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Finite locally-quasiprimitive graphs

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Abstract

A finite graph is said to be locally-quasiprimitive relative to a subgroup G of automorphisms if, for all vertices α, the stabiliser in G of α is quasiprimitive on the set of vertices adjacent to α. (A permutation group is said to be quasiprimitive if all of its non-trivial normal subgroups are transitive.) The graph theoretic condition of local quasiprimitivity is strictly weaker than the conditions of local primitivity and 2-arc transitivity which have been studied previously. It is shown that the family of locally-quasiprimitive graphs is closed under the formation of a certain kind of quotient graph, called a normal quotient, induced by a normal subgroup. Moreover, a locally-quasiprimitive graph is proved to be a multicover of each of its normal quotients. Thus finite locally-quasiprimitive graphs which are minimal in the sense that they have no non-trivial proper normal quotients form an important sub-family, since each finite locally-quasiprimitive graph has at least one such graph as a normal quotient. These minimal graphs in the family are called "basic" locally-quasiprimitive graphs, and their structure is analysed. The process of constructing locally-quasiprimitive graphs with a given locally-quasiprimitive graph Σ as a normal quotient is then considered. It turns out that this can be viewed as a problem of constructing covering graphs of certain multigraphs associated with Σ. Further, it is shown that, under certain conditions, a locally-quasiprimitive graph can be reconstructed from knowledge of two of its normal quotients. Finally a series of open problems is presented.

Keywords: Arc-transitive graph; Quasiprimitive permutation group; Normal quotient; Normal cover

1. Introduction

Perhaps the most celebrated use of group theory in effecting a classification of a class of graphs is that leading to the classification of the finite distance transitive graphs, which is now approaching completion, see [12]. The suggestion that this classification might indeed be feasible comes from early work of Biggs and Smith [2,27] which in a sense reduced the problem to the case of vertex-primitive distance transitive graphs.

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Then work of Saxl, Yokoyama and the second author in [25] further reduced the vertex-primitive classification to the case where the automorphism group is almost simple or affine.

The key step of reducing a classification problem to the vertex-primitive case seems not to be possible for many other interesting families of finite arc-transitive graphs. However, some families of arc-transitive graphs possess a weaker property which still allows certain members of the family to be designated as "basic", and also allows the structure of an arbitrary member of the family to be described in terms of some of the basic members. The largest family of finite arc-transitive graphs which possesses one such property is the family $\mathscr{F}$ of finite, locally-quasiprimitive, arc-transitive graphs, and these graphs are the subject of this paper.

A *graph* $\Gamma=(V,E)$ consists of a set V of vertices and a subset E of unordered pairs from V, called edges. Such a graph is called a *simple graph* since there is at most one edge between each pair of vertices. In Section 3, we need to extend this definition to allow multiple edges. A group G of permutations of a set Δ is said to be *quasiprimitive* if each non-trivial normal subgroup of G is transitive on Δ. For a graph Γ, and a group G acting as a group of automorphisms of Γ (not necessarily faithfully), we say that Γ is *G-arc-transitive* if G acts transitively on the arcs of Γ (*arcs* being ordered pairs of vertices joined by an edge of Γ), and *G-locally-quasiprimitive* if, for each vertex α, the stabiliser G_α is quasiprimitive in its action on the set $\Gamma(\alpha)=\{\beta\colon \{\alpha,\beta\}\in E\}$ of *neighbours* of α in Γ.

The family of finite vertex-transitive locally-quasiprimitive graphs contains several families of arc-transitive graphs which have been studied extensively, for example, 2-arc transitive graphs and locally-primitive graphs, (which are vertex-transitive graphs Γ such that the stabilizer of each vertex α is 2-transitive or primitive on $\Gamma(\alpha)$, respectively). It is well-known that the family of 2-arc-transitive graphs is a proper subset of the family of locally-primitive graphs. Similarly, the family of locally-primitive graphs is a proper subset of the family of locally-quasiprimitive graphs, as the following example demonstrates.

Example 1.1. For each prime $p\geqslant 29$ such that $p\equiv \pm 1 \pmod 5$, there exists a graph Γ of valency 20 with $\mathrm{Aut}\,\Gamma\cong \mathrm{PSL}(2,p)$ acting primitively on vertices such that $\mathrm{Aut}\,\Gamma$ is locally-quasiprimitive but not locally-primitive on Γ. (See Proposition 2.1 for the construction and proof.)

Definition 1.2. Let $\mathscr{F}$ be the family of those graphs Γ which are G-vertex-transitive and G-locally-quasiprimitive for some $G\leqslant \mathrm{Aut}(\Gamma)$. In such a case we say that $\Gamma\in\mathscr{F}$ *with respect to G*.

The fundamental observation about the class $\mathscr{F}$ of finite locally-quasiprimitive, arc-transitive graphs is that it is closed under the formation of a certain type of quotient graph. For $\mathscr{P}$ a partition of the vertex set V of a graph Γ, we define the *quotient graph* $\Gamma_{\mathscr{P}}$ of Γ relative to $\mathscr{P}$ as the graph with vertex set $\mathscr{P}$ such that two parts P,P'

form an edge if and only if there is at least one edge of Γ joining a vertex of P and a vertex of P'. If $\mathscr{P}$ is *G-invariant* for some group G of automorphisms of Γ (that is, G permutes the parts of $\mathscr{P}$ setwise), then the action of G on Γ induces a natural action of G as a group of automorphisms of $\Gamma_{\mathscr{P}}$. In this case, although the property of arc-transitivity is preserved, more restrictive local properties, such as local quasiprimitivity, are not in general inherited by the action of G on the quotient graph. However, local quasiprimitivity is inherited by quotients relative to normal partitions. We call a partition $\mathscr{P}$ of vertices *G-normal relative to N* if N is a normal subgroup of G and $\mathscr{P}$ is the set of N-orbits in V; for such partitions we write $\mathscr{P}=\mathscr{P}_N$, and we write the quotient graph $\Gamma_{\mathscr{P}}$ as Γ_N, and call Γ_N a *normal quotient*, or a *G-normal quotient*, of Γ. When N has more than two orbits in V, not only is Γ_N a G-locally-quasiprimitive graph, but also Γ is a multicover of Γ_N and N is semiregular on vertices. (A graph Γ is said to be a *multicover* of its quotient graph $\Gamma_{\mathscr{P}}$ if, for each edge $\{P,P'\}$ of $\Gamma_{\mathscr{P}}$ and each $\alpha\in P$, the cardinality $|\Gamma(\alpha)\cap P'|>0$. In the case where the cardinality $|\Gamma(\alpha)\cap P'|$ is always 1 we say that Γ is a *cover* of $\Gamma_{\mathscr{P}}$. A permutation group N on a set V is *semiregular* on V if the only element of N which fixes a point of V is the identity. If a group G has an action on a set V then G^V denotes the permutation group induced by G on V.)

Theorem 1.3 ([19, Section 1]). *Let $\Gamma=(V,E)$ be a finite connected G-vertex-transitive, G-locally-quasiprimitive graph of valency v, and let N be a normal subgroup of G. Then one of the following holds.*

(a) *N is transitive on V; or*
(b) *Γ is bipartite and the N-orbits in V are the two parts of the bipartition of Γ; or*
(c) *N has more than two orbits in V, $\Gamma_N=(\mathscr{P}_N,E_N)$ is a connected G-arc-transitive, G-locally-quasiprimitive graph of valency v/k where, for each $\{P,P'\}\in E_N$ and each $\alpha\in P$, $|\Gamma(\alpha)\cap P'|=k$, and Γ is a multicover of Γ_N. Moreover,*
   (i) *N is semiregular on V and is the kernel of the action of G on $\mathscr{P}_N$;*
   (ii) *if $P\in\mathscr{P}_N$ and $\alpha\in P$, then $G_\alpha^{\Gamma(\alpha)}$ acts faithfully on the partition $\mathscr{P}(\alpha):=\{\Gamma(\alpha)\cap P' \mid \{P,P'\}\in E_N\}$ of $\Gamma(\alpha)$, and the permutation groups $G_\alpha^{\mathscr{P}(\alpha)}$ and $G_P^{\Gamma_N(P)}$ are permutationally isomorphic;*
   (iii) *if moreover Γ is G-locally-primitive then Γ is a cover of Γ_N (that is $k=1$) and Γ_N is G-locally-primitive.*

The proof of this result may be found in [19, Lemmas 1.1, 1.4(p), 1.5 and 1.6]. In that paper a multicover was called a pseudocover, but the term multicover has been used more recently, and we believe that it is more appropriate. Theorem 1.3 may be refined as follows, thus identifying certain graphs in $\mathscr{F}$ as candidates for designation as "basic". These are graphs for which the action of the group G on vertices is "close" to being quasiprimitive. They are obtained by taking the normal subgroup N in Theorem 1.3 to be maximal in some sense.

We say that a group G acting on a set V is *bi-quasiprimitive* on V if

(i) G is transitive on V, and
(ii) each normal subgroup of G which acts non-trivially on V has at most two orbits in V, and
(iii) there exists a normal subgroup of G with two orbits in V.

A bi-quasiprimitive group G on V has a system of imprimitivity consisting of two blocks of size $|V|/2$, and hence has a subgroup G^+ of index 2 which fixes the two blocks setwise. Moreover, provided that $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ (acting regularly on a set of four points), then G^+ is the unique subgroup with these properties. A bipartite graph $\Gamma=(V,E)$ is said to be *G-bi-quasiprimitive* if G acts as a group of automorphisms of Γ and G is bi-quasiprimitive on V.

Theorem 1.4. *Let $\Gamma=(V,E)$ be a finite, connected graph of valency v which is G-vertex-transitive and G-locally-quasiprimitive, and let N be a normal subgroup of G which is maximal subject to having more than two orbits in V. Then one of the following holds for the quotient Γ_N.*

(a) *Γ_N is G-quasiprimitive; or*
(b) *Γ and Γ_N are both bipartite, $N \leqslant G^+$, and Γ_N is G-bi-quasiprimitive. Moreover, either $\Gamma=K_{v,v}$, or G^+ acts faithfully on each part of the bipartition $\{\Delta_1,\Delta_2\}$ of Γ. In the latter case, either*
  (i) *G^+ is quasiprimitive on each part of the bipartition of Γ_N; or*
  (ii) *G^+ has two normal subgroups M_1 and M_2 properly containing N which are interchanged by G, are semiregular on V and intransitive on each Δ_i, and are such that, M_1/N and M_2/N are distinct minimal normal subgroups of G^+/N.*

The class $\mathscr{F}$ of finite vertex-transitive, locally-quasiprimitive graphs was first investigated in [20]. At that time the ‘O’Nan-Scott Theorem’ [21] for quasiprimitive groups, which described the possible structures of finite quasiprimitive permutation groups, was not available, and many of the results in [20] constitute precursors for parts of that theorem. Complete bipartite graphs $K_{v,v}$ were singled out in [20, Lemma 1.1]. These certainly arise as examples in Theorem 1.4 (b) as can be seen by taking $G=S_v \,\mathrm{wr}\, S_2$. Moreover in [20] the concept of a *G-irreducible* graph in $\mathscr{F}$ was introduced as a G-vertex-transitive, G-locally-quasiprimitive graph which is not a multicover of any of its *proper* quotient graphs (that is, quotient graphs with more than two vertices) relative to G-invariant partitions. Whether or not a graph is G-irreducible may be difficult to determine because the complete lattice of all G-invariant partitions of V may not be known. It is usually simpler to determine all the G-normal partitions than all the G-invariant ones. Consequently, we define a G-vertex-transitive, G-locally-quasiprimitive graph to be *G-basic* if it is not a multicover of any of its proper G-normal quotients. By Theorem 1.3, every graph in $\mathscr{F}$ has at least one basic normal quotient, and by Theorem 1.4, the basic graphs in $\mathscr{F}$, apart from complete bipartite graphs, arise in three broad categories. A subgroup M of automorphisms of

Γ is said to be *locally-transitive* if M_α is transitive on $\Gamma(\alpha)$ for each vertex α; in this case, if Γ is connected, then either M is vertex-transitive on Γ, or Γ is bipartite and M has as orbits the two parts of the bipartition.

Theorem 1.5. *Let $\Gamma=(V,E)$ be a finite, connected, G-vertex-transitive, G-locally-quasiprimitive graph of valency v, and suppose that Γ is G-basic. If Γ is not bipartite then Γ is G-quasiprimitive. On the other hand, if Γ is bipartite with bipartition $\{\Delta_1,\Delta_2\}$, then Γ is G-bi-quasiprimitive, and either $\Gamma=K_{v,v}$, or G^+ is faithful on each of the Δ_i and one of the following holds*:

(a) *G^+ is quasiprimitive on each of the Δ_i,*
(b) *G^+ has distinct minimal normal subgroups, M_1 and M_2, which are semiregular on V, intransitive on the Δ_i, and interchanged by G; the group $M:=M_1\times M_2$ is normal in G and either*
 (i) *M is regular on each of the Δ_i; or*
 (ii) *M is locally-transitive on Γ, M is the unique minimal normal subgroup of G, $M\cong T^{2k}$ for some non-abelian simple group T and positive integer k, and M_α is a subgroup of a diagonal subgroup of $M=M_1\times M_2$.*

Theorems 1.4 and 1.5 will be proved in Section 2. There many examples known of graphs satisfying Theorem 1.5(a) and (b) (i) (see, for example [12,21,22]). We shall prove in Section 2 that there are also many examples of graphs satisfying Theorem 1.5(b) (ii):

Example 1.6. Let p be a prime, and suppose that T is a finite simple group with a generating set $\{x,y\}$ such that $o(x)=o(y)=p$ and there is no automorphism of T which maps x to y. Then there exists a G-basic, G-locally-primitive graph Γ of valency p satisfying Theorem 1.5(b) (ii) with $M_1\cong M_2\cong T$. (See Proposition 2.2 for the construction and proof.)

A natural problem arising from these results is the problem of constructing finite locally-quasiprimitive graphs as multicovers of a given locally-quasiprimitive graph. A universal construction method for such multicovers will be presented in Section 4. There we define (see Definition 4.1) a *G-extender* of Γ as a certain G-vertex-transitive, G-locally-quasiprimitive graph with multiple edges (where these definitions are appropriately amended to apply to graphs with multiple edges). Trivially Γ is a G-extender of itself, and it turns out that there are only finitely many G-extenders for a given Γ. Moreover, there is an important link between extenders and (locally-quasiprimitive normal) multicovers of Γ which admit an action of the given group G (see Section 3.2).

Theorem 1.7. *Let Γ be a finite, connected, G-vertex-transitive, G-locally-quasiprimitive graph. Then the finite, locally-quasiprimitive, normal multicovers of Γ are precisely* *the simple G-admissible covers of the G-extenders of Γ. Further, from each G-extender arise infinitely many multicovers.*

Remark 1.8. This result demonstrates that the set of all "nice" (that is, G-locally-quasiprimitive normal) multicovers of Γ is partitioned naturally into infinite subsets corresponding to the G-extenders. Those which are covers of Γ correspond to the trivial G-extender Γ.

Although the construction implicit in the statement of Theorem 1.7 produces every finite locally-quasiprimitive multicover of a given H-locally-quasiprimitive graph Σ (see Theorem 4.5), it does not seem to admit a refinement whereby we may specify a group G with a normal subgroup N such that $G/N \cong H$, and construct all G-locally-quasiprimitive graphs Γ such that $\Gamma_N \cong \Sigma$. Some preliminary results along these lines, when we are given G and a pair of intransitive normal subgroups N_1, N_2, will be given in Section 5 where we discuss the problem of reconstructing Γ from a collection of its normal quotients. In the final section we discuss several open problems concerning finite locally-quasiprimitive graphs suggested by the results of this paper.

2. Examples

It is well-known that all arc-transitive-graphs may be constructed by a method introduced by Sabidussi [26]. This is described as follows. For a group G, a core-free subgroup H of G (that is, $\bigcap_{x\in G} H^x = 1$), and a 2-element $g \in G$, we define the *coset graph* $\Gamma(G,H,HgH) = (V,E)$ to have vertex set $V = [G:H] = \{Hx\colon x \in G\}$ and edge set $E = \{\{Hx,Hy\}\colon xy^{-1} \in HgH\}$. We require that $g \notin N_G(H)$, $g^2 \in H$ and $\langle H,g\rangle = G$. Then $\Gamma(G,H,HgH)$ is a connected G-arc-transitive graph where G acts on V by right multiplication. Moreover (see, for example, [16] or [23]), every arc-transitive graph is isomorphic to a coset graph of this type. We use this construction to justify the assertion made in Example 1.1. The *socle* of a finite group G is the product of its minimal normal subgroups, and is denoted by $\mathrm{soc}(G)$.

Proposition 2.1. *Let p be a prime such that $p \geqslant 29$ and $p \equiv \pm 1 \pmod 5$. Let $G \cong \mathrm{PSL}(2,p)$. Then G contains a maximal subgroup $H \cong \mathrm{A}_5$ and an involution $g \in G\setminus H$ such that the orbital graph $\Gamma(G,H,HgH)$ has full automorphism group isomorphic to G, and is locally-quasiprimitive, but not locally-primitive, of valency* 20.

Proof. By [28, p. 416, Ex. 2], G has two conjugacy classes of maximal subgroups isomorphic to A_5, which are fused in $\mathrm{Aut}(G) \cong \mathrm{PGL}(2,p)$. Thus in particular $\mathbf{N}_{\mathrm{Aut}(G)}(H) = H$.

Let Γ be an orbital graph of G with respect to the permutation representation of G on $[G:H]$. Then $G < \mathrm{Sym}([G:H])$ is primitive. If $\mathrm{soc}(\mathrm{Aut}\,\Gamma) \neq G$, then by [15, Theorem], all possibilities for the pair $(G, \mathrm{soc}(\mathrm{Aut}\,\Gamma))$ are listed in [15, Tables II–VI]. Checking these tables, we conclude that there are no possibilities with $G = \mathrm{PSL}(2,p)$

and degree $|G|/|\mathrm{A}_5| = p(p^2-1)/120$. Thus $\mathrm{soc}(\mathrm{Aut}\,\Gamma) = G$. Further, since $\mathbf{N}_{\mathrm{Aut}\,(G)}(H) = H$, it follows that $\mathrm{Aut}\,\Gamma = G$.

Let $a, z \in H$ be such that $o(a) = 5$, $o(z) = 2$ and $\langle a \rangle \rtimes \langle z \rangle \cong \mathrm{D}_{10}$. Then $\mathbf{N}_G(\langle a, z \rangle) = \mathrm{D}_{10}$ or D_{20}, and hence $\mathbf{N}_G(\langle a, z \rangle)$ contains 5 or 11 involutions, respectively. Since 5 divides $p + \epsilon$ where $\epsilon = 1$ or -1, $\mathbf{N}_G(\langle a \rangle) \cong \mathrm{D}_{p+\epsilon}$, and so $\mathbf{N}_G(\langle a \rangle)$ contains at least $(p + \epsilon)/2$ involutions. Therefore, as $(p + \epsilon)/2 > 11$, there exists an involution g in $\mathbf{N}_G(\langle a \rangle) \backslash \mathbf{N}_G(\langle a, z \rangle)$. Then $g \notin H$, and it follows that $H \cap H^g = \langle a \rangle \cong \mathbb{Z}_5$. Let $\Gamma := \Gamma(G, H, HgH)$. Let α be the vertex of Γ corresponding to H, and let β be the vertex of Γ corresponding to H^g. Then $G_\alpha = H$, β is adjacent to α, and $G_{\alpha\beta} = H \cap H^g \cong \mathbb{Z}_5$. Thus $G_{\alpha\beta}$ is not a maximal subgroup of G_α, and so G_α is not primitive on $\Gamma(\alpha)$, that is Γ is not locally-primitive. But since $H \cong \mathrm{A}_5$ is simple, Γ is locally-quasiprimitive. □

Our next example of the Sabidussi construction yields an infinite family of graphs which satisfy Theorem 1.5(b) (ii), and which prove the assertion made in Example 1.6.

Proposition 2.2. *Let p be a prime, and suppose that T is a finite simple group with a generating set $\{x, y\}$ such that $o(x) = o(y) = p$ and there is no automorphism of T which maps x to y. Let $G = T \,\mathrm{wr}\, \mathbb{Z}_2$. Then there exists a G-locally-primitive graph Γ of valency p satisfying Theorem* 1.5(b) (ii) *with $M_1 \cong M_2 \cong T$.*

Proof. Let $M := T_1 \times T_2$ denote the base group of G, where $T_1 \cong T_2 \cong T$. Let H be the subgroup of M generated by the element (x, y). Then $H \cong \mathbb{Z}_p$. Let g be the involution of G such that $(u, v)^g = (v, u)$ for all $(u, v) \in M$. Then g interchanges T_1 and T_2, and $G = M.\langle g \rangle$. Consider the subgroup $N := \langle H, H^g \rangle$ of M. Since N contains both (x, y) and $(x, y)^g = (y, x)$, and since $T = \langle x, y \rangle$, it follows that N projects onto both T_1 and T_2. Suppose that $N \neq M$. Then N is a diagonal subgroup of $T_1 \times T_2$, that is, $N = \{(t, t^\phi) \mid t \in T\}$ for some $\phi \in \mathrm{Aut}\,(T)$. This is not the case since x and y are not conjugate in $\mathrm{Aut}\,(T)$. So $N = M$, and hence $\langle H, g \rangle = G$. Let $\Gamma = \Gamma(G, H, HgH)$. Since $\langle H, g \rangle = G$ and $|H : H \cap H^g| = p$, Γ is a connected graph of valency p. Further, Γ is bipartite, M is intransitive on V, each of the T_i is semiregular on V, and M is locally-primitive on V. □

There are many simple groups T with generating sets satisfying the condition of Proposition 2.2. For example, if $p \geqslant 5$ and $T = \mathrm{A}_{2p}$, then $x = (1, 2, \ldots, p)(p+1, \ldots, 2p)$ and $y = (1, 2, \ldots, p-1, p+1)$ have the desired properties.

3. Proofs of Theorems 1.4 and 1.5

Throughout this section $\Gamma = (V, E)$ will be a finite, connected, G-vertex-transitive, G-locally-quasiprimitive graph of valency v, and N will be a normal subgroup of G which has more than two orbits in V. Thus N is semiregular on V by Theorem 1.3.

Set $G^+ := \langle G_\alpha \mid \alpha \in V \rangle$. Then $|G : G^+| \leqslant 2$ with equality if and only if Γ is bipartite. Whether or not the graphs Γ and Γ_N are bipartite is governed by the following simple criterion.

Lemma 3.1. *Γ_N is bipartite if and only if Γ is bipartite and $N \leqslant G^+$.*

Proof. Since Γ is connected, all edges of Γ join vertices in distinct N-orbits. Thus if Γ_N is bipartite, then Γ is also bipartite and the parts of the bipartition are unions of N-orbits, so in addition $N \leqslant G^+$. The converse implication is clear. □

Thus, for example, if Γ is bipartite but $N \nleqslant G^+$, then the quotient graph Γ_N is not bipartite, and in particular no normal subgroup of G has two orbits on the vertices of Γ_N. In this case, if N is maximal normal subject to having more than two orbits in V, then Γ_N is G-quasiprimitive, and part (a) of Theorem 1.4 holds.

Before proceeding with the proof of Theorem 1.4 we make a few remarks about bipartite graphs Γ which possess a non-bipartite normal quotient Γ_N. It turns out that such a graph Γ also has a normal quotient which is isomorphic to a direct product of K_2 (the complete graph on two vertices) and Γ_N. The *direct product* (see [9, p. 231]) $\Sigma \otimes \Delta$ of graphs $\Sigma = (V_\Sigma, E_\Sigma)$ and $\Delta = (V_\Delta, E_\Delta)$ is the graph with vertex set $V_\Sigma \times V_\Delta$ such that (σ, δ) is joined by an edge to (σ', δ') if and only if $\{\sigma, \sigma'\} \in E_\Sigma$ and $\{\delta, \delta'\} \in E_\Delta$. First we characterise the situation where the graph Γ itself is such a direct product, and then we prove our assertions above as a corollary.

Proposition 3.2. *Suppose that Γ is bipartite. Then the following are equivalent.*

(a) $\mathrm{Aut}(\Gamma)$ *has a subgroup $G_0 = G^+ \times Z \cong G^+ \times \mathbb{Z}_2$ which is vertex-transitive and locally-quasiprimitive on Γ.*
(b) *$\Gamma \cong K_2 \otimes \Delta$, where Δ is G^+-vertex-transitive, G^+-locally-quasiprimitive, and non-bipartite.*

Given G_0 as in (a), *the graph Δ obtained in* (b) *is isomorphic to the G_0-normal quotient Γ_Z.*

Proof. Suppose that $G_0 \leqslant \mathrm{Aut}(\Gamma)$ with G_0 as in part (a), and note that $(G_0)^+ = G^+$, $Z \lhd G_0$ and $G_0/Z \cong G^+$. Then by Theorem 1.3, the G_0-normal quotient $\Gamma_Z = (V_Z, E_Z)$ is G^+-vertex-transitive and G^+-locally-quasiprimitive; and by Lemma 3.1, Γ_Z is not bipartite. Each Z-orbit consists of one vertex from each part of the bipartition $\{\Delta_1, \Delta_2\}$ of Γ. We define a mapping $\varphi : V \to \{1, 2\} \times V_Z$ as follows: for $\alpha \in \Delta_i$ and α in the Z-orbit δ, define $\varphi(\alpha) = (i, \delta)$. Clearly φ is a bijection. Moreover, if $\varphi(\alpha) = (i, \delta)$ and $\varphi(\beta) = (j, \sigma)$, then $\{\alpha, \beta\} \in E$ if and only if $i \neq j$ and $\{\delta, \sigma\} \in E_Z$. Thus φ is an isomorphism from Γ to $K_2 \otimes \Gamma_Z$ and part (b) holds with $\Delta = \Gamma_Z$.

Conversely suppose that $\Gamma \cong K_2 \otimes \Delta$ as in part (b). Then $\mathrm{Aut}\,\Gamma$ contains $\mathrm{Aut}\,K_2 \times \mathrm{Aut}\,\Delta$ which contains $G_0 := \mathrm{Aut}\,K_2 \times G^+ \cong \mathbb{Z}_2 \times G^+$. Also G_0 is vertex-transitive and locally-quasiprimitive on Γ. □

Corollary 3.3. *Suppose that Γ is bipartite and that Γ_N is not bipartite, and let $N^+ = N \cap G^+$. Then $\Gamma_{N^+} \cong K_2 \otimes \Gamma_N$.*

Proof. The subgroup N^+ is normal in G and contained in G^+, and so, by Theorem 1.3 and Lemma 3.1, Γ_{N^+} is an H-vertex-transitive, H-locally-quasiprimitive bipartite graph, where $H := G/N^+$. Now $|N : N^+| = 2$, $H^+ = G^+/N^+$, and $H \cong (N/N^+) \times H^+ \cong \mathbb{Z}_2 \times H^+$. It therefore follows from Proposition 3.2 that $\Gamma_{N^+} \cong K_2 \otimes \Gamma_N$. □

Now we return to the proof of Theorem 1.4. Suppose that N is maximal subject to having more than two orbits in V. From the remarks following Lemma 3.1, we may assume that Γ is bipartite with bipartition $\{\Delta_1, \Delta_2\}$, and that $N < G^+$. Let $g \in G \setminus G^+$; since $g^2 \in G^+$, we may assume that g is a 2-element. By [20, Lemma 1.1], if G^+ acts unfaithfully on Δ_1 or Δ_2, then $\Gamma \cong K_{v,v}$. Hence we may assume that G^+ is faithful on each of the Δ_i.

We claim that every minimal normal subgroup of G is contained in G^+; for suppose that K is a minimal normal subgroup of G with $K \not\leq G^+$. Then, by the minimality of K, $K \cap G^+ = 1$, and hence $|K| = 2$, $\langle K, N \rangle = K \times N$, and the number of orbits in V of $K \times N$ is one half of the number of N-orbits in V. By the maximality of N, $K \times N$ has exactly two orbits in V, but these are not the Δ_i, contradicting Theorem 1.3. This proves the claim.

It now follows that Γ_N is G-bi-quasiprimitive: for if K is normal in G with more than two orbits in the vertex set $\mathscr{P}_N$ of Γ_N, then KN also has more than two orbits in $\mathscr{P}_N$ (since N acts trivially on $\mathscr{P}_N$) and hence more than two orbits in V. Then, by the maximality of N, it follows that $K \subseteq N$ and so K acts trivially on $\mathscr{P}_N$.

If G^+ is quasiprimitive on one (and hence both) of the Δ_i then Theorem 1.4(b) (i) holds, so we may assume that this is not the case. To complete the proof we need to show that part (b) (ii) of Theorem 1.4 holds. It is sufficient to do this in the case where $N = 1$. Thus, we assume that $N = 1$, and we note that, in proving Theorem 1.4 in this case, we also complete the proof of Theorem 1.5. Note that the assumption $N = 1$ means that every non-trivial normal subgroup of G has at most two orbits in V. Since we are assuming that G^+ is not quasiprimitive on Δ_1 and Δ_2, without loss of generality there exists a minimal normal subgroup M_1 of G^+ such that M_1 is intransitive on Δ_1, and hence M_1 has more than two orbits in V. Thus M_1 is not normal in G, and so $M_2 := M_1^g \neq M_1$. Since $g^2 \in G^+$, it follows that $M_2^g = M_1$, so g, and hence also G, interchanges M_1 and M_2. Clearly M_2 is a minimal normal subgroup of G^+, and M_2 is intransitive on Δ_2. By minimality, $M_1 \cap M_2 = 1$, and hence $M := M_1M_2 = M_1 \times M_2$. Also $M \lhd G$, and hence M is transitive on each Δ_i. Suppose first that M is regular on Δ_1. Since G^+ is faithful on Δ_1 it follows that $|M| = |\Delta_1| = |\Delta_2|$ and hence that M is semiregular on V. Therefore each of the M_i is semiregular on V with $|V|/|M_i| = 2|M|/|M_i| > 2$ orbits in V, and Theorem 1.4 (b) (ii) holds. Thus we may assume that M is not regular on Δ_1.

This means in particular that the M_i are non-abelian, for if not, then M would be abelian, and transitive on Δ_1, and hence would be regular on Δ_1, which is not the case. Thus $M_1 \cong M_2 \cong T^k$ for some non-abelian simple group T and some positive integer k. Therefore $M \cong T^{2k}$ and G is transitive on the $2k$ simple direct factors of M, whence M is a minimal normal subgroup of G. Next suppose that G has a minimal normal subgroup K distinct from M. Then, as we observed above, $K \leqslant G^+$, and K is transitive on each of the Δ_i. By minimality, $K \cap M = 1$, so M centralises K. However, the centraliser in the symmetric group on Δ_1 of the transitive group K is semiregular, whereas M is not semiregular on Δ_1. This contradiction shows that M is the unique minimal normal subgroup of G. Since M is not semiregular on V, the hypotheses of [20, Theorem 2.1B] hold, and by this result we have that M_1 and M_2 are semiregular on V. This completes the proof of Theorems 1.4 and 1.5.

4. Constructing multicovers

The process of forming normal quotient graphs produces the subclass of basic graphs of the class $\mathscr{F}$ of finite, vertex-transitive, locally-quasiprimitive graphs. It is of interest to reverse this process.

Suppose that $\tilde{\Gamma}$ is $\tilde{G}$-vertex-transitive and $\tilde{G}$-locally-quasiprimitive, and that N is a normal subgroup of $\tilde{G}$ with more than two orbits on vertices. Let $\Gamma = \tilde{\Gamma}_N$ and $G = \tilde{G}/N$. As observed in part (c) of Theorem 1.3, $\tilde{\Gamma}$ is a multicover of Γ; but it is not necessarily a cover. In this section, we show that $\tilde{\Gamma}$ may be considered as a cover of a graph *with multiple edges*, which we will call a G-extender, and which is closely related to Γ. There are only finitely many G-extenders of Γ, and, most importantly, the set of all G-extenders of Γ is completely determined by certain local properties of the G-action on Γ, namely by the stabilisers in G of a pair of adjacent vertices and the edge between them.

Let Y be a topological space and G a group of homeomorphisms of Y. It is known (see [10] or [18, Proposition 8.2]) that, under certain conditions, Y is a regular covering space of the topological quotient space Y/G. It is essentially this observation, in the combinatorial setting we are interested in, that will form the basis of our method; the quotients Y/G will be the G-extenders. The process can be represented figuratively as below (where "Locally-QP" is an abbreviation for "Locally-quasiprimitive"):

$$\text{Locally-QP Graph} \xrightarrow{\text{Local properties}} \text{Extenders} \xrightarrow{\text{Admissible covers}} \text{Locally-QP Multicovers.}$$

4.1. Defining and constructing extenders

We first define the concept of a *G-extender* Γ' of Γ, together with an action of G on Γ'. Such an extender is a graph which in general will have multiple edges. An automorphism of such a graph is a permutation of the vertex set together with a permutation of the edge set so that vertex-edge incidence is preserved. An arc of such a graph is a directed edge.

Definition 4.1. Let $\Gamma=(V,E)$ be a G-vertex-transitive, G-locally-quasiprimitive graph. A *G-extender* of Γ, sometimes called an *extender* of the pair (Γ,G), is defined as a graph $\Gamma'=(V,E')$ together with an action of G on Γ' (as a group of automorphisms) so that the following conditions are satisfied:

(a) for $\alpha,\beta\in V$, there is an edge between α and β in Γ if and only if there is at least one edge in Γ' incident with α and β.
(b) The action of G on the set V of vertices is the same for both the G-action on Γ and the G-action on Γ'; therefore α^g may be unambiguously defined for $\alpha\in V$ and $g\in G$.
(c) G acts transitively on the arcs of Γ'.

In addition, we say that Γ' is a *G-locally-quasiprimitive G-extender* if the stabiliser G_α of $\alpha\in V$ acts quasiprimitively on the edges of Γ' incident with α.

We will say an edge $\mathscr{E}$ of Γ' *lies above* an edge $\boldsymbol{\epsilon}$ of Γ if $\mathscr{E}$ and $\boldsymbol{\epsilon}$ are incident with the same vertices of V. Let $\alpha,\beta\in V$ be adjacent in Γ and let $\boldsymbol{\epsilon}$ be the edge between them. Let $G_{\boldsymbol{\epsilon}}$ be the stabiliser of $\boldsymbol{\epsilon}$ in G, and let $G_{\alpha\beta}$ be (as before) the subgroup of G fixing both α and β. Let $\mathscr{E}$ be any edge between α and β in Γ' (so $\mathscr{E}$ lies above $\boldsymbol{\epsilon}$) and let $H=G_{\mathscr{E}}$, the stabiliser of $\mathscr{E}$. Then $H\leqslant G_{\boldsymbol{\epsilon}}$. Since the action of G on the arcs, and therefore the edges, of Γ' is transitive, we may identify the edge set of Γ' with the right cosets of H in G. Further the "edge" Hx of Γ' lies above the edge $\boldsymbol{\epsilon}^x$ of Γ, and the two vertices incident with this edge are α^x and β^x. Hence, given Γ and G, the graph structure of Γ' is entirely specified by H, and clearly knowledge of H determines the action of G (by right multiplication) on the edges of Γ'. Finally, arc-transitivity of G on Γ' is expressed by the statement that H is not contained in $G_{\alpha\beta}$; for once it is known that G is transitive on the edges of Γ', the only additional condition required for arc-transitivity is that there exists an element $g\in G$ which fixes the edge $\mathscr{E}$ but switches the two vertices incident with $\mathscr{E}$.

These remarks demonstrate that every extender Γ' may be constructed by the method we describe below.

Construction 4.2. Let Γ be a connected G-arc-transitive graph. Let $\boldsymbol{\epsilon}$ be an edge of Γ incident with vertices α and β. Let H be a proper subgroup of $G_{\boldsymbol{\epsilon}}$ not contained in $G_{\alpha\beta}$.

The edge set E' of the extender Γ' is defined as the set of right cosets of H in G, so G acts transitively on E' by right multiplication. Also, for $x\in G$, the edge Hx of Γ' lies above the edge $\boldsymbol{\epsilon}^x$ of Γ, thus determining the vertices with which it is incident.

Further, Γ' is G-locally-quasiprimitive if and only if G_α is quasiprimitive on the right cosets of $G_{\alpha\beta}\cap H$ in G_α.

All the claims here have already been addressed, except the last on quasiprimitivity. This follows easily, however, since $G_{\alpha\beta}\cap H$ is the stabiliser in G_α of the edge H, that is, of the coset $H\cdot 1$. The Orbit-Stabiliser theorem then shows that the action of G_α on the edges of Γ' incident with α is permutationally isomorphic to its action (by right multiplication) on the right cosets of $G_{\alpha\beta}\cap H$ in G_α.

The existence and analysis of G-extenders is thereby reduced to a purely group-theoretic problem. Most importantly, this is a group theoretic problem which takes as input purely *local* information: point, edge and arc stabilisers corresponding to a pair of adjacent vertices; the entirety of G does not enter into the calculation at all. It is also clear that Γ will have only finitely many G-extenders. We demonstrate this by the following example, where the graph Γ is the complete graph K_n and the group $G=S_n$.

Example 4.3. Take $\Gamma=K_n$ and $G=S_n$. Identify the vertex set V with $\{1,2,\ldots,n\}$, and take $\alpha=1,\beta=2$ in the above construction. Then for $\epsilon=\{\alpha,\beta\}$, we have $G_\alpha=\mathrm{Sym}(\{2,3,\ldots,n\})\cong S_{n-1}$, and $G_{\boldsymbol{\epsilon}}=\langle(12)\rangle\times\mathrm{Sym}(\{3,\ldots,n\})\cong S_2\times S_{n-2}$. Suppose, first, that $n\geqslant 6$. Then there exist G-locally-quasiprimitive extenders: take the subgroup H to be $\langle(12)\rangle\times P$, where P is any subgroup of $\mathrm{Sym}\{3,4,\ldots,n\}$ that is not contained in the alternating group A_n. Certainly H is not contained in $G_{\alpha\beta}$. Also $G_\alpha\cong S_{n-1}$ is quasiprimitive in its action, by right multiplication, on the set of right cosets in G_α of $G_{\alpha\beta}\cap H=P$ since P is not contained in A_n.

If $n=5$ the situation is different: there are no non-trivial G-locally-quasiprimitive extenders. In fact, up to conjugacy, we have two possibilities for H of order 6, namely,

$$H_1=\langle(12)\rangle\times\langle(345)\rangle,\ H_2=\langle(12)(34),(345)\rangle,$$

leading to G-extenders Γ' with edges of multiplicity 2; one possibility corresponding to a G-extender Γ' with edges of multiplicity 3, namely, $H_3=\langle(12),(34)\rangle$; and two possibilities where the G-extender Γ' has edges with multiplicity 6, namely $H_4=\langle(12)(34)\rangle$, and $H_5=\langle(12)\rangle$. In none of these cases is the G-extender G-locally-quasiprimitive because the normal elementary abelian subgroup of G_α of order 4 is, in each case, intransitive on the (multiple) edges incident with α (as there are in each case at least 8 such edges).

4.2. Covers and multicovers

Let us first introduce the concept of a covering graph $(\tilde{\Gamma},p)$ of a graph Γ, where both $\tilde{\Gamma}$ and Γ may have multiple edges. We will assume the graphs do not have loops. A *simple* graph is then a graph with no multiple edges.

Let $\Delta=(V_1,E_1)$ and $\Xi=(V_2,E_2)$ be graphs, possibly with multiple edges. Because of the possibility of multiple edges, we define a *homomorphism* $f\colon\Delta\to\Xi$ to consist of a pair of maps $f_V:V_1\to V_2$ and $f_E:E_1\to E_2$ which preserve vertex-edge incidence.

A *covering graph* of Γ is a graph $\tilde{\Gamma}$, together with a graph homomorphism $p:\tilde{\Gamma}\to\Gamma$ such that p is surjective (both as a map between vertex-sets and between edge-sets); and for each vertex $\tilde{\alpha}$ of $\tilde{\Gamma}$, p maps the edges incident with $\tilde{\alpha}$ *bijectively* onto the edges incident with $(\tilde{\alpha})p$. We sometimes refer to the map p as a *covering*. It is possible that the graph $\tilde{\Gamma}$ is simple even when Γ is not. (We could at this point also define a multicover by replacing the condition of "local bijectivity" with "local surjectivity.") We will call a covering graph $(\tilde{\Gamma},p)$ *simple* when $\tilde{\Gamma}$ is simple. Unless otherwise stated,

$$\begin{tikzcd}
\tilde{\Gamma} \arrow[r, "\tilde{g}"] \arrow[d, "p"'] & \tilde{\Gamma} \arrow[d, "p"] \\
\Gamma \arrow[r, "g"] & \Gamma
\end{tikzcd}$$

Fig. 1. The definition of a lift.

in this section we will assume all graphs to be connected. In particular all the covering graphs we consider will be connected.

An automorphism g of Γ is said to *lift* if there is an automorphism $\tilde{g}$ of $\tilde{\Gamma}$ such that the diagram in Fig. 1 commutes.

The set N of lifts of the identity automorphism forms a group under composition, and the covering is said to be *regular* or *normal* if N acts transitively on each vertex fibre $(\alpha)p^{-1}$. It is known that N acts *semiregularly* on the vertex set of $\tilde{\Gamma}$; N is called the *covering group* of the covering graph $(\tilde{\Gamma}, p)$.

Now suppose that G is a group of automorphisms of Γ. The covering graph $(\tilde{\Gamma}, p)$ is said to be *G-admissible* if the covering is regular and each automorphism $g \in G$ lifts to an automorphism of $\tilde{\Gamma}$. Let $\tilde{G}$ denote the lifted group; that is, the set of all lifts of elements of G. Admissible coverings have been studied previously; see [4,17] or [29]. We will require the following simple lemma on G-admissible coverings.

Lemma 4.4. *Let $(\tilde{\Gamma}, p)$ be a G-admissible covering graph of Γ; let $\tilde{\alpha}$ be a vertex of $\tilde{\Gamma}$ such that $(\tilde{\alpha})p = \alpha$. Then the actions of $\tilde{G}_{\tilde{\alpha}}$ on the edges of $\tilde{\Gamma}$ incident with $\tilde{\alpha}$, and of G_α on the edges of Γ incident with α are permutationally isomorphic.*

Proof. The projection p gives a bijection between the edges of $\tilde{\Gamma}$ adjacent with $\tilde{\alpha}$ and the edges of Γ adjacent with α. There is a natural homomorphism θ from $\tilde{G}_{\tilde{\alpha}}$ to G_α, which is compatible with p (this follows from the definition of lifting). The kernel of θ is trivial as the covering group N is semiregular. Further, given $g \in G_\alpha$, there exists a lift $\tilde{g} \in \tilde{G}$ of g. By regularity of the covering there exists $n \in N$ such that $\tilde{\alpha}^n = \tilde{\alpha}^{\tilde{g}}$, and it follows that $\tilde{g}n^{-1}$ is a lift of g contained in $\tilde{G}_{\tilde{\alpha}}$. Hence θ is surjective also. $\square$

We introduce a definition for ease of notation in stating Theorem 4.5 below. Let Γ be a G-locally-quasiprimitive graph. A *locally-quasiprimitive normal multicover, associated with* (Γ, G), is defined as a graph $\tilde{\Gamma}$ admitting a vertex-transitive, locally-quasiprimitive action of a group $\tilde{G}$, such that, for some normal subgroup N of $\tilde{G}$, we have $\tilde{\Gamma}_N = \Gamma$, and the group induced by $\tilde{G}$ on $\tilde{\Gamma}_N$ is $\tilde{G}/N \cong G$. Our main theorem reduces the problem of finding multicovers to the much better understood problem of finding admissible coverings.

Theorem 4.5. *Let Γ be a G-locally-quasiprimitive graph, and Γ' a G-locally-quasiprimitive G-extender of Γ. If $(\tilde{\Gamma}, p)$ is a simple G-admissible cover of Γ' and $\tilde{G}$ is the lifted group, then $\tilde{\Gamma}$ is a locally-quasiprimitive normal multicover associated with (Γ, G). Conversely, any locally-quasiprimitive normal multicover $(\tilde{\Gamma}, \tilde{G})$ associated with (Γ, G) arises in this way.*

Proof. Firstly, suppose $(\tilde{\Gamma}, p)$ is a simple G-admissible covering graph of the G-extender Γ'. Then, by Lemma 4.4, $\tilde{\Gamma}$ is $\tilde{G}$-locally-quasiprimitive, where $\tilde{G}$ is the lifted group. Let N be the covering group. Using the definition of G-admissibility and the vertex-transitivity of G on Γ, we easily check that $\tilde{G}$ is also vertex-transitive on $\tilde{\Gamma}$. So $\tilde{\Gamma}$ is a $\tilde{G}$-vertex-transitive, $\tilde{G}$-locally-quasiprimitive graph. Moreover, it is easy to check that $\tilde{\Gamma}_N$ is isomorphic to Γ. (In fact, this quotient graph is just a version of Γ' with the multiple edges coalesced, that is, Γ.) Also, almost by definition of the covering group, we have $\tilde{G}/N \cong G$. It follows that $(\tilde{\Gamma}, \tilde{G})$ is a locally-quasiprimitive multicover associated with (Γ, G).

Conversely, suppose we are given a $\tilde{G}$-locally-quasiprimitive graph $\tilde{\Gamma}$ and $N \lhd \tilde{G}$, so that $\tilde{\Gamma}$ is a multicover of the graph $\Gamma = \tilde{\Gamma}_N$. Let $G = \tilde{G}/N$. We wish to show that $\tilde{\Gamma}$ is a G-admissible cover of some G-locally-quasiprimitive G-extender Γ' of Γ.

Let Γ' be the graph with vertices the N-orbits on the vertex set of $\tilde{\Gamma}$, with edges the N-orbits on edges of $\tilde{\Gamma}$, and with incidence induced from Γ. The multiplicity of the edges of Γ', that is, the number of Γ'-edges incident with each pair of adjacent vertices of Γ', is the constant k of Theorem 1.3 (c). Moreover, the group $G = \tilde{G}/N$ acts naturally on Γ'.

We claim that Γ' is a G-extender of Γ. Firstly, it follows from the definitions that the vertex sets and the G-actions on the vertex sets are the same. Secondly, two vertices α, β of Γ' are joined by at least one edge in Γ' if and only if there is at least one edge between the corresponding N-orbits in $\tilde{\Gamma}$; that is, if and only α and β are joined by an edge in Γ. The fact that G acts arc-transitively on Γ' follows from the corresponding statement for the action of $\tilde{G}$ on $\tilde{\Gamma}$.

We claim now that $\tilde{\Gamma}$ is a G-admissible cover of Γ' with respect to the natural map $p : \alpha \mapsto \alpha^N$. The covering part follows in a straightforward way from the definition of Γ', and the fact that, by Theorem 1.3, N is semiregular. The G-admissibility of p (that is, the fact that p is regular and every element of G has a lift) follows from the definitions. Finally, the G-local-quasiprimitivity of Γ' now follows from Lemma 4.4 and the fact that $\tilde{\Gamma}$ is $\tilde{G}$-locally-quasiprimitive. □

Topologically, the multigraph Γ' defined in the proof above is the quotient space $\tilde{\Gamma}/N$, but we avoided this terminology so as not to confuse Γ' with the quotient graph $\tilde{\Gamma}_N$. It is important to note that Γ' is not in general the same as the quotient graph $\tilde{\Gamma}_N$. Indeed, these are the same if and only if $\tilde{\Gamma}$ is a cover of $\tilde{\Gamma}_N$.

Using this result, the reconstruction process decomposes naturally into the process of finding extenders, and that of constructing their regular coverings. The latter question, fortunately, is well-investigated. Some slight complications arise here from the fact that we are taking coverings of graphs with multiple edges, and we wish to end up with coverings which are simple graphs. It is easy to translate this condition into either the language of “voltage assignments” or into a condition on subgroups of the fundamental group. Then, a careful modification of the covering graph construction of Biggs allows us to construct the simple covers we need, and obtain the following important result.

Proposition 4.6. *To each extender Γ' there corresponds at least one, and in fact infinitely many, simple G-admissible covers.*

Proof. We use a construction of Biggs given in [2, Theorem 19.5 and the preceding remarks], or more precisely its natural generalisation to graphs with multiple edges. We take connected components to ensure that we get connected covering graphs. Using the terminology of Biggs, the reason the resulting graph is *simple* is the following: the K-chain ϕ constructed before [2, Theorem 19.5] assigns different values to different arcs with the same beginning and end vertices. In fact, if a is any arc, the only other arc a' with $\phi(a)=\phi(a')$ is the reverse of a.

If Γ' is any graph, possibly with multiple edges, and G a group of automorphisms of Γ', this construction may be used to construct a simple, G-admissible covering graph Γ'_1. Let G_1 be the group of all lifts of G to Γ'_1. Then any G_1-admissible covering graph of Γ'_1 is in a natural way a simple, G-admissible cover of Γ'. It is well known that there are infinitely many of the former; for instance one may repeat Biggs' construction infinitely many times as in Corollary 19.6 of his book. (See [2, Definition 4.4] for the definition of the terms rank and co-rank used in [2, Corollary 19.6]; the co-rank is always greater than 1 for the non-trivial graphs we consider, since they are vertex-transitive and so contain a cycle.) Consequently we find infinitely many simple, G-admissible covers of Γ'. $\square$

Theorem 1.7 in the introduction follows from Theorem 4.5 and Proposition 4.6.

5. Reconstruction

Let Γ be a connected G-vertex-transitive, G-locally-quasiprimitive graph. Here, we consider the problem of reconstructing the pair (Γ, G) from some of the G-normal quotient graphs of Γ. Let $\{N_i\}_{i\in I}$ be a family of intransitive normal subgroups of G, and for $i\in I$, denote by Γ_i the quotient graph Γ_{N_i} of Γ relative to the partition into N_i-orbits.

Question 5.1. Given the group G, the quotient graphs Γ_i $(i\in I)$, and the G-actions on the Γ_i, under what circumstances can we reconstruct Γ and the G-action on Γ? In particular, what extra information is required to make reconstruction possible?

The essence of this question of reconstructing $\Gamma=(V,E)$ from a given collection Γ_i $(i\in I)$ lies in the case where there are just two intransitive normal subgroups

N_1 and N_2. From such information it is impossible to determine more than the quotient graph $\Gamma_{N_1\cap N_2}$, and so we shall assume that $I=\{1,2\}$ and that $N_1\cap N_2=1$. We denote by M the product $M:=N_1N_2$. By Theorem 1.3, either M has at most two orbits on V, or M is semiregular and intransitive on V. If M is not semiregular then it acts locally-transitively on Γ, and in this case the techniques we develop give no information. We shall examine the case where M is semiregular and intransitive on V.

Fixing notation. Γ, G, N_i, M and Γ_i will be as stated above with M semiregular and intransitive on V. The vertex and edge sets of Γ (respectively Γ_i) will be denoted by V and E (respectively, V_i and E_i). Vertices will be denoted by Greek letters, and we reserve the letter ϵ for denoting edges. Edges and vertices belonging to Γ_i will be subscripted with an i. We will often identify V_i with the set of N_i-orbits on V. It should be noted that a corresponding identification cannot be effected for edges, in general. (However, in the special case where Γ *covers* Γ_i, we can identify E_i with the set of N_i-orbits on E.)

The natural quotient map from Γ to Γ_i will be denoted by ϕ_i, so $\phi_i: V\to V_i$ is given by $\alpha\mapsto\alpha_i$ where $\alpha_i=\alpha^{N_i}=\{\alpha^n\mid n\in N\}$; and ϕ_i induces a natural map $E\to E_i$. We shall denote by Φ the product map $\Phi: V\to V_1\times V_2$, given by $\alpha\mapsto(\alpha_1,\alpha_2)$ where $\alpha_i=(\alpha)\phi_i=\alpha^{N_i}$. Then, similarly, Φ induces a natural map $E\to E_1\times E_2$. Now G acts naturally on both $V_1\times V_2$ and $E_1\times E_2$ (by $(\alpha_1,\alpha_2)^g=(\alpha_1^g,\alpha_2^g)$ and similarly for edges), and Φ commutes with these G-actions in the sense that $(\alpha^g)\Phi=((\alpha)\Phi)^g$.

We define $\mathscr{V}$ and $\mathscr{E}$ to be the images of V and E, respectively under Φ. Further, we will say that vertices $\alpha_1\in V_1$ and $\alpha_2\in V_2$ are *cognate* if $(\alpha_1,\alpha_2)\in\mathscr{V}$, and similarly edges $\epsilon_1\in E_1$ and $\epsilon_2\in E_2$ will be called *cognate* if $(\epsilon_1,\epsilon_2)\in\mathscr{E}$. Since $\mathscr{V}$ and $\mathscr{E}$ are both G-orbits, it follows that a single pair of cognate edges determines $\mathscr{E}$ and a single pair of cognate vertices determines $\mathscr{V}$.

First we prove a simple lemma which demonstrates that the vertex set V may be identified with the set $\mathscr{V}$ of cognate vertex pairs when M is semiregular. Note that in the lemma we *do* identify V_i with the set of N_i-orbits in V.

Lemma 5.2. *Choose $\alpha\in V$ and set $H:=G_\alpha$, $H_1:=HN_1$ and $H_2:=HN_2$.*

(a) *A pair $(\alpha_1,\alpha_2)\in V_1\times V_2$ lies in $\mathscr{V}$ if and only if $\bigcup_{m\in M}\alpha_1^m=\bigcup_{m\in M}\alpha_2^m$.*
(b) *If M is semiregular, then for all $\omega\in V$, $\omega^{N_1}\cap\omega^{N_2}=\{\omega\}$. Moreover, if M is semiregular and intransitive and $v\in\Gamma(\omega)$, then $\omega^M\cap v^M=\emptyset$.*
(c) *$H_1\cap H_2=H$ if and only if M is semiregular.*
(d) *Φ is injective if and only if M is semiregular.*

Proof. Let $(\alpha_1,\alpha_2)\in V_1\times V_2$, say $\alpha_i=\beta_i^{N_i}$ for some $\beta_i\in V$, $i=1,2$. By definition, (α_1,α_2) lies in $\mathscr{V}$ if and only if, for some $\beta\in V$, we have $\beta^{N_i}=\beta_i^{N_i}$ for $i=1,2$. If the latter condition is true then clearly $\bigcup_{m\in M}\alpha_1^m=\beta^M=\bigcup_{m\in M}\alpha_2^m$. Conversely, if $\bigcup_{m\in M}\alpha_1^m=\bigcup_{m\in M}\alpha_2^m$, then $\beta_1=\beta_2^m$ for some $m=n_1n_2\in M$ and, writing $\beta:=\beta_1^{n_1^{-1}}=\beta_2^{n_2}$, we have $\beta_i^{N_i}=\beta^{N_i}$ for $i=1,2$. This establishes part (a).

The proof of part (b) is straightforward and is omitted. For part (c), we observe that certainly $H \subseteq H_1 \cap H_2$. Moreover, for $h_i \in H, n_i \in N_i$ $(i = 1, 2)$, we have $n_1h_1 = n_2h_2 \in H_1 \cap H_2$ if and only if $n_2^{-1}n_1 = h_2h_1^{-1} \in M \cap H$, and $M \cap H = 1$ if and only if M is semiregular. Thus M is semiregular if and only if $H_1 \cap H_2 \subseteq H$, and hence $H_1 \cap H_2 = H$, proving (c). It follows immediately from part (b) that, if M is semiregular, then Φ is injective. To complete the proof of part (d), suppose that Φ is injective. Then as $H_1 \cap H_2$ stabilises $(\alpha)\Phi = (\alpha_1, \alpha_2)$, it also stabilises α, that is $H_1 \cap H_2 \leqslant H$. Hence $H_1 \cap H_2 = H$, and so by part (c), M is semiregular. □

The thrust of this lemma is that M is semiregular if (and only if) the vertex set V of Γ is determined by the vertex sets of Γ_1 and Γ_2. Strictly speaking, we require also a pair of cognate vertices. However, the arc-transitive graphs $\Gamma, \Gamma_1, \Gamma_2$ are often given as coset graphs (as defined at the beginning of Section 2). The problem is to determine $\Gamma = \Gamma(G, H, HgH)$, given $\Gamma_i = \Gamma(G, H_i, H_ig_iH_i)$ for $i = 1, 2$, where the H_i are as in the lemma above and the elements g, g_1, g_2 are 2-elements. Thus the vertices $\alpha_1 := H_1$ and $\alpha_2 := H_2$ are given to us, and (α_1, α_2) is a pair of cognate vertices.

The main problem is that of determining the edge set E for Γ from the edge sets for Γ_1 and Γ_2, or equivalently, determining the double coset HgH from the given double cosets $H_1g_1H_1$ and $H_2g_2H_2$. In Theorem 5.7, we will prove that, given as additional information a pair of cognate edges of Γ_1 and Γ_2, we can do this and thereby reconstruct Γ. The technical information we require is contained in the next lemma.

Lemma 5.3. *Suppose that M is semiregular and intransitive.*

(a) *Let α and β be adjacent vertices of Γ. Then $\{\alpha, \beta\} = (\alpha^{N_1} \cup \beta^{N_1}) \cap (\alpha^{N_2} \cup \beta^{N_2})$.*
(b) *The map Φ induces a bijective map $E \to \mathscr{E}$.*
(c) *An edge $\epsilon \in E$ is incident with $\alpha \in V$ if and only if $(\epsilon)\phi_i$ is incident with $(\alpha)\phi_i$ for $i = 1, 2$. Further, if α and β are the two vertices incident with ϵ, then $(\alpha)\phi_i \neq (\beta)\phi_i$ for $i = 1, 2$.*

Proof. Clearly $(\alpha^{N_1} \cup \beta^{N_1}) \cap (\alpha^{N_2} \cup \beta^{N_2}) = \{\alpha, \beta\} \cup (\alpha^{N_1} \cap \beta^{N_2}) \cup (\alpha^{N_2} \cap \beta^{N_1})$. However, if we have an equality $\alpha^{n_1} = \beta^{n_2}$, it then follows that $\alpha^{n_1n_2^{-1}} = \beta$, which contradicts Lemma 5.2 (b). Thus $\alpha^{N_1} \cap \beta^{N_2} = \emptyset$. Similarly it follows that $\alpha^{N_2} \cap \beta^{N_1}$ is empty. Thus part (a) is proved, and part (b) follows from it. The first statement of part (c) is a consequence of part (a) applied to the two vertices α and β incident with ϵ. The second statement of (c) follows from Lemma 5.2 (b), since if $(\alpha)\phi_i = (\beta)\phi_i$ then α and β are in the same N_i-orbit. □

Lemmas 5.2 and 5.3 yield immediately the following theorem.

Theorem 5.4. *Suppose that M is semiregular and intransitive on V. Define Γ^* as the graph with vertex set $\mathscr{V}$, and edge set $\mathscr{E}$ (that is, (α_1, α_2) is adjacent to (β_1, β_2) if* *and only if* $\{\alpha_i, \beta_i\} \in E_i$ *for* $i = 1, 2$). *Then the map* Φ *defines an isomorphism from* Γ *to* Γ^* *which respects the* G*-action on each.*

In view of this theorem, it follows that if we could determine the sets $\mathscr{V}$ and $\mathscr{E}$ from our knowledge of Γ_1, Γ_2 and the G-actions on these, then Γ would be determined. It seems, however, that it is difficult in general to obtain precise information about edges. One of the reasons is that, for a normal subgroup N, the edges of the resulting quotient graph Γ_N cannot in general be identified with the N-orbits on edges of Γ. We lose more information about the edges than we do for vertices.

We may obtain reasonable results, however, in an important special case, namely, when Γ *covers* Γ_M. This will always be the case if we are dealing with G-locally-primitive graphs and M has more than two orbits; and it is an important special case. In this case, the edges of the quotient graph Γ_M *can* be identified with the M-orbits on the edges of Γ; using this, we will be able to give a reasonable condition under which we can reconstruct the graph Γ.

In order to prove Theorem 5.7, we construct two graphs isomorphic to Γ_M using only the information available to us. Denote by Γ_{12} the quotient graph of Γ_1 modulo the orbits in V_1 of the normal subgroup N_2; and by Γ_{21} the quotient graph of Γ_2 modulo the orbits in V_2 of the normal subgroup N_1. It is clear that $\Gamma_{12} \cong \Gamma_{21} \cong \Gamma_M$. In fact, there is a natural isomorphism $f\colon \Gamma_{12} \to \Gamma_{21}$ which is induced by the identity automorphism of Γ. This is illustrated in Fig. 2, where $p_{12} : \alpha_1 \mapsto \alpha_1^{N_2}$ and $p_{21} : \alpha_2 \mapsto \alpha_2^{N_1}$ are the natural projection maps. It is easy to see that this natural

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
\Gamma_1 \arrow[d, "p_{12}"'] & & \Gamma_2 \arrow[d, "p_{21}"] \\
\Gamma_{12} \arrow[rr, "f"] \arrow[dr] & & \Gamma_{21} \arrow[dl] \\
 & \Gamma_M &
\end{tikzcd}$$

Fig. 2.

isomorphism may be described as follows: map the N_2-orbit $\alpha_1^{N_2}$ in V_1 to the N_1-orbit $\alpha_2^{N_1}$ in V_2, where α_1 and α_2 are cognate vertices.

It should be noted, however, that f cannot in general be determined purely from the given information about G, Γ_1 and Γ_2. Nevertheless, we have the following result. If a group G acts on sets Ω_1 and Ω_2, then a map $f\colon \Omega_1 \to \Omega_2$ is said to *preserve these G-actions* if $(\omega_1^g)f = ((\omega_1)f)^g$, for all $\omega_1 \in \Omega_1, g \in G$.

Lemma 5.5. *Suppose* $C_{\mathrm{Aut}(\Gamma_M)}(G/M) = 1$. *Then* f *is the only isomorphism from* Γ_{12} *to* Γ_{21} *that preserves the* (G/M)*-action on each. This condition holds, in particular, if distinct vertices of* Γ_M *have distinct stabilisers in* G/M.

Proof. Suppose $g : \Gamma_{12} \to \Gamma_{21}$ is an isomorphism that preserves the (G/M)-actions on Γ_{12} and Γ_{21}. Since f has the same property, we can write $g = hf$, where h is an automorphism of Γ_{12} that preserves the (G/M)-actions. The latter condition is equivalent to requiring h to lie in $C_{\mathrm{Aut}(\Gamma_{12})}(G/M)$. The final assertion is true since $C_{\mathrm{Aut}(\Gamma_{12})}(G/M)$ acts faithfully and semiregularly on the vertex set of Γ_{12}, and $C_{\mathrm{Aut}(\Gamma_{12})}(G/M)$ fixes setwise the fixed point sets of the stabiliser subgroups in G/M. $\square$

From the commutative diagram above, we see that two vertices $\alpha_1 \in V_1$ and $\alpha_2 \in V_2$ have the same image in Γ_M if and only if their images in Γ_{12} and Γ_{21} are the same, when the latter two graphs are identified by means of f. That is to say, with p_{12} and p_{21} defined as in the diagram, we have $((\alpha_1)p_{12})f = (\alpha_2)p_{21}$. It follows from Lemma 5.5 that, if $C_{\mathrm{Aut}\,\Gamma_M}(G/M) = 1$, then we can determine the natural isomorphism f, since f is characterised in terms of known information, namely the (G/M)-actions on Γ_{12} and Γ_{21}. Hence, in this case, we can then determine from our given information whether two vertices of V_1 and V_2 have the same image in Γ_M, and similarly whether two edges from E_1 and E_2 have the same image in Γ_M. We combine this information with the following result.

Proposition 5.6. *Suppose that* Γ *covers* Γ_M. *Then a pair of vertices* $\alpha_1 \in V_1, \alpha_2 \in V_2$ (*or edges* $\epsilon_1 \in E_1$ *and* $\epsilon_2 \in E_2$) *are cognate if and only if their images in* Γ_M *are equal.*

Proof. In both cases, the “only if” part of this statement is trivial. The “if” statement for vertices follows from Lemma 5.2 (a). Observe that since Γ covers Γ_M, it also covers Γ_{N_i}. It therefore follows that the edges of Γ_{N_i} (respectively of Γ_M) can be identified with the N_i-orbits (respectively the M-orbits) on the edges of Γ. This is the critical fact which ensures that the statement for edges follows in the same way as the statement for vertices. $\square$

It is now clear that, if $C_{\mathrm{Aut}(\Gamma_M)}(G/M) = 1$, then we can determine $\mathscr{V}$ and $\mathscr{E}$ purely from information about Γ_1 and Γ_2. From Theorem 5.4 it follows that we can reconstruct the graph Γ.

Theorem 5.7. *Suppose that $\Gamma = (V, E)$ is a G-vertex-transitive, G-locally-quasiprimitive graph, and that N_1, N_2 are distinct intransitive normal subgroups of G such that*

(a) $N_1 \cap N_2 = 1$ *and* $M := N_1N_2$ *has more than two orbits in* V,
(b) Γ *covers* Γ_M, *and*
(c) $C_{\mathrm{Aut}(\Gamma_M)}(G/M) = 1$.

Then Γ can be reconstructed from the two quotient graphs Γ_{N_1} and Γ_{N_2}, and is isomorphic to the graph Γ^* defined in Theorem 5.4.

Note that if (a) holds then, by Theorem 1.3, M is semiregular on V. The condition on centralisers in part (c) holds, in particular, when distinct vertices of Γ_M have distinct stabilisers in G/M. Also the condition in (b) holds in particular when Γ is G-locally primitive (see Theorem 1.3).

6. Problems

The results of this paper suggest a structured approach to investigating the graphs in the family $\mathscr{F}$ of finite graphs which admit a group acting transitively and locally-quasiprimitively on vertices. First more detailed information about the basic locally-quasiprimitive graphs in $\mathscr{F}$ would be useful.

Problem 6.1. Analyse further the structure of G-basic, G-vertex-transitive, G-locally-quasiprimitive graphs.

Much work on this problem has been undertaken already for the subfamily of non-bipartite G-basic graphs which are $(G, 2)$-arc transitive (see [1,5,6,11,13,21,22]; a survey of these results is given in [24]). The most important tool currently available for this investigation is the 'O'Nan-Scott' Theorem [21] for finite quasiprimitive permutation groups. This can be used to analyse the non-bipartite G-basic graphs. However, we are lacking a similar group theoretic result for analysing the bipartite examples.

Problem 6.2. Describe the finite bi-quasiprimitive permutation groups (in a manner similar to the O'Nan-Scott Theorem).

The problem of reconstructing Γ from information about all its basic normal quotient graphs Γ_{N_i}, $(i = 1, \ldots, r$, say) remains open, and of fundamental importance. The maximum amount of information we could expect to retrieve about Γ from these quotients would relate only to the graph Γ_N where $N := \bigcap_{i=1}^{r} N_i$.

Problem 6.3. Suppose that Γ is a finite graph which is G-vertex-transitive and G-locally-quasiprimitive, and that $\Gamma_{N_1}, \ldots, \Gamma_{N_r}$ are quotients relative to normal subgroups N_i of G

such that $\bigcap_{i=1}^{r} N_i = 1$. What extra information is needed in order to reconstruct Γ from these normal quotients? In particular, what is required if the graphs Γ_{N_i} are G-basic?

A preliminary result was given in the previous section for the case $r = 2$. A more complete solution to Problem 6.3, or to the following natural extension of it, would be welcome.

Problem 6.4. Suppose that Γ is a finite graph which is G-vertex-transitive and G-locally-quasiprimitive, with G-normal quotient Γ_N. What extra information is needed to reconstruct Γ from Γ_N? For example, under what circumstances is Γ determined by Γ_N together with the bipartite graph induced on the union of two adjacent N-orbits?

Since quasiprimitivity is not necessarily inherited by overgroups, we need to address the following problem.

Problem 6.5. Under what circumstances can we guarantee that a graph $\Gamma \in \mathscr{F}$ is $\mathrm{Aut}(\Gamma)$-locally-quasiprimitive? In particular, when is this true for the basic graphs in $\mathscr{F}$?

Problem 6.6. Suppose $\Gamma \in \mathscr{F}$ with respect to G, and Γ is G-basic. Under what circumstances is Γ also $\mathrm{Aut}(\Gamma)$-basic?

This problem has already received some attention in the case of 2-arc transitive graphs (see [14]) and almost simple locally-primitive graphs (see [7,8]). Finally, we note the following conjecture.

Conjecture 6.7. There is a function f on the natural numbers such that, for a natural number k, if $\Gamma \in \mathscr{F}$ and Γ has valency k, then the cardinality of a vertex stabilizer in $\mathrm{Aut}(\Gamma)$ is at most $f(k)$.

This conjecture is analogous to a conjecture made by Weiss [30] in 1978 for finite locally-primitive graphs, and the task of proving Weiss's conjecture for non-bipartite graphs has been reduced in [3] to proving it in the case where $\mathrm{Aut}(\Gamma)$ is an almost simple group (that is, $T \leqslant \mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}\, T$ for some non-abelian simple group T). Using the approach of this paper to describing graphs in $\mathscr{F}$, it may be possible to reduce the proofs of both this conjecture and the Weiss Conjecture to the case where $\mathrm{Aut}(\Gamma)$ is almost simple (whether or not the graphs are bipartite). Certainly one need only consider basic graphs Γ by Theorem 1.4.

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