each 2-arc (B_0,B_1,B_2) in $\bar{A}_2(\Gamma_{\mathscr{B}})$ determines a unique sequence $B_0,B_1,B_2,\ldots,B_i,B_{i+1},\ldots$ of blocks of $\mathscr{B}$ such that $(B_{i-1},B_i,B_{i+1}),(B_{i+1},B_i,B_{i-1})\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and "B_iB_{i-1}" is adjacent to "$B_{i+1}B_{i+2}$" for each $i\geqslant 1$. Our assumption on labels of adjacent vertices of Γ implies that any four consecutive blocks in this sequence are pairwise distinct. Since we have only a finite number of blocks in $\mathscr{B}$, this sequence must contain repeated terms. Let B_n be the first block in the sequence which coincides with one of the preceding blocks. Then $n\geqslant 4$ and we claim that B_n must coincide with B_0. Suppose to the contrary that $B_n=B_m$ for some integer m with $m\geqslant 1$. Then, since $\bar{A}_2(\Gamma_{\mathscr{B}})$ is a G-orbit on $A_2(\Gamma_{\mathscr{B}})$ (Theorem 4.1(b)), there exists $x\in G$ such that $(B_m^x,B_{m+1}^x,B_{m+2}^x)=(B_0,B_1,B_2)$. By the construction above, one can see that the sequence determined by $(B_m^x,B_{m+1}^x,B_{m+2}^x)$ is $B_m^x,B_{m+1}^x,B_{m+2}^x,\ldots,B_{m+i}^x,\ldots$. So by the uniqueness of the sequence determined by (B_0,B_1,B_2) we must have $B_{m+i}^x=B_i$ for each $i\geqslant 0$. In particular, we have $B_n^x=B_{m+(n-m)}^x=B_{n-m}$. On the other hand, $B_n=B_m$ implies that $B_n^x=B_m^x=B_0$. Thus we have $B_{n-m}=B_0$, which contradicts the minimality of m. So B_n must coincide with B_0 and we get an n-cycle $O(B_0,B_1,B_2):=(B_0,B_1,B_2,\ldots,B_{n-1},B_0)$ of $\Gamma_{\mathscr{B}}$. Note that $(B_2,B_1,B_0)\in\bar{A}_2(\Gamma_{\mathscr{B}})$ implies that there exists a unique block $C\in\Gamma_{\mathscr{B}}(B_0)$ such that "B_0C" is the unique vertex in B_0 adjacent to "B_1B_2". So we have $(C,B_0,B_1)\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and, by the construction above, the sequence determined by (C,B_0,B_1) is $C,B_0,B_1,B_2,\ldots,B_i,\ldots$. Since the first repeated block in this sequence is C, as shown above, we must have $C=B_{n-1}$ and hence $(B_{n-1},B_0,B_1),(B_1,B_0,B_{n-1})\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and "B_0B_{n-1}" is adjacent to "B_1B_2". In a similar way, one can show that $(B_{n-2},B_{n-1},B_0),(B_0,B_{n-1},B_{n-2})\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and "$B_{n-1}B_{n-2}$" is adjacent to "B_0B_1". Therefore, reading the subscripts modulo n (here and in the remainder of the proof), we have $(B_{i-1},B_i,B_{i+1}),(B_{i+1},B_i,B_{i-1})\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and "B_iB_{i-1}" is adjacent to "$B_{i+1}B_{i+2}$" for each $i\geqslant 1$. Hence n must be an even integer and, by definition, all these 2-arcs contained in $O(B_0,B_1,B_2)$ determine the same n-cycle, namely $O(B_0,B_1,B_2)$. By Theorem 4.1(b) any 2-arc in $\bar{A}_2(\Gamma_{\mathscr{B}})$ has the form (B_0^x,B_1^x,B_2^x) for some $x\in G$, and by definition we have $O(B_0^x,B_1^x,B_2^x)=(B_0^x,B_1^x,B_2^x,\ldots,B_{n-1}^x,B_0^x)=(O(B_0,B_1,B_2))^x$. This implies that $\mathscr{O}:=\{O(C,B,D):(C,B,D)\in\bar{A}_2(\Gamma_{\mathscr{B}})\}$ is a G-orbit on n-cycles of $\Gamma_{\mathscr{B}}$. Note that, for a given 2-arc (C,B,D) of $\bar{A}_2(\Gamma_{\mathscr{B}})$, $O(C,B,D)$ is the unique n-cycle in $\mathscr{O}$ containing (C,B,D). So (a) and (b) are true. If "BC","DE" are adjacent in Γ, then $(C,B,D)\in\bar{A}_2(\Gamma_{\mathscr{B}})$ and by the argument above (C,B,D,E) is a 3-arc contained in $O(C,B,D)$. Conversely, from the definition of the n-cycles in $\mathscr{O}$, for each 3-arc (C,B,D,E) contained in an n-cycle of $\mathscr{O}$, "BC","DE" are adjacent in Γ and hence (c) follows. $\square$

For a G-symmetric graph Γ admitting a nontrivial G-invariant partition $\mathscr{B}$ of block size $v\geqslant 3$, if girth$(\Gamma_{\mathscr{B}})\geqslant 4$ and $\mathscr{D}(B)$ contains no repeated blocks, and if Γ *almost covers* $\Gamma_{\mathscr{B}}$ in the sense that $\Gamma[B,C]\cong(v-1)K_2$, then the conditions of Theorem

5.3 are satisfied. In this case, we have $\bar{A}_2(\Gamma_{\mathscr{B}}) = A_2(\Gamma_{\mathscr{B}})$ by Theorem 4.1(c), and hence Theorem 5.3 implies that $\Gamma_{\mathscr{B}}$ is a near n-gonal graph with respect to $\mathscr{C}$, that is, each 2-arc of $\Gamma_{\mathscr{B}}$ is contained in a unique n-cycle of $\mathscr{C}$. Near-polygonal graphs were introduced in [7], and a systematic study of almost covers of 2-arc transitive near-polygonal graphs was conducted in [12].

6. Three-arc graphs and the reconstruction of Γ

Let Σ be a regular graph of valency $v \geqslant 2$ and Δ a self-paired subset of $A_3(\Sigma)$. As pointed out in [6, Section 6], if $G \leqslant \mathrm{Aut}(\Sigma)$ leaves Δ invariant, then G preserves the adjacency relation of the 3-arc graph $\mathrm{Arc}_\Delta(\Sigma)$ and hence induces a faithful action as a group of automorphisms of $\mathrm{Arc}_\Delta(\Sigma)$. (See the introduction for the definition of a 3-arc graph.) Moreover, the vertex set $A(\Sigma)$ of $\mathrm{Arc}_\Delta(\Sigma)$ admits the following three G-invariant partitions:

(i) $\mathscr{B}(\Sigma) := \{B(\sigma)\colon \sigma \in V(\Sigma)\}$, where $B(\sigma) := \{(\sigma,\tau)\colon \tau \in \Sigma(\sigma)\}$;
(ii) $\mathscr{B}^*(\Sigma) := \{B^*(\sigma)\colon \sigma \in V(\Sigma)\}$, where $B^*(\sigma) := \{(\tau,\sigma)\colon \tau \in \Sigma(\sigma)\}$;
(iii) $\mathscr{P}(\Sigma) := \{\{(\sigma,\tau),(\tau,\sigma)\}\colon (\sigma,\tau) \in A(\Sigma)\}$.

Lemma 6.1 (Li et al. [6, Lemma 3]). *Let Σ, Δ be as above, and let $G \leqslant \mathrm{Aut}(\Sigma)$ leave Δ invariant. Then*

(a) *G is transitive on the vertices of $\mathrm{Arc}_\Delta(\Sigma)$ if and only if Σ is G-symmetric.*
(b) *G is transitive on the arcs of $\mathrm{Arc}_\Delta(\Sigma)$ if and only if G is transitive on Δ.*
(c) *For $\sigma \in V(\Sigma), G_\sigma = G_{B(\sigma)} = G_{B^*(\sigma)}$, and the actions of G_σ on $\Sigma(\sigma)$, $B(\sigma)$ and $B^*(\sigma)$ are permutationally equivalent.*

In fact, the action of G_σ on $\Sigma(\sigma)$ is permutationally equivalent to the actions of G_σ on $B(\sigma)$, $B^*(\sigma)$ with respect to the bijections defined by $\sigma' \mapsto (\sigma,\sigma')$, $\sigma' \mapsto (\sigma',\sigma)$, for $\sigma' \in \Sigma(\sigma)$, respectively. Thus, if Σ is a G-symmetric graph and Δ is a self-paired G-orbit on $A_3(\Sigma)$, then by Lemma 6.1, $\mathrm{Arc}_\Delta(\Sigma)$ is a G-symmetric graph satisfying (PE) for the G-invariant partition $\mathscr{B}(\Sigma)$. By using the labelling technique developed in Section 3, we now prove that any G-symmetric graph Γ satisfying (PE) has this form, namely Γ is isomorphic to a 3-arc graph of $\Gamma_{\mathscr{B}}$ relative to a certain self-paired G-orbit on $A_3(\Gamma_{\mathscr{B}})$. Therefore, such a graph Γ can be reconstructed from the quotient $\Gamma_{\mathscr{B}}$ and the action of G on $\mathscr{B}$.

Theorem 6.2. *Suppose that Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that, for some $B \in \mathscr{B}$, the actions of G_B on B and $\Gamma_{\mathscr{B}}(B)$ are permutationally equivalent, so the vertices of Γ are labelled by ordered pairs of adjacent blocks of $\Gamma_{\mathscr{B}}$. Then $\Gamma \cong \mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$ for Δ the (self-paired) G-orbit on $A_3(\Gamma_{\mathscr{B}})$ containing the 3-arc (C,B,D,E), where ("BC","DE") is an arc of Γ.*

Conversely, for any G-symmetric graph Σ and any self-paired G-orbit Δ on $A_3(\Sigma)$, the triple $(\Gamma, G, \mathscr{B})$, where $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$ and $\mathscr{B} = \mathscr{B}(\Sigma)$, satisfies all the conditions above. Moreover, we have $\Gamma_{\mathscr{B}} \cong \Sigma$.

Proof. Let Γ, G and $\mathscr{B}$ be as in the first part of the theorem. Let ("BC","DE") be a fixed arc of Γ. Then by Theorem 3.2(d), (C,B,D,E) is a 3-arc of $\Gamma_{\mathscr{B}}$. Let Δ be the G-orbit on $A_3(\Gamma_{\mathscr{B}})$ containing (C,B,D,E). Since Γ is G-symmetric, there exists $x \in G$ such that ("BC","DE")^x = ("DE","BC"). So $(E,D,B,C) = (C,B,D,E)^x \in \Delta$ by (1), and hence Δ is self-paired. Again by the G-symmetry of Γ and (1), we have: $(C_1,B_1,D_1,E_1) \in \Delta \Leftrightarrow$ there exists $x \in G$ such that $(C_1,B_1,D_1,E_1) = (C,B,D,E)^x \Leftrightarrow$ there exists $x \in G$ such that ("B_1C_1","D_1E_1") = ("BC","DE")^x $\Leftrightarrow$ ("B_1C_1","D_1E_1") $\in A(\Gamma)$. Therefore, the mapping "B_1C_1" $\mapsto (B_1,C_1)$, for "B_1C_1" $\in V(\Gamma)$, establishes a graph isomorphism from Γ to $\mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$.

Now suppose Σ is a G-symmetric graph and Δ is a self-paired G-orbit on $A_3(\Sigma)$, and let $(\tau,\sigma,\sigma',\tau') \in \Delta$. Then from Lemma 6.1, $\Gamma := \mathrm{Arc}_\Delta(\Sigma)$ is a G-symmetric graph with $\mathscr{B} := \mathscr{B}(\Sigma)$ a G-invariant partition of $V(\Gamma)$. If $B(\alpha)$ and $B(\alpha')$ are adjacent blocks of $\mathscr{B}$, then there exist $(\alpha,\beta) \in B(\alpha)$ and $(\alpha',\beta') \in B(\alpha')$ such that $(\alpha,\beta),(\alpha',\beta')$ are adjacent in Γ, and hence $(\beta,\alpha,\alpha',\beta') \in \Delta$. In particular, we have $(\alpha,\alpha') \in A(\Sigma)$. Conversely, suppose $(\alpha,\alpha') \in A(\Sigma)$. Then since Σ is G-symmetric there exists $x \in G$ such that $(\sigma,\sigma')^x = (\alpha,\alpha')$. Set $\tau^x = \beta$ and $(\tau')^x = \beta'$, then $(\beta,\alpha,\alpha',\beta') = (\tau,\sigma,\sigma',\tau')^x \in \Delta$. So $(\alpha,\beta) \in B(\alpha)$ is adjacent to $(\alpha',\beta') \in B(\alpha')$ in Γ and hence $B(\alpha)$ and $B(\alpha')$ are adjacent blocks of $\mathscr{B}$. Thus, $\alpha \mapsto B(\alpha)$ defines an isomorphism from Σ to $\Gamma_{\mathscr{B}}$. From Lemma 6.1(c), the actions of $G_{B(\sigma)}$ on $B(\sigma)$ and $\Sigma(\sigma)$ are permutationally equivalent with respect to the bijection $(\sigma,\sigma') \mapsto \sigma'$. So the actions of $G_{B(\sigma)}$ on $B(\sigma)$ and $\Gamma_{\mathscr{B}}(B(\sigma))$ are permutationally equivalent with respect to the bijection $\rho : (\sigma,\sigma') \mapsto B(\sigma')$. $\square$

Remark 6.3. (a) Theorem 6.2 is a counterpart of [6, Theorems 10(a) and (b) and 11], where $\Gamma_{\mathscr{B}}$ and Σ are assumed to be $(G,2)$-arc transitive and similar result is proved.

(b) From Theorem 5.3(c) and the proof above one can see that, under the assumptions of Theorem 5.3, the self-paired G-orbit Δ on $A_3(\Gamma_{\mathscr{B}})$ such that $\mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}}) \cong \Gamma$ is precisely the set of all 3-arcs of $\Gamma_{\mathscr{B}}$ contained in some n-cycle of $\mathscr{O}$.

Conversely, if, for a G-symmetric graph Σ, there exist an even integer $n \geqslant 4$ and a G-orbit $\mathscr{O}$ on n-cycles of Σ such that each 2-arc of Σ is contained in at most one n-cycle of $\mathscr{O}$, and that the set of 2-arcs of Σ contained in some n-cycle of $\mathscr{O}$ is a G-orbit on $A_2(\Sigma)$, then one can check that the following (i)–(iii) hold:

(i) the set Δ of 3-arcs of Σ contained in some n-cycle of $\mathscr{O}$ is a self-paired G-orbit on $A_3(\Sigma)$, and thus $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$ is well-defined;

(ii) for $\mathscr{B} = \mathscr{B}(\Sigma)$, the bijection ρ from $B(\sigma)$ to $\Gamma_{\mathscr{B}}(B(\sigma))$ defined at the end of the proof of Theorem 6.2 is incidence-preserving; and

(iii) $\Gamma[B(\sigma), B(\sigma')] \cong kK_2$ for adjacent blocks $B(\sigma)$, $B(\sigma')$ of $\mathscr{B}$.

These results together give the inverse of Theorem 5.3.

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