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No-hole 2-distant colorings for Cayley graphs on finitely generated abelian groups

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Abstract

A no-hole 2-distant coloring of a graph Γ is an assignment c of nonnegative integers to the vertices of Γ such that $|c(v)-c(w)| \geqslant 2$ for any two adjacent vertices v and w, and the integers used are consecutive. Whenever such a coloring exists, define $\mathrm{nsp}(\Gamma)$ to be the minimum difference (over all c) between the largest and smallest integers used. In this paper we study the no-hole 2-distant coloring problem for Cayley graphs over finitely generated abelian groups. We give sufficient conditions for the existence of no-hole 2-distant colorings of such graphs, and obtain upper bounds for the minimum span $\mathrm{nsp}(\Gamma)$ by using a group-theoretic approach.

Keywords: Channel assignment; *T*-coloring; No-hole 2-distant coloring; Cayley graph

1. Introduction

In a cellular radio system the communication area is usually divided into regions, within which a transmitter serves customers in the region. Two transmitters may interfere with each other due to various reasons such as geographic proximity and meteorological factors. A fundamental problem [11] is to assign channels (non-negative integers) to the transmitters such that the bandwidth used is minimized while the interference is avoided as much as possible. In the past more than two decades, researchers have formulated various mathematical models for this problem, among them the *T*-coloring model has been widely studied [25] since early 1980s. Given a set *T* of non-negative integers containing 0, a *T*-coloring is an assignment of channels to the transmitters under the constraint that if two transmitters interfere

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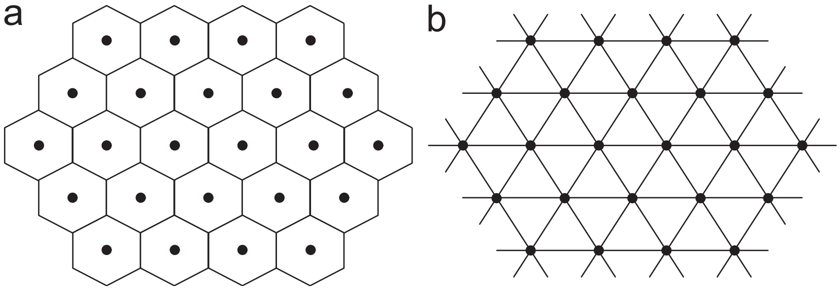

Fig. 1. (a) Part of a hexagonal system (dots represent transmitters); (b) the corresponding interference graph.

then the difference between their channels does not fall into the forbidden set T. This problem can be formulated [6,25] as the following coloring problem for the interference graph, which is defined to have transmitters as vertices such that two vertices are adjacent if and only if the corresponding transmitters interfere with each other (see Fig. 1 for an illustration).

More explicitly, given a graph $\Gamma = (V(\Gamma), E(\Gamma))$ and a set T of non-negative integers containing 0, a *T-coloring* of Γ is a mapping $c : V(\Gamma) \to \{0, 1, 2, \ldots\}$ such that

$$|c(u) - c(v)| \notin T \quad \text{if } uv \in E(\Gamma).$$

In the case when $c(V(\Gamma)) := \{c(v) : v \in V(\Gamma)\}$ consists of consecutive integers, a T-coloring c is called a *no-hole T-coloring* of Γ. The *span* of a T-coloring c is the difference between the largest and smallest colors used in $c(V(\Gamma))$. The *T-span* of Γ, denoted by $\mathrm{sp}_T(\Gamma)$, is the minimum span over all possible T-colorings of Γ. For various graphs and forbidden sets, this invariant $\mathrm{sp}_T(\Gamma)$ has been studied extensively by several authors; see e.g. [4,6,7,9,13,18–20,23–25,31]. It is known [6] that, for any graph Γ and forbidden set T, a T-coloring of Γ always exists. However, a no-hole T-coloring does not hold this property. (For instance, take $T = \{0, 1\}$ and $\Gamma = K_2$.) The *no-hole T-span* of Γ, denoted by $\mathrm{nsp}_T(\Gamma)$, is defined to be the minimum span of a no-hole T-coloring of Γ if such a coloring exists, and ∞ otherwise.

For an integer $r \geqslant 0$, a no-hole $\{0, 1, 2, \ldots, r\}$-coloring is also called [27] an *N_r-coloring*; that is, an N_r-coloring of a graph Γ is a mapping $c : V(\Gamma) \to \{0, 1, 2, \ldots\}$ such that $c(V(\Gamma))$ is a set of consecutive integers and

$$|c(u) - c(v)| \geqslant r + 1 \quad \text{if } uv \in E(\Gamma).$$

The no-hole $\{0, 1, 2, \ldots, r\}$-span of Γ is usually denoted by $\mathrm{nsp}_r(\Gamma)$. In the case where $r = 1$, an N_1-coloring is called an *N-coloring* [27] or a *no-hole* 2-*distant coloring* [26], and we use $\mathrm{nsp}(\Gamma)$ in place of $\mathrm{nsp}_1(\Gamma)$. In general, a T-coloring with $T = \{0, 1\}$ will be called a 2-*distant coloring*, and any unused color between 0 and the maximum color used and is called a *hole*.

Besides potential applications, the N_r-coloring problem and in particular the no-hole 2-distant coloring problem are interesting from a pure combinatorial point of view. The main concerns are existence of such colorings and exact values or estimations of $\mathrm{nsp}_r(\Gamma)$ and $\mathrm{nsp}(\Gamma)$. The existence of N-colorings and N_r-colorings was studied by Roberts [26] and Sakai and Wang [27], respectively, for certain special graphs such as paths, cycles, bipartite graphs and 1-unit sphere graphs. In the case when such a coloring exists, they also gave upper and lower bounds for $\mathrm{nsp}(\Gamma)$ and $\mathrm{nsp}_r(\Gamma)$, respectively. The exact values of $\mathrm{nsp}_T(\Gamma)$ were studied by Liu and Yeh in [21], where they proved the following result: if T is r-initial (that is, $T = \{0, 1, 2, \ldots, r\} \cup A$ where A contains no multiple of $(r+1)$) or $T = \{0, a, a+1, a+2, \ldots, b\}$ for some integers $a, b \geqslant 1$, then for any sufficiently large n there exists a graph Γ with n vertices such that $\mathrm{nsp}_T(\Gamma) = n - 1$. In [2], Chang et al. gave the exact values of $\mathrm{nsp}_r(\Gamma)$ for all bipartite graphs with at least $r - 2$ isolated vertices, and determined completely the exact values of $\mathrm{nsp}_2(\Gamma)$ for all bipartite graphs. In [5] the authors of the present paper proved among other things that a Hamming graph $H_{q_1,q_2,\ldots,q_d} := K_{q_1} \Box K_{q_2} \Box \cdots \Box K_{q_d}$ (Cartesian product) admits a no-hole 2-distant coloring if and only if it is not isomorphic to K_4, and moreover they obtained the exact value of $\mathrm{nsp}(H_{q_1,q_2,\ldots,q_d})$. The reader is referred to [3,15,21,28,29] for other results concerning no-hole 2-distant colorings and the minimum span $\mathrm{nsp}(\Gamma)$.

In this paper we will study no-hole 2-distant colorings of finite and countable Cayley graphs on abelian groups. Such graphs Γ are precisely Cayley graphs on finitely generated abelian groups G, which are either finite or countably infinite [30, Section 5.4.2]. We will provide generic approaches to obtaining no-hole 2-distant colorings of Γ, and give upper bounds for $\mathrm{nsp}(\Gamma)$. Our main results, Theorems 1–3 in the next section, indicate a strong connection between the existence of a no-hole 2-distant coloring of Γ and that of a subgroup of G with certain properties. Moreover, the upper bounds obtained suggest a close relationship between $\mathrm{nsp}(\Gamma)$ and the chromatic number of a certain subgraph of Γ. All these results will be established under a rather general setting, and they can be applied to various Cayley graphs on finitely generated abelian groups. Due to limited space we will only apply them to a certain family of circulant graphs in this paper, and this will be discussed in the last section (Theorems 7 and 8).

The technique that we are going to use is a combination of group-theoretic and combinatorial methods. It bears some similarity with the methodology developed in [33,35], where a group-theoretic approach was initiated by the third-named author in order to study the $L(j,k)$-labelling problem (see [1] for a recent survey) for Cayley graphs over finite abelian groups. However, in the present paper the no-hole condition increases complication, and thus more sophisticated application of elementary group theory becomes necessary. The methodology developed in this paper can be refined to obtain a generic upper bound for the chromatic number of any Cayley graph on a finitely generated group. See Theorem 9 for details.

2. Main results and notation

In this section we will introduce the notation that we need and present the main results of the paper. Proofs of such results, Theorems 1–3, will be given in Section 3.

We will follow standard terminology and notation for graphs [32] and groups [30]. For abelian groups G, we will write the operation as addition. Accordingly, we use 0 to denote the identity element of G, $-g$ the inverse of an element g, ng the element $g+g+\cdots+g$ (n terms) and $(-n)g$ the element $-(ng)$ for any positive integer n, and $0g$ the group identity 0. For a subset Y of G, $\langle Y\rangle := \left\{\sum_{i=1}^{t} n_i y_i : n_i \in \mathbb{Z}, y_i \in Y, t \geqslant 1\right\}$ is the subgroup of G generated by Y. We always use G for a *finitely generated group*, that is, $G=\langle Y\rangle$ for some finite $Y \subseteq G$; thus, G is a finite or countably infinite group [30, Section 5.4.2]. A subset X of G with $0 \notin X$ is called a *Cayley set* of G if it is closed under taking inverse, that is, $x \in X$ implies $-x \in X$. For such an X, the *Cayley graph* of G with respect to X, denoted by $\Gamma(G,X)$, is the graph with vertex set G such that $g,h \in G$ are adjacent if and only if $g-h \in X$. Thus, $g,h \in G$ are joined by a path of $\Gamma(G,X)$ if and only if $g-h \in \langle X\rangle$; in particular, $\Gamma(G,X)$ is a connected graph if and only if X is a *generating set* of G, that is, $\langle X\rangle = G$. For convenience we will allow X to be the empty set; in this case $\Gamma(G,X)$ is the graph of isolated vertices. We will only consider finite Cayley sets X of G, so that $\Gamma(G,X)$ has finite degree $|X|$. This is the case in particular when G is a finite group.

In practice the communication area is often divided into regular lattices [8,12] such as triangular, square and hexagonal lattices. Mathematically, this means that the Euclidean plane $\mathbb{R}^2$ is covered by one of these lattices, and the actual communication area occupies part of it. Fig. 1 shows part of the hexagonal lattice, together with the corresponding interference graph, which is the triangular lattice. Similarly, the interference graph of the triangular lattice is the hexagonal lattice, and that of the square lattice is the square lattice. The channel assignment problem for these lattices was studied in [12], and homogeneous channel assignments on hexagonal systems were explored in [8]. We observe that these lattices are all countable Cayley graphs on finitely generated abelian groups. (A *countable graph* is an infinite graph whose vertex and edge sets are countably infinite sets.) For example, letting $\mathbf{e}_1=(1,0)$, $\mathbf{e}_2=(0,1)$ and $\mathbf{g}=(\frac{1}{2},\sqrt{3}/2)$, the triangular lattice $\triangle$ is the Cayley graph of the abelian group $G=\langle\{\mathbf{e}_1,\mathbf{g}\}\rangle=\{m\mathbf{e}_1+n\mathbf{g} : m,n\in\mathbb{Z}\}$ with respect to the Cayley set $\{\mathbf{e}_1,-\mathbf{e}_1,\mathbf{g},-\mathbf{g},\mathbf{g}-\mathbf{e}_1,-\mathbf{g}+\mathbf{e}_1\}$, and the square lattice $\square$ is the Cayley graph of $G=\langle\{\mathbf{e}_1,\mathbf{e}_2\}\rangle=\{m\mathbf{e}_1+n\mathbf{e}_2 : m,n\in\mathbb{Z}\}$ with respect to $\{\mathbf{e}_1,-\mathbf{e}_1,\mathbf{e}_2,-\mathbf{e}_2\}$. Note that the interference graph of any cellular radio system is planar provided that only interference from geographic proximity is taken into account. Thus, from [16, Theorem 2] it follows that such an interference graph, regular or not, is a subgraph of some planar Cayley graph.

We first record the following result, in which "−" means set-theoretic subtraction.

Theorem 1. *Let G be a finite abelian group and X a Cayley set of G. Then $\Gamma(G,X)$ admits a no-hole 2-distant coloring if and only if $\langle G-X\rangle = G$.*

This result can be easily derived from the hamiltonicity of Cayley graphs on finite abelian groups. Note that it does not tell us useful information about the minimum span. In Theorems 2 and 3 we will give upper bounds for $\mathrm{nsp}(\Gamma(G,X))$, for both finite and countable $\Gamma(G,X)$, under certain conditions.

For a subgroup H of G, $H+g$ denotes the coset of g in H, and $G/H=\{H+g : g\in G\}$ the quotient group of G by H. The cardinality of G/H, denoted by $|G:H|$, is called the index of H in G. For two subsets Y, W of G, define $Y+W:=\{y+w : y\in Y, w\in W\}$; in particular, denote $2Y:=Y+Y$. The next result shows that, if $\langle G-(H+X)\rangle=G$ for some H disjoint from X, then $\Gamma(G,X)$ admits a no-hole 2-distant coloring with the property that each color is used by the same number (cardinal) of vertices. We call such a coloring having this advantageous property a *homogeneous* (or *balanced*) no-hole 2-distant coloring.

Theorem 2. *Let G be a finitely generated abelian group and X a finite Cayley set of G. Suppose that there exists a subgroup H of G with finite index $|G:H|$ such that $\langle G-(H+X)\rangle=G$ and $H\cap X=\emptyset$. Then $\Gamma(G,X)$ admits a homogeneous no-hole 2-distant coloring under which all elements in the same coset of H receive the same color, and moreover*

$$\mathrm{nsp}(\Gamma(G,X))\leqslant|G:H|-1. \tag{1}$$

This theorem applies to both finite and countable $\Gamma(G,X)$ with X finite. A homogeneous no-hole 2-distant coloring of $\Gamma(G,X)$ using $|G:H|-1$ colors will be given explicitly during the proof of Theorem 2. In the case when G is finite, by taking $H=\{0\}$ in Theorem 2, we get the sufficiency part of Theorem 1: if $\langle G-X\rangle=G$, then $\Gamma(G,X)$ admits a no-hole 2-distant coloring.

For a finite or countable graph $\Gamma=(V(\Gamma),E(\Gamma))$, the *chromatic number* $\chi(\Gamma)$ of Γ is the least cardinal k such that Γ can be colored *properly* by k colors, that is, adjacent vertices receive different colors. A *hamiltonian path* of a finite graph Γ is a path $P=v_1,v_2,\ldots,v_{|V(\Gamma)|}$ containing all vertices of Γ. If, for some integer $\ell\geqslant1$, v_i and v_j are adjacent in Γ whenever $|i-j|\leqslant\ell$, then P is called a *hamiltonian ℓ-path*. Define $\rho(\Gamma)$ to be the largest $\ell\geqslant1$ such that a hamiltonian ℓ-path of Γ exists; and set $\rho(\Gamma)=0$ if Γ contains no hamiltonian path. For positive integers k and ℓ, denote

$$n(k,\ell):=\begin{cases}\lceil(2k-1)/\ell\rceil, & \ell \text{ is even},\\ \lceil 2k/(\ell+1)\rceil, & \ell \text{ is odd}.\end{cases}$$

Let G be a finitely generated abelian group, and H a subgroup of G with $|G:H|$ finite. For any $Y\subseteq G$, define

$$Y/H:=\{H+y : y\in Y\}.$$

Note that Y/H is not necessarily a subgroup of the quotient group G/H, and that $H+g\in Y/H$ does not imply $g\in Y$. For a finite Cayley set X of G, denote

$$X_0:=X\cup\{0\},\quad X_1:=G-(H+X_0). \tag{2}$$

In (d) of Lemma 6 we will show that X_1/H is a Cayley set of G/H. Thus, $\Gamma(G/H,X_1/H)$ is a finite Cayley graph since $|G:H|$ is finite. Define

$$\rho(X,H):=\rho(\Gamma(G/H,X_1/H)).$$

Then $\rho(X,H)\geqslant1$ if $\langle G-(H+X)\rangle=G$ (Lemmas 5 and 6(d)). Since X is finite, $H\cap X$ is a finite Cayley set of H (Lemma 6(a)), and hence defines the Cayley graph $\Gamma(H,H\cap X)$ of finite degree $|H\cap X|$. If G is countable, then $\Gamma(H,H\cap X)$ is a countable graph as $|G:H|$ is finite. By Brooks' theorem all finite subgraphs of $\Gamma(H,H\cap X)$ have chromatic number at most $|H\cap X|+1$. Hence

$$\chi(\Gamma(H,H\cap X))\leqslant|H\cap X|+1 \tag{3}$$

by the compactness theorem of de Bruijn and Erdős (see [14, Theorem 1]). Of course, if G is finite, then the validity of (3) is guaranteed by Brooks' theorem directly.

Theorem 3. *Let G be a finitely generated abelian group and X a finite Cayley set of G. Suppose that there exists a subgroup H of G such that $\langle G-(H+X)\rangle=G$ and $|G:H|$ is finite and even. Then $\Gamma(G,X)$ admits a no-hole 2-distant coloring. Moreover, for each integer ℓ between 1 and $\rho(X,H)$, we have*

$$\mathrm{nsp}(\Gamma(G,X))\leqslant(|G:H|-2)n(\chi,\ell)+2\chi-1, \tag{4}$$

where $\chi=\chi(\Gamma(H,H\cap X))$.

3	1	4	2	0	3	1	4
0	3	1	4	2	0	3	1
2	0	3	1	4	2	0	3
4	2	0	3	1	4	2	0
1	4	2	0	3	1	4	2
3	1	4	2	0	3	1	4
0	3	1	4	2	0	3	1
2	0	3	1	4	2	0	3

Fig. 2. An optimal homogeneous no-hole 2-distant colorings of $\square$.

Using (3) and noting $n(\chi, \ell) \leqslant \chi$, we get the following.

Corollary 4. *Under the same assumptions as in Theorem* 3, *we have*

$$\mathrm{nsp}(\Gamma(G, X)) \leqslant |G : H|\chi - 1. \tag{5}$$

In particular,

$$\mathrm{nsp}(\Gamma(G, X)) \leqslant |G : H|(|H \cap X| + 1) - 1. \tag{6}$$

Remark 1. The condition $\langle G - (H + X)\rangle = G$ cannot be removed from Theorems 2 and 3 for otherwise the results are not guaranteed. For example, let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the group of integers modulo 4, and let $X = \{1, 3\}$ and $H = \{0\}$. Then $|G : H| = 4$ is even but $G - (H + X) = \{0, 2\}$ is not a generating set of G. In this case $\Gamma(G, X)$ is the cycle of length 4, which does not admit any no-hole 2-distant coloring.

The bounds (1), (4)–(6) are generic in nature. Thus, it is unrealistic to expect them to be universally tight in all occasions, because the class of Cayley graphs on finitely generated abelian groups is very large. Nevertheless, by judiciously choosing H, they can give good bounds or even the exact values of $\mathrm{nsp}(\Gamma(G, X))$ in some cases, as shown in the following example. The inclusion of this simple example is merely for illustrative purpose since the same result $\mathrm{nsp}(\square) = 4$ can be obtained without using our group-theoretic approach.

Example 1. As mentioned earlier in this section the square lattice $\square = \Gamma(G, X)$ is the Cayley graph of $G = \mathbb{Z} \times \mathbb{Z}$ with respect to $X = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Let $H = \{(m, n) : m, n \in \mathbb{Z}, 5 \mid (m + n)\}$. Then H is a subgroup of G, and $G/H = \{H, H + (1, 0), H + (2, 0), H + (3, 0), H + (4, 0)\}$. Clearly, we have $H \cap X = \emptyset$. Also, since, say, $(1, 2), (3, 7) \notin H + X$ and $\langle (1, 2), (3, 7)\rangle = G$, we have $\langle G - (H + X)\rangle = G$. So $\mathrm{nsp}(\square) \leqslant |G : H| - 1 = 4$ by (1). However, one can easily see that 4 is also a lower bound for $\mathrm{nsp}(\square)$. Hence $\mathrm{nsp}(\square) = 4$. Following our general scheme that will be given in the proof of Theorems 2 and 3, the coloring under which all elements of H are colored by 0, $H + (2, 0)$ by 1, $H + (4, 0)$ by 2, $H + (1, 0)$ by 3, $H + (3, 0)$ by 4 is a homogeneous no-hole 2-distant coloring of $\square$ using 4 colors (see Fig. 2).

3. Proofs of Theorems 1–3

We will exploit the following known result [22, Corollary 3.2].

Lemma 5. *Every connected Cayley graph on a finite abelian group contains a hamiltonian path.*

For a finite graph Γ, we use $\overline{\Gamma}$ to denote the *complement* of Γ, that is, the graph with the same vertex set as Γ in which two vertices are adjacent precisely when they are not adjacent in Γ. One can see [27, Corollary 2.4] that Γ admits a no-hole 2-distant coloring if and only if $\overline{\Gamma}$ has a hamiltonian path.

Proof of Theorem 1. Since $\overline{\Gamma(G,X)}=\Gamma(G,G-X_0)$, by Lemma 5, $\overline{\Gamma(G,X)}$ contains a hamiltonian path if and only if $\langle G-X_0\rangle=G$. Noting that $\langle G-X_0\rangle=\langle G-X\rangle$, from the aforementioned result it follows that $\Gamma(G,X)$ admits a no-hole 2-distant coloring if and only if $\langle G-X\rangle=G$. $\square$

For a partition $\mathscr{P}$ of the vertex set of a graph Γ, the *quotient graph* $\Gamma_{\mathscr{P}}$ of Γ with respect to $\mathscr{P}$ is the graph with vertex set $\mathscr{P}$ in which two parts $B,C\in\mathscr{P}$ are adjacent if and only if there exists at least one edge of Γ joining a vertex in B to a vertex in C. In the case where each part of $\mathscr{P}$ is an independent set with k vertices and the subgraph induced by two adjacent parts is a bipartite graph of degree d, for some integers k and d with $1\leqslant d\leqslant k$, the graph Γ is said to be a (k,d)-*multicover* of the quotient $\Gamma_{\mathscr{P}}$. Multicovers have been studied in the context of algebraic graph theory; see for example [17,34].

In the following we assume that H is a subgroup of G with $|G:H|$ finite. For any $Z\subseteq G$, by the definition of Z/H, we have

$$H+g\in Z/H \quad\Leftrightarrow\quad (H+g)\cap Z\neq\emptyset. \tag{7}$$

We will use the same notation G/H to denote the partition of the abelian group G with parts the cosets $H+g$ of H in G. Thus, for any finite Cayley set X of G, $\Gamma(G,X)_{G/H}$ is the quotient graph of $\Gamma(G,X)$ with respect to this partition. Define

$$X_2=X-H.$$

Since $H\in X/H$ if and only if $H\cap X\neq\emptyset$ by (7), we have

$$X_2/H=\begin{cases}X/H, & H\cap X=\emptyset,\\ X/H-\{H\}, & H\cap X\neq\emptyset.\end{cases}$$

Hence

$$|X_2/H|=|X/H|-\delta(X,H),$$

where

$$\delta(X,H):=\begin{cases}0, & H\cap X=\emptyset,\\ 1, & H\cap X\neq\emptyset.\end{cases}$$

Since X is finite and closed under taking inverse, $2X$ is a finite subset of G containing 0. Thus,

$$\eta(X,H):=|H\cap(2X)|$$

defines a positive integer.

To prove Theorems 2 and 3 we will need the following lemma, which might be useful in other occasions regarding coset partitions of Cayley graphs.

Lemma 6. *Let G be a finitely generated abelian group, X a finite Cayley set of G, and H a subgroup of G with finite index $|G:H|$. Let $\Gamma=\Gamma(G,X)$, and let $\eta=\eta(X,H)$ and $\delta=\delta(X,H)$ be as above. Then the following* (a)–(e) *hold.*

(a) *$H\cap X$ is a Cayley set of H, and for each $H+g$, the subgraph $\Gamma[H+g]$ of Γ induced by $H+g$ is isomorphic to $\Gamma(H,H\cap X)$. In particular, $H\cap X=\emptyset$ if and only if each $H+g$ is an independent set of Γ.*
(b) *X_2 is a Cayley set of G, and $\Gamma(G,X_2)$ is the graph obtained from Γ by deleting the edges of $\Gamma[H+g]$, for all $H+g\in G/H$.*
(c) *The mapping defined by $x\mapsto H+x$, $x\in X$, is an η to* 1 *mapping from X to X/H.*
(d) *X_1/H and X_2/H are Cayley sets of G/H, and $\Gamma(G/H,X_1/H)$ and $\Gamma(G/H,X_2/H)$ are complementary graphs with degrees $|G:H|-|X|/\eta+\delta-1$ and $|X|/\eta-\delta$, respectively.*
(e) *$\Gamma_{G/H}\cong\Gamma(G/H,X_2/H)$, and $\Gamma(G,X_2)$ is an $(|H|,\eta)$-multicover of $\Gamma(G/H,X_2/H)$.*

Proof. (a) It is easy to see that $H\cap X$ is a Cayley set of H, and that $h\mapsto h+g$, $h\in H$, defines an isomorphism from the Cayley graph $\Gamma(H,H\cap X)$ to the induced subgraph $\Gamma[H+g]$ of Γ. Clearly, we have: $H\cap X=\emptyset\Leftrightarrow\Gamma(H,H\cap X)$ is an edgeless graph $\Leftrightarrow$ each $H+g$ is an independent set of Γ.

(b) Since both X and H are closed under taking inverse, it follows that X_2 is a Cayley set of G. Two elements v, w of G are adjacent in $\Gamma(G, X_2)$ if and only if they are not in the same coset of H and $v - w \in X$, and this is true if and only if $\{v, w\}$ is an edge of the graph obtained from Γ by deleting the edges of all $\Gamma[H + g]$, for $H + g \in G/H$.

(c) Clearly, $x \mapsto H + x$ defines a surjective mapping from X to X/H. For $v, w \in X$, $H + v = H + w$ if and only if $v - w \in H \cap (2X)$. From this (c) follows immediately.

(d) Since $-(H + x) = H + (-x)$ and X is closed under taking inverse, it follows that X/H is closed under taking inverse, and hence X_2/H is a Cayley set of G/H. From (7) we have: $H + g \in X_1/H \Leftrightarrow (H + g) \cap X_1 \neq \emptyset \Leftrightarrow$ there exists $h \in H$ such that $h + g \notin H + X_0 \Leftrightarrow g \notin H + X_0 \Leftrightarrow g \notin H$ and $g \notin H + X \Leftrightarrow H + g \neq H$ and $H + g \notin X/H$ $\Leftrightarrow H + g \neq H$ and $H + g \notin X_2/H$. In other words, $\{X_1/H, X_2/H\}$ is a partition of $G/H - \{H\}$. Hence X_1/H is a Cayley set of G/H, and $\Gamma(G/H, X_1/H)$ and $\Gamma(G/H, X_2/H)$ are complementary Cayley graphs. From (c) we have $|X_2/H| = |X|/\eta - \delta$, and so the degrees of $\Gamma(G/H, X_1/H)$ and $\Gamma(G/H, X_2/H)$ are as stated.

(e) For distinct $H+v, H+w \in G/H$, $H+v$ and $H+w$ are adjacent in $\Gamma(G/H, X_2/H) \Leftrightarrow (H+v)-(H+w) \in X_2/H$ $\Leftrightarrow v - w = h + x$ for some $x \in X_2$ and $h \in H \Leftrightarrow v - (h + w) = x$ for some $x \in X_2$ and $h \in H$ (note that G is abelian by our assumption) $\Leftrightarrow v \in H + v$ and $h + w \in H + w$ are adjacent in $\Gamma(G, X_2)$ for some $h \in H \Leftrightarrow g + v \in H + v$ and $g + h + w \in H + w$ are adjacent in $\Gamma(G, X_2)$ for some $h \in H$ and any $g \in H \Leftrightarrow H + v$ and $H + w$ are adjacent in the quotient graph $(\Gamma(G, X_2))_{G/H}$. Since $\Gamma_{G/H} \cong (\Gamma(G, X_2))_{G/H}$ by (b), we have $\Gamma_{G/H} \cong \Gamma(G/H, X_2/H)$. Moreover, from the arguments above we can see that, for distinct $H + v$ and $H + w$ adjacent in $\Gamma(G/H, X_2/H)$, each element of $H + v$ is adjacent to at least one element of $H + w$ in Γ; and, assuming that v and w are adjacent in Γ, v and $h + w$ are adjacent in $\Gamma \Leftrightarrow v - (h + w) \in X \Leftrightarrow h \in H \cap (2X)$.Thus, each element of $H + v$ is adjacent to exactly η elements of $H + w$ in Γ. In other words, $\Gamma(G, X_2)$ is an $(|H|, \eta)$-multicover of $\Gamma(G/H, X_2/H)$. $\square$

Remark 2. In the case where X in Lemma 6 is a generating set of G, one can verify that $\Gamma(G/H, X_1/H)$ is the underlying simple graph of the Schreier coset graph with respect to (G, H, X). (For any group G, any subgraph H of G, and any generating set X of G, the *Schreier coset graph* (see e.g. [10]) with respect to (G, H, X) is the directed graph with vertex set $G/H = \{Hg : g \in G\}$ and arcs (Hg, Hgx) for all $Hg \in G/H$ and $x \in X$. Note that in a Schreier coset graph loops and multiple arcs are allowed.)

Recall that a 2-distant coloring of a graph Γ is a mapping $c : V(\Gamma) \to \{0, 1, 2, \ldots\}$ such that $|c(u) - c(v)| \geqslant 2$ for any two adjacent vertices u, v of Γ.

Proof of Theorem 3. Let $\Gamma := \Gamma(G, X)$ and $r := |G : H|$. By Lemma 6(a), for each coset $H + g$, the subgraph $\Gamma[H + g]$ of Γ induced by $H + g$ is isomorphic to $\Gamma(H, H \cap X)$. Hence $\Gamma[H + g]$ has chromatic number χ, which is finite by (3) and the finiteness of X. For any integer $k \geqslant 0$, and any proper coloring of $\Gamma[H + g]$ with χ colors, the coloring under which the vertices in the ith color class are labelled $k + 2(i - 1)$, for $i = 1, 2, \ldots, \chi$, is a 2-distant coloring of $\Gamma[H + g]$. In the following, we will use this fact repeatedly, and we will say without further explanation that the vertices of $\Gamma[H + g]$ are colored by $k, k + 2, \ldots, k + 2\chi - 2$.

Since $\langle G - (H + X)\rangle = G$ by our assumption, we have

$$\langle X_1/H\rangle = \langle X_1\rangle/H = \langle G - (H + X_0)\rangle/H = \langle G - (H + X)\rangle/H = G/H.$$

Note that X_1/H is a Cayley set of G/H by Lemma 6(d), and that $|G : H|$ is finite by our assumption. Hence $\Gamma(G/H, X_1/H)$ is a connected Cayley graph over the finite abelian group G/H, and contains a hamiltonian path by Lemma 5. Thus, we have $\rho(X, H) \geqslant 1$. If $r = 2$, say $G/H = \{H, H + g\}$, then from (d) of Lemma 6, $\Gamma(G/H, X_2/H)$ is the graph consisting of two isolated vertices, namely H and $H + g$. Hence, from (e) of the same lemma, there is no edge of Γ between H and $H + g$. Thus, from the discussion in the previous paragraph, the coloring under which the elements of H are colored by $0, 2, \ldots, 2(\chi - 1)$ and that of $H + g$ by $1, 3, \ldots, 2\chi - 1$ is a no-hole 2-distant coloring of Γ. In this case we have $\mathrm{nsp}(\Gamma) \leqslant 2\chi - 1$ and hence (4) is true.

Since r is even by our assumption, in the remaining part of the proof we assume that $r \geqslant 4$. Since $\rho(X, H) \geqslant 1$, for each $1 \leqslant \ell \leqslant \rho(X, H)$, by the definition of $\rho(X, H)$, $\Gamma(G/H, X_1/H)$ contains a hamiltonian ℓ-path, say $H + g_0, H + g_1, \ldots, H + g_{r-1}$. Thus, from (d) and (e) of Lemma 6, for any i, j with $1 \leqslant |i - j| \leqslant \ell$ there is no edge of Γ between $H + g_i$ and $H + g_j$. Now set $n = n(\chi, \ell)$ and define a coloring c of Γ in the following way: for $0 \leqslant i \leqslant r - 1$, if i is even then color the vertices of $H + g_i$ by $ni, ni + 2, \ldots, ni + 2\chi - 2$, and if i is odd then color the vertices of $H + g_i$ by $n(i-1)+1, n(i-1)+3, \ldots, n(i-1)+2\chi-1$. Thus, for even integers i, the vertices of $(H+g_i) \cup (H+g_{i+1})$ use all colors in the integer interval $[ni, ni+2\chi-1]$. Since $n \leqslant \chi$ from the definition of $n(\chi, \ell)$, we have $n(i+2) \leqslant ni+2\chi$ and hence the union of $[ni, ni+2\chi-1]$ and $[n(i+2), n(i+2)+2\chi-1]$ is an integer interval, namely $[ni, n(i+2)+2\chi-1]$. Since this is true for $i=0, 2, 4, \ldots, r-2$, and since r is even, c is a no-hole coloring of Γ using all colors in the interval $[0, n(r-2)+2\chi-1]$. By our convention at the beginning of the proof, we have $|c(v)-c(w)| \geqslant 2$ for any two adjacent vertices v, w in the same coset $H+g_i$. In other words, the 2-distant condition is satisfied by each "slice" $\Gamma[H+g_i]$. Thus, to prove that c is a 2-distant coloring it suffices to show that, if two vertices, say, $v \in H+g_i$ and $w \in H+g_j$, are adjacent in Γ, then $|c(v)-c(w)| \geqslant 2$. In fact, from the discussion above we have $|i-j| > \ell$ in this case, and hence

$$|c(v)-c(w)| \geqslant \min\{|c(\hat{v})-c(\hat{w})| : \hat{v} \in H+g_0, \hat{w} \in H+g_{\ell+1}\} \tag{8}$$

by the definition of our coloring c. If ℓ is odd, then the vertices of $H+g_{\ell+1}$ are colored $n(\ell+1), n(\ell+1)+2, \ldots, n(\ell+1)+2\chi-2$ and hence the right-hand side of (8) is $n(\ell+1)-2\chi+2$; whilst if ℓ is even, then the vertices of $H+g_{\ell+1}$ are colored $n\ell+1, n\ell+3, \ldots, n\ell+2\chi-1$ and hence the right-hand side of (8) is $n\ell-2\chi+3$. In both cases, the right-hand side of (8) is at least 2 by the definition of $n(\chi, \ell)$, and hence $|c(v)-c(w)| \geqslant 2$. Therefore, c is a no-hole 2-distant coloring of Γ with span $n(r-2)+2\chi-1$, and the proof is complete. □

Proof of Theorem 2. This can be conducted in a similar way as above. Since $H \cap X = \emptyset$ by our assumption, $\Gamma(H, H \cap X)$ is an edgeless graph and hence $\chi = \chi(\Gamma(H, H \cap X)) = 1$. Using the notation in the proof of Theorem 3, we define c to be a coloring of $\Gamma := \Gamma(G, X)$ such that all vertices in $H+g_i$ are colored by i, for $0 \leqslant i \leqslant r-1$. This coloring is a 2-distant coloring because each coset $H+g_i$ is an independent set of Γ (Lemma 6(a)) and there is no edge between $H+g_{i-1}$ and $H+g_i$ for $1 \leqslant i \leqslant r-1$. Obviously, c is no-hole and its span is $\mathrm{sp}(\Gamma; c) = r-1$. Moreover, each color i is used by $|H|$ vertices, where $|H|$ is the cardinality of H. This completes the proof. □

4. A special case: circular graphs

The simplest finitely generated abelian groups are the additive groups $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ of integers modulo n, where $n \geqslant 1$ is an integer. In this section we will apply the general results in Section 2 to Cayley graphs on $\mathbb{Z}_n$, which are called *circulant graphs*. For $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, $S \cup (-S)$ is a Cayley set of $\mathbb{Z}_n$, where $-S = \{n-w : w \in S\}$. Hence it defines a circulant graph $\Gamma(\mathbb{Z}_n, S \cup (-S))$, which we denote by $C(n, S)$, More explicitly,

$$C(n, S) := (\mathbb{Z}_n, \{\{v, v \pm w\} : v \in \mathbb{Z}_n, w \in S\}),$$

where the operation is modulo n. Note that $C(n, \emptyset)$ is the graph of n isolated vertices. Define

$$\phi(S) := |\{w \in S : \gcd(n, w) = 1\}|,$$

$$\mu(S) := \begin{cases} 0, & w \neq n/2 \text{ for all } w \in S, \\ 1 & \text{otherwise,} \end{cases}$$

where $\gcd(\cdot, \cdot)$ denotes the greatest common divisor. Note that $\mu(S) = 1$ occurs only when n is even, and that $\phi(n) = \phi(\{1, 2, \ldots, n\}) = |\{1 \leqslant w \leqslant n : \gcd(n, w) = 1\}|$ is the Euler's function. The following theorem gives a simple criterion on the existence of a no-hole 2-distant coloring for any circulant graph.

Theorem 7. *Let $n \geqslant 2$ be an integer and $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Then $C(n, S)$ admits a no-hole 2-distant coloring if and only if*

$$2\phi(S) - \mu(S) < \phi(n).$$

Proof. The quantity $2\phi(S) - \mu(S)$ is the number of integers $w \in S \cup (-S)$ with $\gcd(n, w) = 1$. Note that an integer $w \in \mathbb{Z}_n$ generates $\mathbb{Z}_n$ if and only if $\gcd(n, w) = 1$; hence $\mathbb{Z}_n$ has $\phi(n)$ generators. Combining this with Theorem 1, we have: $C(n, S)$ admits a no-hole 2-distant coloring $\Leftrightarrow$ $\mathbb{Z}_n - (S \cup (-S))$ contains a generator of $\mathbb{Z}_n$ $\Leftrightarrow$ $\phi(n) > 2\phi(S) - \mu(S)$. □

Theorem 7 paves the way towards investigating the no-hole 2-distant coloring problem for various families of circulant graphs. However, due to space limit in this paper we would rather restrict to circulant graphs of the form $C(2p^r, S)$, where p is a prime, $r \geqslant 2$, and $S \subseteq \{1, 2, \dots, p^r\}$. For each $1 \leqslant t \leqslant r-1$, define $k_t(S)$ to be the maximum cardinality of a subset W of S such that each element of W is odd and not a multiple of p, and that $v \neq w$, $v + w \neq 2p^r \pmod{2p^t}$ for distinct $v, w \in W$. Thus, we have a decreasing sequence

$$k_1(S) \geqslant \cdots \geqslant k_{r-1}(S).$$

In the particular case where all odd integers in S are multiples of p, we have $k_t(S) = 0$ for all t. Define

$$W_t := \left\{ \frac{w}{2p^t} : w \in S,\ 2p^t \text{ divides } w \right\}.$$

Then $W_t \subseteq \{1, 2, \dots, p^{r-t}\}$, and hence W_t defines the circulant graph $C(p^{r-t}, W_t)$.

Theorem 8. *Let p be a prime and $r \geqslant 2$ an integer, and let $S \subseteq \{1, 2, \dots, p^r\}$. If there exists at least one t, $1 \leqslant t \leqslant r-1$, such that $k_t(S) < p^{t-1}(p-1)/2$, then $C(2p^r, S)$ admits a no-hole 2-distant coloring and*

$$\operatorname{nsp}(C(2p^r, S)) \leqslant 2p^t\chi - 1 \tag{9}$$

where $\chi = \chi(C(p^{r-t}, W_t))$. In particular, if none of the elements of S is a multiple of $2p^t$, then

$$\operatorname{nsp}(C(2p^r, S)) \leqslant 2p^t - 1. \tag{10}$$

Proof. Let $S = \{s_1, s_2, \dots, s_k\}$, where $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant p^r$. Denote $X = S \cup (-S) = \{s_j, 2p^r - s_j : j = 1, 2, \dots, k\}$. Let $H = \langle 2p^t \rangle = \{2vp^t : 0 \leqslant v \leqslant p^{r-t} - 1\}$ be the unique subgroup of $\mathbb{Z}_{2p^r}$ with order p^{r-t}. Then $|\mathbb{Z}_{2p^r} : H| = 2p^t$ and $H + X = \bigcup_{j=1}^{k}(Y_j \cup Z_j)$, where $Y_j := \{2vp^t + s_j : 0 \leqslant v \leqslant p^{r-t} - 1\}$ and $Z_j := \{2p^r + 2vp^t - s_j : 0 \leqslant v \leqslant p^{r-t} - 1\}$. For distinct i and j, $Y_i = Y_j$ if and only if $s_i \equiv s_j \pmod{2p^t}$, which is true if and only if $Z_i = Z_j$. Similarly, $Y_i = Z_j$ if and only if $s_i + s_j \equiv 2p^r \pmod{2p^t}$, which is true if and only if $Y_j = Z_i$. Note that $w \in \mathbb{Z}_{2p^r} - \{0\}$ generates $\mathbb{Z}_{2p^r}$ if and only if $\gcd(2p^r, w) = 1$, that is, w is odd and not a multiple of p. The number of such integers w is given by $\phi(2p^r) = p^{r-1}(p-1)$. Clearly, an element $2vp^t + s_j$ of Y_j is odd and not a multiple of p if and only if s_j is odd and not a multiple of p, and in this case every element of Y_j is odd and not a multiple of p. In other words, either all elements of Y_j are generators of $\mathbb{Z}_{2p^r}$, or none of them is a generator of $\mathbb{Z}_{2p^r}$. Similar statement is true for Z_j. Thus, from the definition of $k_t(S)$ and the discussion above, it follows that the number of integers in $H + X$ which are odd and not multiples of p is equal to $2k_t(S)p^{r-t}$, which is strictly less than $p^{r-1}(p-1)$ since $k_t(S) < p^{t-1}(p-1)/2$ by our assumption. Hence $\mathbb{Z}_{2p^r} - (H + X)$ contains at least one generator of $\mathbb{Z}_{2p^r}$. Thus, by Theorem 3, $C(2p^r, S)$ admits a no-hole 2-distant coloring and $\operatorname{nsp}(C(2p^r, S)) \leqslant 2p^t\chi - 1$, where χ is the chromatic number of the Cayley graph $\Gamma(H, H \cap X)$. However, we have $\Gamma(H, H \cap X) \cong C(p^{r-t}, W_t)$ via the bijection $2vp^t \mapsto v$ from H to $\mathbb{Z}_{p^{r-1}}$. Hence χ is equal to the chromatic number of $C(p^{r-t}, W_t)$ and (9) is proved. In the particular case where no element of S is a multiple of $2p^t$, we have $W_t = \emptyset$, $\chi = 1$ and (9) gives rise to (10). $\square$

Remark 3. The condition $k_t(S) < p^{t-1}(p-1)/2$ in Theorem 8 cannot be weakened in general. For example, let $S = \{1, 2, \dots, 2p-1\}$, where p is an odd prime, and let $r = 2$ and $t = 1$. Then $k_1(S) \geqslant (p-1)/2$ and $W_1 = \emptyset$. Since $\mathbb{Z}_{2p^2} - (S \cup (-S))$ contains generators of $\mathbb{Z}_{2p^2}$, say $2p+1$, by Theorem 3, $C(2p^2, S)$ admits a no-hole 2-distant coloring. Since $\{0, 1, \dots, 2p-1\}$ is a clique of $C(2p^2, S)$, its $2p$ vertices must receive colors with mutual difference at least 2 under any 2-distant coloring of $C(2p^2, S)$. Thus, we have $\operatorname{nsp}(C(2p^2, S)) \geqslant 4p$, which is much bigger than $2p - 1$ (the right-hand side of (10)). In other words, the upper bound (10) does not apply to this S.

5. Concluding remarks

For some concrete Cayley graphs, a lot of space is left to improve our bounds and to tune our method in order to suit situations not covered by Theorems 2 and 3. In particular, applications of our method to circulant graphs not satisfying the conditions of Theorem 8 will be interesting and promising.

The method developed in this paper can be also refined to study 2-distant colorings (not necessarily no-hole) of Cayley graphs on finitely generated abelian groups. Let $\operatorname{sp}(\Gamma)$ denote the minimum span over all 2-distant colorings of a graph Γ. Then $\operatorname{sp}(\Gamma) = 2(\chi(\Gamma) - 1)$ ([6,18]), where as before χ denotes the chromatics number. Similar to the proof of Theorem 2 one can prove the following result, which can be viewed as a generalisation of Theorem 2.

Theorem 9. *Let G be a finitely generated abelian group and X a finite Cayley set of G. Then, for any subgroup H of G such that $|G:H|$ is finite and $H \cap X = \emptyset$, we have*

$$\chi(\Gamma(G, X)) \leqslant \frac{1}{2}\left(|G:H| + |G:\langle G-(H+X)\rangle|\right). \tag{11}$$

Moreover, $\Gamma(G, X)$ admits a 2-distant coloring which uses $|G:H| + |G:\langle G-(H+X)\rangle| - 2$ colors and contains $|G:\langle G-(H+X)\rangle| - 1$ holes.

Due to the generic nature one can easily construct examples for which the upper bound in (11) is far away from the chromatic number. On the other hand one can equally easily find examples for which (11) is attained. (For example, taking $G = \mathbb{Z}_4$, $X = \{1, 3\}$, $H = \{0, 2\}$, we have $\Gamma(G, X) \cong K_{2,2}$ and both sides of (11) are equal to 2.) In general, our knowledge about the chromatic number of Cayley graphs is very limited, and most likely (11) is the best that we can hope under the condition of Theorem 9.

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