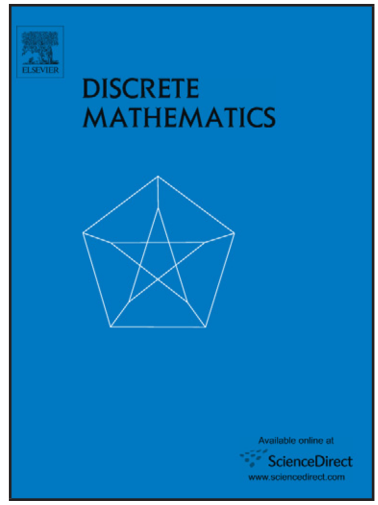

Large forbidden trade volumes and edge packings of random graphs

Nicholas C. Wormald, SanmingZhou

PII: S0012-365X(07)00404-9
DOI: doi:10.1016/j.disc.2006.06.035
Reference: DISC 6865

To appear in: *Discrete Mathematics*

Received date: 20 August 2003
Revised date: 10 September 2004
Accepted date: 21 June 2006

Cite this article as: Nicholas C. Wormald and SanmingZhou, Large forbidden trade volumes and edge packings of random graphs, *Discrete Mathematics* (2007), doi:10.1016/j.disc.2006.06.035

Large forbidden trade volumes and edge packings of random graphs

Nicholas C. Wormald*
Department of Combinatorics and Optimization
University of Waterloo
Waterloo ON, Canada
nwormald@uwaterloo.ca

Sanming Zhou†
Department of Mathematics and Statistics
The University of Melbourne
VIC 3010, Australia
smzhou@ms.unimelb.edu.au

Abstract

Let G be a graph. A G-trade of volume m is a pair $(\mathcal{T}, \mathcal{T}')$, where each of $\mathcal{T}$ and $\mathcal{T}'$ consists of m graphs, pairwise edge-disjoint, isomorphic to G, such that $\mathcal{T} \cap \mathcal{T}' = \emptyset$ and the union of the edge sets of the graphs in $\mathcal{T}$ is identical to the union of the edge sets of the graphs in $\mathcal{T}'$. Let $X(G)$ be the set of non-negative integers m such that no G-trade of volume m exists. In this paper we prove that, for $G \in \mathcal{G}(n, \frac{1}{2})$, $\{1, 2, \ldots, \lceil cn/\log n \rceil\} \subseteq X(G)$ holds asymptotically almost surely, where $c = \log(4/3)/88$.

Keywords: Trade spectrum; Trade volume; Random graph; Block design

Let $G = (V(G), E(G))$ be a simple graph. A *G-decomposition* of a simple graph $H = (V(H), E(H))$ is a set $\mathcal{T} = \{G_i : 1 \le i \le m\}$ of graphs such that $G_i \cong G$, $1 \le i \le m$, and $\{E(G_i) : 1 \le i \le m\}$ is a partition of $E(H)$. A *G-trade of volume m* is a pair $(\mathcal{T}, \mathcal{T}')$, where each of $\mathcal{T}$ and $\mathcal{T}'$ is a G-decomposition of the same simple graph H such that $|\mathcal{T}| = |\mathcal{T}'| = m$ and $\mathcal{T} \cap \mathcal{T}' = \emptyset$. The *trade spectrum* of G, denoted $TS(G)$, is defined to be the set of integers m such that a G-trade of volume m exists. From this definition it follows that $0 \in TS(G)$, and $1 \in TS(G)$ if and only if G contains at least one isolated vertex. Denote by $X(G)$

*Research supported by the Australian Research Council, the Canada Research Chairs Program and NSERC. Research partly carried out while the author was at the Department of Mathematics and Statistics, University of Melbourne.

†Research supported by a Discovery Project Grant from the Australian Research Council.

the set of *forbidden trade volumes*, that is, the set of non-negative integers m such that no G-trade of volume m exists. Then

$$X(G) = \{0, 1, 2, \ldots\} \setminus TS(G).$$

The concept of a G-trade originated from design theory. There, a (v, k, t)-trade of volume m is defined to be a pair $(\mathbf{T}, \mathbf{T}')$, where $\mathbf{T}, \mathbf{T}'$ are collections of m blocks of size k chosen from a fixed v-set such that $\mathbf{T} \cap \mathbf{T}' = \emptyset$ and each t-subset of the v-set occurs in precisely the same number of blocks of $\mathbf{T}$ as of $\mathbf{T}'$. Trades in the design theory setting are useful for changing designs into other designs, and the trade spectrum has implications for the applicability of such a construction. It also constrains the amount of common structure (i.e. blocks in the block design case) that is possible between two structures with identical parameters, which relates to the "intersection problem" in design theory. Such questions have prompted analogous questions about trade volumes in graphs. Note that if we identify a complete graph K_v with a v-set and identifying a complete subgraph K_k of K_v with a block of size k, a K_k-trade of volume m such that the underlying graph $H = K_v$ is exactly a $(v, k, 2)$-trade. The reader is referred to [2, 3, 4, 5, 7] for recent results on trade spectra of graphs.

One can see that $TS(G)$ is *additive*, that is, if $m_1, \ldots, m_k \in TS(G)$, then $\sum_{i=1}^{k} c_i m_i \in TS(G)$ for any non-negative integers c_i. Thus, $X(G) = \emptyset$ if and only if [2, Lemma 2.1] G contains isolated vertices. Also, if $2, 3 \in TS(G)$, then $X(G) = \{1\}$ since any integer no less than 2 can be written as $2c_1 + 3c_2$ for some c_1 and c_2. In general, Billington and Hoffman [2] proved that $X(G) \subseteq \{1, 2\}$ holds for several families of graphs. Also, they show [2, Theorem 3.2] that, for any graph $G \neq K_2$, $2s, 3s \notin X(G)$ holds for any integer $s \geq \delta(G)$, where $\delta(G)$ is the minimum degree of G. As a consequence all integers large enough, say, no less than $5\delta(G)+2$, are not in $X(G)$ (see [2, Theorem 3.2] for details). That is, graphs with small minimum degree cannot have large forbidden trade volumes. On the other hand, for complete graphs K_n of order n, we have $\{1, 2, \ldots, 2n-3\} \subseteq X(K_n)$ [2, Lemma 4.1], and hence the forbidden trade volumes increase with the order. Complete graphs are the only known graphs with this property. Billington [1] asked whether there exist non-complete graphs G of order n such that the forbidden trade volumes of G increase with n. In this paper we answer this question affirmatively for random graphs.

As usual we use $\mathcal{G}(n, \frac{1}{2})$ to denote the probability space of random graphs of order n with any two vertices being adjacent with probability $1/2$. For a sequence of probability spaces Ω_n, $n \geq 1$, an event A_n of Ω_n occurs *asymptotically almost* *surely*, abbreviated to *a.a.s.* in the following, if $\lim_{n\to\infty} \mathbf{P}(A_n) = 1$. Set

$$c = \frac{\log(4/3)}{88}.$$

Our main result is the following theorem.

Theorem 1 *For $G \in \mathcal{G}(n, \frac{1}{2})$, a.a.s.*

$$\{1, 2, \ldots, \lceil cn/\log n\rceil\} \subseteq X(G). \tag{1}$$

In order to prove this we introduce the following two concepts. A graph $G = (V(G), E(G))$ of order n is called *j-non-meshing*, for some integer j with $2 \le j \le n$, if every way of identifying j vertices of one copy of G with j vertices of another copy of G gives a graph with multiple edges. In other words, G is j-non-meshing if, for any two graphs G_1 and G_2 isomorphic to G and having j vertices in common, there exist $u, v \in V(G_1) \cap V(G_2)$ such that u and v are adjacent in both G_1 and G_2. For example, K_n is j-non-meshing for $2 \le j \le n$. For a graph G, a subset K of $V(G)$ is *G-defining* if there exists no non-identity permutation σ of $V(G)$ such that, for all $u \in K$ and $v \in V(G)$, $uv \in E(G)$ if and only if $\sigma^{-1}(u)\sigma^{-1}(v) \in E(G)$. Denote

$$j_0(n) = \frac{8\log n}{\log(4/3)}.$$

Lemma 1 *Asymptotically almost surely, $G \in \mathcal{G}(n, \frac{1}{2})$ is j-non-meshing for all j with $j_0(n) < j \le n$.*

Proof. Let J be a subset of $V(G)$ with $|J| = j$. Let $A(J)$ be the event that there exists an injection σ from J to $V(G)$ such that for all pairs $\{u, v\}$ of distinct vertices u, v in J,

$$\text{either } uv \notin E(G) \text{ or } \sigma(u)\sigma(v) \notin E(G). \tag{2}$$

For a fixed pair $\{u, v\}$, the probability that (2) holds is 1/2 when $\{\sigma(u), \sigma(v)\} = \{u, v\}$ (as this can only happen if $uv \notin E(G)$) and 3/4 otherwise. However, these events are not independent for different pairs $\{u, v\}$. If $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$ is a set of distinct pairs of vertices in J such that $\sigma(u_i) = u_{i+1}$ and $\sigma(v_i) = v_{i+1}$ for $1 \le i < k$ ($k \ge 2$), we say that these pairs are *associated* by σ. For all of these pairs to satisfy (2), it is necessary that no two consecutive pairs in the sequence $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$ are edges of G. (The extra condition on the image of $\{u_k, v_k\}$ under σ gives no improvement, as it turns out, since it may happen that $\{u_1, v_1\} = \{u_k, v_k\}$, and $k = 2$ is the value of k which determines the final result.) The probability that this happens is 3/4 for $k = 2$, whilst for $k \ge 3$ it is

$$2^{-k} \sum_{i=0}^{k/2} \binom{k-i+1}{i} \le 2^{-k} \lfloor k/2 + 1 \rfloor 2^{0.724k} < \left(\frac{3}{4}\right)^{k/2}.$$

The middle step here follows on noting that the binomial is increasing in i for $i \le \frac{k}{2} - \frac{k}{2\sqrt{5}}$, and the last step follows by calculus and checking the small values of k. The pairs of vertices in J can be partitioned into maximal associated sets, and the event considered above is, for a maximal associated set, independent of all other pairs of vertices in J. Thus, for a given injection σ from J to $V(G)$, the probability that σ satisfies (2) for all $\binom{j}{2}$ pairs of vertices in J is at most $(3/4)^{j(j-1)/4}$. Thus, $\mathbf{P}(A(J)) \le [n]_j (3/4)^{j(j-1)/4}$, where $[n]_j = n(n-1)\cdots(n-j+1)$. Consequently, if X_j is the number of sets J with $|J| = j$ such that $A(J)$ holds,

$$\mathbf{E}(X_j) \le \binom{n}{j} [n]_j \left(\frac{3}{4}\right)^{j(j-1)/4} \le \frac{n^{2j}}{j!} \left(\frac{3}{4}\right)^{j(j-1)/4} = \frac{e^{(2\log n + (j-1)\log(3/4)/4)j}}{j!} \tag{3}$$

which is $O(1/j!)$ since $j > j_0(n)$. Thus $\mathbf{E}\left(\sum_{j>j_0} X_j\right) = o(1)$ using linearity of expectation. So by the first moment principle, $\mathbf{P}\left(\sum_{j>j_0} X_j \ge 1\right) = o(1)$, and the result follows. ∎

Lemma 2 *Let $G \in \mathcal{G}(n, \frac{1}{2})$. Then a.a.s. all subsets $K \subseteq V(G)$ with $|K| \ge 10n/11$ are G-defining.*

Proof. Let $K \subseteq V(G)$ with $|K| = k \ge 10n/11$. Suppose that σ is a non-identity permutation on $V(G)$ with support R, i.e., $R = \{v \in V(G) : \sigma(v) \ne v\}$, and let $r = |R|$. Then σ induces a permutation σ^* on the set of unordered pairs $\{u, v\}$ of distinct vertices in $V(G)$, defined by $\sigma^*(\{u, v\}) = \{\sigma(u), \sigma(v)\}$. Let S be the set of unordered pairs $\{u, v\}$ not fixed (as an unordered pair) by σ^* and with at least one of u, v in K. That is,

$$S = \{\{u, v\} : u, v \in V(G), \{u, v\} \cap K \ne \emptyset, \{\sigma(u), \sigma(v)\} \ne \{u, v\}\}.$$

Let $i = |K \cap R|$. The number of unordered pairs $\{u, v\}$ with one of u, v in K and the other in R is $i(k-i) + k(r-i) + \binom{i}{2}$. All these unordered pairs are in S, except for at most $r/2$ which correspond to transpositions in σ. So we have

$$|S| \ge kr - \frac{i(i+1)}{2} - \frac{r}{2} \ge \frac{(k-2)r}{2} \tag{4}$$

using $i \le r$ and $i + 1 \le k + 1$.

The permutation σ^* induces a digraph on the set of unordered pairs of distinct vertices of G, in which there is an arc from $\{u, v\}$ to $\{u', v'\}$ if and only if $\sigma^*(\{u, v\}) = \{u', v'\}$. The sub-digraph D of this digraph induced by S consists of directed paths, and directed cycles of length at least 2. Let d be the number of such cycles, so that $d \le |S|/2$. Suppose that for all $\{u, v\} \in S$ we have

$$uv \in E(G) \text{ if and only if } \sigma^{-1}(u)\sigma^{-1}(v) \in E(G). \tag{5}$$

Suppose all edges uv of G with $\{u,v\} \notin S$ are given. Then the number of possibilities for assigning edges of G to these paths and cycles is 2^d, because the edges in paths of D are determined by (5) and previously chosen edges of G, and for each cycle of D there are two possibilities (either all the edges are

present, or none). The probability that $G \in \mathcal{G}(n, \frac{1}{2})$ satisfies (5) is thus at most $2^{d-|S|} \le 2^{-|S|/2} \le 2^{-(k-2)r/4}$ by (4).

There are $\binom{n}{k}$ subsets $K \subseteq V(G)$ with $|K| = k$ and at most $\binom{n}{r} r! < n^r$ permutations σ as above (note that $r \ge 2$ by its definition). Since $k \ge 10n/11$ we have by Stirling's formula for $i!$ that

$$\binom{n}{k} \le \binom{n}{\lceil 10n/11 \rceil} = O(n^{-1/2})(11/10^{10/11})^n < 1.36^n$$

for sufficiently large n. So the probability that $G \in \mathcal{G}(n, \frac{1}{2})$ satisifies (5) for some K and σ is at most

$$1.36^n n^r 2^{-(k-2)r/4} = 1.36^n 2^{-(k-2-4\log n)r/4} \le 1.36^n 2^{-(5/11-\varepsilon)n}$$

for all $\varepsilon > 0$ using $r \ge 2$. Since $2^{5/11} > 1.37$, the sum of this expression over all $k \ge 10n/11$ and $r \ge 2$ goes to zero, and the lemma is proved. ∎

We will use the two lemmas above in the proof of Theorem 1. We will also use the following known results, see e.g. [6, Lemma 2.1]. For a graph G and $v \in V(G)$, denote by $N_G(v)$ the set of neighbours of v in G, and $d(v) = |N_G(v)|$ the degree of v.

Lemma 3 *Let $G \in \mathcal{G}(n, \frac{1}{2})$ and $0 < \varepsilon < 1/10$. Then the following hold a.a.s.*

(a) $|d(v) - n/2| < \varepsilon n$ *for all* $v \in V(G)$*;*

(b) *for all* $u, v \in V(G)$, $\big||N_G(u) \cap N_G(v)| - n/4\big| < \varepsilon n$.

Proof of Theorem 1. Select a graph G on n vertices satisfying all of the properties in Lemmas 1 to 3 which are asserted to hold a.a.s. We prove that (1) holds for such a graph G. It then follows by Lemmas 1 to 3 that a random graph $G \in \mathcal{G}(n, \frac{1}{2})$ satisfies (1) a.a.s. Let $m \le n/(11 j_0(n)) = cn/\log n$. To prove that there is no G-trade of volume m, it suffices to show that, for any two G-decompositions

$$\mathcal{T} = \{G_i : 1 \le i \le m\}, \ \ \mathcal{T}' = \{G'_i : 1 \le i \le m\}$$

of a simple graph H, we have $G_1 = G'_i$ for some i.

Since H is simple and, by Lemma 1, G is j-non-meshing for any $j > j_0(n)$, G_1 has at most $j_0(n)$ vertices in common with each of G_i, for $i = 2, \ldots, m$. Hence there are at most $mj_0(n) \leq n/11$ vertices in $V(G_1) \cap (\bigcup_{i=2}^m V(G_i))$. Denote by K the set of all other vertices of G_1, that is, $K = V(G_1) \setminus (\bigcup_{i=2}^m V(G_i))$. Then $|K| \geq 10n/11$. Note that, by the definition of K, any edge of H incident with a vertex in K must be in G_1. Hence $d_H(v) = d_{G_1}(v)$ for all $v \in K$, and in particular $d_H(v)$ is close to $n/2$ by Lemma 3(a). For distinct vertices $u \in K$ and $v \in K$, let G'_i and G'_j be graphs in $\mathcal{T}'$ containing u and v, respectively. Then i is unique since otherwise $d_H(u)$ would be close to n by Lemma 3(a), a contradiction. Similarly, j is unique. Also, $N_H(u) \cap N_H(v) = N_{G_1}(u) \cap N_{G_1}(v)$ and so by Lemma 3(b) it follows that $|N_H(u) \cap N_H(v)|$ is close to $n/4$. On the other hand, it is easily seen that $N_H(u) \cap N_H(v) \subseteq V(G'_i) \cap V(G'_j)$. Furthermore, if $i \neq j$, we have by Lemma 1 that $|V(G'_i) \cap V(G'_j)| \leq j_0(n) << n/4$. Thus, we must have $i = j$. Since this is true for all $u, v \in K$, we conclude that $K \subseteq V(G'_i)$ for some i. Moreover, since all vertices of $G_1 - K$ have degree at least $(1/2 - \varepsilon)n$ in G_1 by the statement in Lemma 3(a), a vertex v of $H - K$ has neighbours in K if and only if v is in G_1. The only if part follows immediately by the definition of K, but it also follows by the lower bound on the degree of vertices of G_1 and the size of K. It then follows that the same statement holds for G'_i (since, as seen above, for each vertex in K, $V(G'_i)$ is the unique graph in the second G-decomposition which contains the vertex), from which it follows that $V(G_1) = V(G'_i)$.

Since $G_1 \cong G \cong G'_i$, there exists a permutation σ of $V(G_1)$ which induces an isomorphism from G_1 to G'_1. Thus, $uv \in E(G)$ if and only if $\sigma^{-1}(u)\sigma^{-1}(v) \in E(G)$ for $u, v \in K$. However, as $|K| \geq 10n/11$, K is G-defining by the statement in Lemma 2. So σ must be the identity permutation. Hence $G_1 = G'_i$ and we are done. ∎

Concluding remarks

It would be interesting to know how much the interval of values in Theorem 1 can be increased without making the theorem false. Clearly the upper end of the interval can be increased, since we made no attempt to obtain the best possible constant in Lemma 1; the difficulties with cycles in σ of length 2 will not be typical. On the other hand, the upper end must be less than n, by Lemma 3 and the above-mentioned result that $2\delta(G) \notin X(G)$. Moreover, this upper bound can be decreased a little since the minimum degree of a random graph is $n/2 - \Theta(\sqrt{n \log n})$.

References

[1] E. J. Billington, private communication.

[2] E. J. Billington and D. G. Hoffman, Trades and graphs, Graphs Combin. 17 (2001) 39–54.

[3] E. J. Billington and D. G. Hoffman, Trade spectra of complete partite graphs, Discrete Math. 250 (2002) 23–39.

[4] D. Bryant, M. Grannell, T. Griggs, B. Maenhaut, On the volume of 4-cycle trades, Graphs Combin. 19 (2003) 53–63.

[5] Y. Caro and R. Yuster, Intersecting designs, J. Combin. Theory Ser. A 89 (2000) 113-125.

[6] A. M. Frieze and B. Reed, Probabilistic analysis of algorithms, in: M. Habib, et al., (Eds.), Probabilistic Methods for Algorithmic Discrete Mathematics, Algorithms and Combinatorics 16, Springer, Berlin, 1998, pp. 36–92.

[7] B. Maenhaut, On the volume of 5-cycle trades, Graphs Combin. 17 (2001) 315–328.