Minimum partition of an independence system into independent sets

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Abstract

Given an independence system $(E, \mathcal{P})$, the Minimum Partition Problem (MPP) seeks a partition of E into the least number of independent sets. This notion provides a unifying framework for a number of combinatorial optimisation problems, including various conditional colouring problems for graphs. The smallest integer n such that E can be partitioned into n independent sets is called the $\mathcal{P}$-chromatic number of E. In this article we study MPP and the $\mathcal{P}$-chromatic number with emphasis on connections with a few other well-studied optimisation problems. In particular, we show that the $\mathcal{P}$-chromatic number of E is equal to the domination number of a split graph associated with $(E, \mathcal{P})$. With the help of this connection we give a few upper bounds on the $\mathcal{P}$-chromatic number of E in terms of some basic invariants of $(E, \mathcal{P})$.

Keywords: independence system; hereditary property; conditional colouring; conditional chromatic number; minimum set cover problem; domination number; connected domination number; clique domination number; split graph

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1 Introduction

A *set system* $(S, \mathcal{A})$ is a finite set S together with a family $\mathcal{A}$ of subsets of S. An *independence system* is a set system $(E, \mathcal{P})$ such that $Y \subseteq X, X \in \mathcal{P}$ implies $Y \in \mathcal{P}$. A subset X of E is *independent* if $X \in \mathcal{P}$ and *dependent* otherwise. Thus the set of dependent sets of $(E, \mathcal{P})$ is $\mathcal{Q} = 2^E \setminus \mathcal{P}$, and $(E, \mathcal{Q})$ is a *dependence system* in the sense that $Y \subseteq X, Y \in \mathcal{Q}$ implies $X \in \mathcal{Q}$. Throughout this article we assume without mentioning explicitly that $\mathcal{P} \neq \emptyset$ and $\mathcal{Q} \neq \emptyset$, so that $\emptyset \in \mathcal{P}$ and $E \in \mathcal{Q}$. For $X \subseteq E$, a *base* of X is a maximal (with respect to set-theoretic inclusion) independent set of $(E, \mathcal{P})$ contained in X, and a *base* of $(E, \mathcal{P})$ is a base of E. A *circuit* of X is a minimal (with respect to set-theoretic inclusion) dependent set of $(E, \mathcal{P})$ containing X, and a *circuit* of $(E, \mathcal{P})$ is a circuit of $\emptyset$. We use $\mathcal{B}$ and $\mathcal{C}$ to denote the sets of bases and circuits of $(E, \mathcal{P})$, respectively.

Independence systems have been studied extensively, especially in the context of combinatorial optimisation [2, 11, 16, 20, 23, 26, 27, 28, 30, 32, 34, 39, 40, 42, 43, 44, 46]. Much work in this area has been focused on the fundamental *maximum independent set problem*,

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which seeks an independent set with maximum weight in a given weighted independence system [12, 13, 17, 19, 26, 27, 28, 30, 35]. In this article we will investigate the following problem.

> **Minimum Partition Problem (MPP)** Given an independence system $(E, \mathcal{P})$, partition E into minimum number of independent sets of $(E, \mathcal{P})$.

This problem is **NP**-complete since it contains the ordinary graph colouring problem as a special case. Define the *chromatic number* of $(E, \mathcal{P})$, $\chi(E, \mathcal{P})$, to be the smallest integer $n \geq 1$ such that E can be partitioned into n independent sets. *To avoid redundant elements, we assume without mentioning explicitly that each $x \in E$ is contained in at least one independent set.* Under this assumption, $\{\{x\} : x \in E\}$ is a partition of E into independent sets and hence $\chi(E, \mathcal{P})$ is well-defined.

The importance of MPP lies in that it provides a unifying framework for a number of combinatorial problems. At least six other languages can be used to describe MPP. First, in the language of algebraic topology an independence system is an abstract simplicial complex [36] with independent sets as faces, and vice versa. Thus, MPP seeks a partition of the vertex set E into minimum number of faces of the simplicial complex $(E, \mathcal{P})$, and $\chi(E, \mathcal{P})$ is the minimum number of faces in such a partition. Partitions of this kind without involving minimisation were studied by Fisk in [21]. Second, MPP is equivalent to the ordinary vertex colouring problem [3, Chapter 4] for the hypergraph $(E, \mathcal{C})$, and thus $\chi(E, \mathcal{P})$ is equal to the chromatic number of $(E, \mathcal{C})$. In the literature an independence system is also called a hereditary system. In fact, we may identify $\mathcal{P}$ with the property that is possessed precisely by independent sets of $(E, \mathcal{P})$. Then $\mathcal{P}$ is a *hereditary property* associated with the subsets of E. Conversely, any hereditary property $\mathcal{P}$ associated with the subsets of a finite set E gives rise to an independence system, namely $(E, \mathcal{P})$ with $\mathcal{P}$ identified with the family of subsets of E possessing $\mathcal{P}$. It is from this viewpoint that MPP arises naturally. Clearly, a partition of E into n independent sets of $(E, \mathcal{P})$ can be identified with a *colouring* $\pi : E \to \{1, 2, \ldots, n\}$ of E such that, for each colour $i \in \{1, 2, \ldots, n\}$, the *colour class* $\pi^{-1}(i) := \{x \in E : \pi(x) = i\}$ has property $\mathcal{P}$, and vice versa. In the following we will mainly use this language of colourings, which is the third language for MPP. The colouring π is called a *$\mathcal{P}$-n-colouring* of E, or a *$\mathcal{P}$-colouring* of E if the number of colours used is unknown or is less important in the context. Thus $\chi(E, \mathcal{P})$ is the least number of colours required by a $\mathcal{P}$-colouring of E, and so is also called the *$\mathcal{P}$-chromatic number* of E. For any subset X of E, $\mathcal{P}$ induces a hereditary property associated with the subsets of X, which corresponds to the *induced independence system* $(X, \mathcal{P}_X)$ of $(E, \mathcal{P})$, where $\mathcal{P}_X$ is the family of independent sets of $(E, \mathcal{P})$ contained in X. Thus, the $\mathcal{P}_X$-chromatic number $\chi(X, \mathcal{P}_X)$ is well-defined. For brevity we will use $\chi(X, \mathcal{P})$ instead of $\chi(X, \mathcal{P}_X)$, and call it the *$\mathcal{P}$-chromatic number* of X. Similarly, a $\mathcal{P}$-colouring of X is meant a $\mathcal{P}_X$-colouring of X.

Combinatorialists and graph theorists have long been studying $\chi(E, \mathcal{P})$ for various independence systems $(E, \mathcal{P})$. In particular, enormous work has been done when $E = V(G)$ or $E(G)$ for a graph $G = (V(G), E(G))$ and $\mathcal{P}$ is a hereditary graphical property. (In these situations a subset X of $V(G)$ or $E(G)$ is said to have property $\mathcal{P}$ if the subgraph $G[X]$ of G induced by X possesses $\mathcal{P}$.) A large number of invariants for graphs can be expressed as $\chi(E, \mathcal{P})$; see e.g. [53, Table 1] and [1, 6, 18, 24, 47, 48, 52, 54] and the references cited therein. For example, if $\mathcal{P}$ is the property of being a vertex independent set, then $\chi(V(G), \mathcal{P})$ is the ordinary chromatic number of G; if $\mathcal{P}$ is the property of being an edge independent set, then $\chi(E(G), \mathcal{P})$ is the edge chromatic number of G; if $\mathcal{P}$ is the property of being a forest, then $\chi(E(G), \mathcal{P})$ is the arboricity of G. A number of results for $\chi(E(G), \mathcal{P})$ concerning individual graphical properties $\mathcal{P}$ exist in the literature. In [6, 7] Brown and Corneil studied the $\mathcal{P}$-chromatic number $\chi(E, \mathcal{P})$ of a graph G, where $E = V(G)$ and $\mathcal{P}$ is a hereditary property for graphs. For general independence systems $(E, \mathcal{P})$, Cockayne, Miller and Prins [11] proved that, if $(E, \mathcal{P})$ admits complete $\mathcal{P}$-n-colourings for $n = k, \ell$, where $k < \ell$, then it admits such a colouring for every n between k and ℓ, where a $\mathcal{P}$-n-colouring of E is said to be *complete* if the union of any two colour classes is not an independent set. In [32] Ivančo gave the dual of this result for dependence systems, and in [31] he compared $\chi(E, \mathcal{P})$ with other invariants associated with $(E, \mathcal{P})$. In [53] the author obtained interpolation theorems for $\chi(E, \mathcal{P})$ and a few other invariants for $(E, \mathcal{P})$, and in [54] he gave a sequential algorithm for $\mathcal{P}$-colouring E and obtained a Welsh-Powell type upper bound [51] for $\chi(E, \mathcal{P})$. It is well-known that an independence system $(E, \mathcal{P})$ is a matroid [50] in the case where for any $X \subseteq E$ the bases of X all have the same cardinality, which is called the *rank* [50] of X and denoted by $\rho(X)$. In this case MPP was studied by Edmonds in [15], where he proved the celebrated result $\chi(E, \mathcal{P}) = \max_{\emptyset \neq X \subseteq E} \lceil |X|/\rho(X) \rceil$ for any matroid $(E, \mathcal{P})$. For an arbitrary independence system $(E, \mathcal{P})$, it is hard to obtain even a good estimate of $\chi(E, \mathcal{P})$: the class of independence systems is too broad to sustain deep results.

This article is an attempt towards understanding $\mathcal{P}$-colourings and $\mathcal{P}$-chromatic numbers with emphasis on connections between MPP and a few other well-studied problems. In §2 we give a structure theorem for $\mathcal{P}$-n-critical sets, which is a generalisation of [6, Theorem 2.5] for conditional colourings of graphs. In §3 we observe that MPP is equivalent to the minimum set cover problems for $(E, \mathcal{P})$ and $(E, \mathcal{B})$, and that it can be reduced to the maximum independent set problem for a derived independence system; thus we have the fourth and the fifth languages for MPP. These observations lead to two greedy algorithms for MPP by invoking known algorithms for the minimum set cover and maximum independent set problems. In §4 we introduce a split graph associated with $(E, \mathcal{P})$ and prove that its domination number is equal to $\chi(E, \mathcal{P})$; hence the sixth language. With the help of this connection we then give a few upper bounds for $\chi(E, \mathcal{P})$ in terms of some basic parameters of $(E, \mathcal{P})$.

The reader is referred to [5], [50] and [37, 49] for notation and terminology concerning graphs, matroids and algorithms, respectively. *Unless stated otherwise, throughout the article* $(E, \mathcal{P})$ *is an arbitrary independence system.* Sometimes it is convenient to view $(E, \mathcal{B})$ as an incidence structure [4] with point set E, block set $\mathcal{B}$ and incidence relation the usual containment. Thus, the *flags* of $(E, \mathcal{B})$ are (x, B), where $x \in E, B \in \mathcal{B}$ with $x \in B$. For a fixed $x \in E$, let $\mathcal{B}_x$ denote the set of bases B of $(E, \mathcal{P})$ such that (x, B) is a flag. Using notation from design theory [4], define

$$v = |E|, \;\; b = |\mathcal{B}|, \;\; r(x) = |\mathcal{B}_x|, \;\; f = \sum_{x \in E} r(x) \tag{1}$$

$$\mathcal{B} = \{B_1, B_2, \ldots, B_b\}, \;\; \mathcal{B}_x = \{B_{1,x}, B_{2,x}, \ldots, B_{r(x),x}\}$$

where $x \in E$ and $B \in \mathcal{B}$. Call $\rho(E, \mathcal{P}) = \max\{|B| : B \in \mathcal{B}\}$ the *rank* of $(E, \mathcal{P})$. Note that $f = \sum_{B \in \mathcal{B}} |B|$ by double counting.

2 Structure of critical sets

A subset X of E is said to be $\mathcal{P}$-n-*critical* if $\chi(X, \mathcal{P}) = n$ but $\chi(X \setminus \{x\}, \mathcal{P}) < n$ for any $x \in X$. It is expected that investigation of $\mathcal{P}$-critical sets will help understand the $\mathcal{P}$-chromatic number, as is the case for the ordinary chromatic number [5] and various conditional chromatic numbers (see e.g. [6, 7]) of graphs. In the case of $\mathcal{P}$-colourings for graphs, a structure theorem for $\mathcal{P}$-

critical subsets of the vertex set was obtained in [6, Theorem 2.5]. In this section we extend this result to any independence system $(E, \mathcal{P})$ by using similar techniques.

Lemma 2.1 *If $\chi(E, \mathcal{P}) = n$, then E contains a $\mathcal{P}$-m-critical subset for every m with $1 \le m \le n$.*

Proof Let $\mathcal{A} = \{X \subseteq E : \chi(X, \mathcal{P}) = n\}$. Then $\mathcal{A} \neq \emptyset$ as $E \in \mathcal{A}$, and the minimal (with respect to set-theoretic inclusion) members of $\mathcal{A}$ are $\mathcal{P}$-n-critical. Thus, E contains at least one $\mathcal{P}$-n-critical subset, say X_1. If $n = 1$, we are done; otherwise define $\mathcal{A}_1 = \{X \subseteq X_1 : \chi(X, \mathcal{P}) = n-1\}$. Then $\mathcal{A}_1 \neq \emptyset$ since $X_1 \setminus \{x_1\} \in \mathcal{A}_1$ for any $x_1 \in X_1$, and the minimal members of $\mathcal{A}_1$ are $\mathcal{P}$-$(n-1)$-critical. Continuing this process one can show that E contains a $\mathcal{P}$-m-critical subset for each m between 1 and n. $\square$

Lemma 2.2 *If E is $\mathcal{P}$-n-critical, then $\chi(E \setminus X, \mathcal{P}) = n - 1$ for any $X \in \mathcal{P}$ with $X \neq \emptyset$.*

Proof Since E is $\mathcal{P}$-n-critical, we have $\chi(E \setminus X, \mathcal{P}) \le n - 1$. On the other hand, since $\chi(E, \mathcal{P}) = n$ and $X \neq \emptyset$ is an independent set, we have $\chi(E \setminus X, \mathcal{P}) \ge n - 1$ for otherwise a $\mathcal{P}$-$(n-2)$-colouring of $E \setminus X$ and the additional colour class X would form a $\mathcal{P}$-$(n-1)$-colouring of E, a contradiction. $\square$

Lemma 2.3 *Suppose $\chi(E, \mathcal{P}) = n$. Then for any $x \in E$ the following statements are equivalent:*

(i) $\chi(E \setminus \{x\}, \mathcal{P}) = n - 1$;

(ii) *x is in every subset X of E with $\chi(X, \mathcal{P}) = n$;*

(iii) *x is in every $\mathcal{P}$-n-critical subset of E.*

Proof (i) $\Longrightarrow$ (ii) Suppose $\chi(E \setminus \{x\}, \mathcal{P}) = n-1$. If $x \notin X$ for some $X \subseteq E$ with $\chi(X, \mathcal{P}) = n$, then $\chi(E \setminus \{x\}, \mathcal{P}) \ge \chi(X \setminus \{x\}, \mathcal{P}) = \chi(X, \mathcal{P}) = n$, a contradiction.

(ii) $\Longrightarrow$ (iii) Obvious.

(iii) $\Longrightarrow$ (i) Suppose x is in every $\mathcal{P}$-n-critical subset of E. If $\chi(E \setminus \{x\}, \mathcal{P}) \neq n - 1$, then $\chi(E \setminus \{x\}, \mathcal{P}) = n$ as $\chi(E, \mathcal{P}) = n$, and hence $E \setminus \{x\}$ contains a $\mathcal{P}$-n-critical set by Lemma 2.1. This contradicts our assumption and hence $\chi(E \setminus \{x\}, \mathcal{P}) = n - 1$. $\square$

The main result in this section is the following proposition.

Proposition 2.4 *For any $n \ge 2$ the following statements are equivalent:*

(i) *E is $\mathcal{P}$-n-critical;*

(ii) *for any $x \in E$ and each i with $1 \le i \le r(x)$, $E \setminus B_{i,x}$ contains a $\mathcal{P}$-$(n-1)$-critical subset $C_{i,x}$, and moreover $E \setminus \{x\} = \bigcup_{i=1}^{r(x)} C_{i,x}$;*

(iii) *for each i with $1 \le i \le b$, $E \setminus B_i$ contains a $\mathcal{P}$-$(n-1)$-critical subset C_i, and moreover $E = \bigcup_{i=1}^{b} C_i$.*

Proof (i) $\Longrightarrow$ (ii) Suppose E is $\mathcal{P}$-n-critical and let $x \in E$. From Lemmas 2.1 and 2.2, $E \setminus B_{i,x}$ contains a $\mathcal{P}$-$(n-1)$-critical subset $C_{i,x}$ for each i, $1 \le i \le r(x)$. Let $y \in E \setminus \{x\}$. Then $\chi(E \setminus \{y\}, \mathcal{P}) = n - 1$ by Lemma 2.2. Let π be a $\mathcal{P}$-$(n-1)$-colouring of $E \setminus \{y\}$, and let $Z = \{z \in E \setminus \{y\} : \pi(z) = \pi(x)\}$ be the colour class of π containing x. Clearly, $\chi(E \setminus (\{y\} \cup$

$Z), \mathcal{P}) \geq n-2$ for otherwise $\chi(E \setminus \{y\}, \mathcal{P})$ would be smaller than $n-1$. On the other hand, the restriction of π to $E \setminus (\{y\} \cup Z)$ gives rise to a $\mathcal{P}$-$(n-2)$-colouring of $E \setminus (\{y\} \cup Z)$. Thus, $\chi(E \setminus (\{y\} \cup Z), \mathcal{P}) = n-2$. Since Z is an independent set containing x, we have $Z \subseteq B_{j,x}$ for some j. Furthermore, $y \notin B_{j,x}$ for otherwise we would have $E \setminus B_{j,x} \subseteq E \setminus (\{y\} \cup Z)$ and hence $\chi(E \setminus (\{y\} \cup Z), \mathcal{P}) \geq \chi(E \setminus B_{j,x}, \mathcal{P}) = n-1$ by Lemma 2.2, a contradiction. By Lemma 2.2 we have $\chi(E \setminus Z, \mathcal{P}) = n-1$. Since $\chi((E \setminus Z) \setminus \{y\}, \mathcal{P}) = n-2$ as proved above, by Lemma 2.3 y is in every $\mathcal{P}$-$(n-1)$-critical subset of $E \setminus Z$. Thus $y \in C_{j,x}$, where $C_{j,x}$ is a $\mathcal{P}$-$(n-1)$-critical set contained in $E \setminus B_{j,x} \subseteq E \setminus Z$. Since $y \in E \setminus \{x\}$ is arbitrary, we conclude that $E \setminus \{x\} = \bigcup_{i=1}^{r(x)} C_{i,x}$.

(ii) $\Longrightarrow$ (iii) Note that $|E| \geq 2$ as $n \geq 2$. Since each $B_i \in \mathcal{B}$ is also a member of $\mathcal{B}_x$ for $x \in B_i$, it follows from (ii) that $E \setminus B_i$ contains a $\mathcal{P}$-$(n-1)$-critical subset C_i for each i, $1 \leq i \leq b$. For any $x \in E$ and each i with $1 \leq i \leq r(x)$, we have $B_{i,x} = B_j \in \mathcal{B}$ for some j. Setting $C_{i,x} = C_j$, from (ii) we have $E = \bigcup_{x \in E}(E \setminus \{x\}) = \bigcup_{x \in E}\left(\bigcup_{i=1}^{r(x)} C_{i,x}\right) \subseteq \bigcup_{i=1}^{b} C_i$, which implies $E = \bigcup_{i=1}^{b} C_i$.

(iii) $\Longrightarrow$ (i) Let us first prove $\chi(E, \mathcal{P}) \geq n$. Suppose otherwise, and let π be a $\mathcal{P}$-$(n-1)$-colouring of E and Z a colour class of π. Then $\chi(E \setminus Z, \mathcal{P}) \leq n-2$. Since Z is an independent set, it is contained in a base of $(E, \mathcal{P})$, say, $Z \subseteq B_j$. Thus, $\chi(E \setminus B_j, \mathcal{P}) \leq \chi(E \setminus Z, \mathcal{P}) \leq n-2$, which contradicts the assumption that $E \setminus B_j$ contains a $\mathcal{P}$-$(n-1)$-critical set. Therefore, we have $\chi(E, \mathcal{P}) \geq n$ and it suffices to show $\chi(E \setminus \{x\}, \mathcal{P}) \leq n-1$ for each $x \in E$.

For any $x \in E$, we claim that there exists j such that x is in every $\mathcal{P}$-$(n-1)$-critical set of $E \setminus B_j$. Suppose otherwise, then we can choose, for each i with $1 \leq i \leq b$, a $\mathcal{P}$-$(n-1)$-critical set C_i of $E \setminus B_i$ such that $x \notin C_i$ and hence $x \notin \bigcup_{i=1}^{b} C_i = E$, a contradiction. By Lemma 2.3 we have $\chi(E \setminus (\{x\} \cup B_j), \mathcal{P}) = n-2$ and therefore $\chi(E \setminus \{x\}, \mathcal{P}) \leq n-1$. □

Proposition 2.4 generalises both [6, Theorem 2.5] and [48, Proposition 8]. It also implies the following generalisation of [6, Corollary 2.6].

Corollary 2.5 *Suppose that $X \subseteq E$ is $\mathcal{P}$-n-critical. Then, for any m with $1 \leq m \leq n-1$ and any $n-m+1$ pairwise distinct vertices $x, x_1, \ldots, x_{n-m}$ of X, there exists a $\mathcal{P}$-m-critical subset of X which contains x but none of $x_1, \ldots, x_{n-m}$.*

Proof We use induction on $n-m$. By Proposition 2.4, if X is a $\mathcal{P}$-n-critical set, then for any $x_1 \in X$, $X \setminus \{x_1\}$ can be covered by $\mathcal{P}$-$(n-1)$-critical subsets. That is, any $x \in X \setminus \{x_1\}$ is in a $\mathcal{P}$-$(n-1)$-critical subset of $X \setminus \{x_1\}$. Hence the statement is true for $n-m=1$. Suppose that the statement is true for $n-m$ with $1 \leq n-m \leq n-2$. Then, for pairwise distinct vertices $x, x_1, \ldots, x_{n-m}, x_{n-m+1}$ of X, there exists a $\mathcal{P}$-m-critical subset Y of X containing x but none of $x_1, \ldots, x_{n-m}$. If $x_{n-m+1} \notin Y$, then by Lemma 2.1 there exists a $\mathcal{P}$-$(m-1)$-critical subset of Y, and any such subset contains x but none of $x_1, \ldots, x_{n-m}, x_{n-m+1}$. If $x_{n-m+1} \in Y$, then by Proposition 2.4 we can take a $\mathcal{P}$-$(m-1)$-critical subset of Y which contains x but not x_{n-m+1}. In either case we got a $\mathcal{P}$-$(m-1)$-critical subset of X containing x but none of $x_1, \ldots, x_{n-m}, x_{n-m+1}$. Hence the statement is true for $n-m+1$, and the proof is complete. □

3 Greedy algorithms

This section is largely expository. We will show that two well-known greedy algorithms for the minimum set cover and maximum independent set problems can be applied to MPP.

For a set system $(S, \mathcal{A})$, an $\mathcal{A}$*-cover* of S (or a cover of S by $\mathcal{A}$) is a subset $\mathcal{J}$ of $\mathcal{A}$ such that $\bigcup_{X \in \mathcal{J}} X = S$. Given $(S, \mathcal{A})$ and a weight function $c : \mathcal{A} \to \mathbb{R}^+$, the *minimum weight set cover problem* [49] seeks an $\mathcal{A}$-cover $\mathcal{J}$ of S with minimum weight $c(\mathcal{J})$, where $c(\mathcal{J}) = \sum_{X \in \mathcal{J}} c(X)$. A number of combinatorial optimisation problems take the form of this fundamental problem [37, 49]. In the case where each member of $\mathcal{A}$ has a unit weight, the problem is called the *minimum set cover problem* and we use $\psi(S, \mathcal{A})$ to denote the minimum cardinality of an $\mathcal{A}$-cover of S. Johnson [33] and Lovász [41] proposed a greedy algorithm for this problem, and Chvátal [9] generalised it to the weighted case. In each iteration the algorithm picks up a member X of $\mathcal{A}$ such that the "average cost" $c(X)/|X \setminus C|$ at which it covers new elements is as small as possible, where C is the set of elements of S already covered before the beginning of the iteration. Chvátal [9] proved that this algorithm is an $H(m)$-factor approximation algorithm (see also [37, 49]), where $m = \max_{X \in \mathcal{A}} |X|$ and $H(m) = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. This simple algorithm is essentially the best one can hope as explained in [37, 49].

Given $(E, \mathcal{P})$, trivially any $\mathcal{P}$-colouring of E is a $\mathcal{P}$-cover of E. Conversely, a $\mathcal{P}$-cover $\{X_1, X_2, \ldots, X_n\}$ of E gives rise to a $\mathcal{P}$-colouring $\{E_1, E_2, \ldots, E_n\}$ of E, where $E_1 = X_1$ and $E_i = X_i \setminus (\bigcup_{j=1}^{i-1} X_j)$ for $2 \le i \le n$. (It may happen that $E_i = \emptyset$ for some i, and hence we obtain a $\mathcal{P}$-colouring of E using at most n colours.) Thus, $\chi(E, \mathcal{P})$ is equal to the minimum number of independent sets in a $\mathcal{P}$-cover of E. Moreover, a solution to MPP for $(E, \mathcal{P})$ gives rise to a solution to the minimum set cover problem for $(E, \mathcal{P})$, and vice versa. Therefore, the two problems are equivalent. Similarly, since any independent set of $(E, \mathcal{P})$ can be extended to a base of $(E, \mathcal{P})$, MPP is equivalent to the minimum set cover problem for $(E, \mathcal{B})$.

Lemma 3.1 *MPP for $(E, \mathcal{P})$ is equivalent to the minimum set cover problem for $(E, \mathcal{P})$, which in turn is equivalent to the minimum set cover problem for $(E, \mathcal{B})$. Thus,*

$$\chi(E, \mathcal{P}) = \psi(E, \mathcal{P}) = \psi(E, \mathcal{B}). \tag{2}$$

This lemma together with its justification enables us to translate the greedy algorithm [37, 49] for the minimum set cover problem into the following algorithm for MPP.

Algorithm 3.2 (Greedy Set Cover Algorithm)
Input: An independence system $(E, \mathcal{P})$.
Output: A $\mathcal{P}$-colouring of E.

(1) Set $\mathcal{J} := \emptyset$ and $C := \emptyset$ initially.

(2) **While** $C \neq E$ **do**

Choose $X \in \mathcal{B} \setminus \mathcal{J}$ such that $|X \setminus C|$ is as large as possible;
set $\mathcal{J} := \mathcal{J} \cup \{X\}$ and $C := C \cup X$.

(3) Let $\mathcal{J} = \{X_1, X_2, \ldots, X_n\}$ be obtained from step (2) when it terminates (that is, $C = E$), where the subsets X_i are the bases added to $\mathcal{J}$ sequentially.

Set $E_i = X_i \setminus (\bigcup_{j=1}^{i-1} X_j)$, $1 \le i \le n$;
output the $\mathcal{P}$-colouring $\{E_1, E_2, \ldots, E_n\}$.

From the discussion above, Algorithm 3.2 is an $H(\rho)$-factor approximation algorithm for MPP with running time $O(vb)$, where $\rho = \rho(E, \mathcal{P})$. It is essentially the best one can hope for an arbitrary independence system.

The second algorithm that we will present is based on the well-known greedy algorithm for the maximum independent set problem. To this end we will reduce further the minimum set cover problem for $(E, \mathcal{B})$ to the maximum (cardinality) independent set problem for the independence system consisting of those subsets of $\mathcal{B}$ which are not $\mathcal{B}$-covers of E. Denote by $\mathbf{J}$ the set of such $\mathcal{B}$-covers of E. It is clear that, for any $\mathcal{J} \in \mathbf{J}$ and $\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{B}$, $\mathcal{J}'$ is also a $\mathcal{B}$-cover of E. Hence $(\mathcal{B}, \mathbf{J})$ is a dependence system, or equivalently $(\mathcal{B}, 2^{\mathcal{B}} \setminus \mathbf{J})$ is an independence system. The minimum set cover problem for $(E, \mathcal{B})$ is then equivalent to the problem of finding a dependent set of $(\mathcal{B}, \mathbf{J})$ with minimum cardinality, or the problem of finding an independent set of $(\mathcal{B}, 2^{\mathcal{B}} \setminus \mathbf{J})$ with maximum cardinality. Thus, we may apply the "best-in" greedy algorithm [37] to $(\mathcal{B}, 2^{\mathcal{B}} \setminus \mathbf{J})$ or equivalently the "worst-out" greedy algorithm [37] to $(\mathcal{B}, \mathbf{J})$, and this gives the following algorithm.

Algorithm 3.3 (Worst-out Greedy Algorithm)
Input: An independence system $(E, \mathcal{P})$.
Output: A $\mathcal{P}$-colouring of E.

(0) Order the bases $B_1, B_2, \ldots, B_b$ of $(E, \mathcal{P})$ such that $|B_1| \leq |B_2| \leq \cdots \leq |B_b|$.
(1) Set $\mathcal{J} := \{B_1, B_2, \ldots, B_b\}$ initially.
(2) For each $i = 1, 2, \ldots, b$,
    if $\mathcal{J} \setminus \{B_i\} \in \mathbf{J}$, then set $\mathcal{J} := \mathcal{J} \setminus \{B_i\}$;
    otherwise output $\mathcal{J}$ and go to step (3).
(3) Let $\mathcal{J} = \{B_{i_1}, B_{i_2}, \ldots, B_{i_n}\}$ be obtained from step (2) when it terminates.
    Set $E_t = B_{i_t} \setminus (\bigcup_{j=1}^{t-1} B_{i_j})$, $1 \leq t \leq n$;
    output the $\mathcal{P}$-colouring $\{E_1, E_2, \ldots, E_n\}$.

This algorithm is genuinely simple and one may come to it without bothering $(\mathcal{B}, \mathbf{J})$ or $(\mathcal{B}, 2^{\mathcal{B}} \setminus \mathbf{J})$. The nontrivial thing is to analyse its performance, and for this we invoke a recent result of [30]. Note that, for $\mathcal{A} \subseteq \mathcal{B}$, a circuit of $\mathcal{A}$ in $(\mathcal{B}, 2^{\mathcal{B}} \setminus \mathbf{J})$ is a minimal (with respect to inclusion) $\mathcal{B}$-cover of E containing $\mathcal{A}$. Let $g_u(\mathcal{A})$ and $g_l(\mathcal{A})$ be, respectively, the maximum and minimum cardinalities of such a circuit of $\mathcal{A}$. Define

$$c(E, \mathcal{P}) = \max_{\mathcal{A} \subseteq \mathcal{B}} \frac{g_u(\mathcal{A}) - |\mathcal{A}|}{g_l(\mathcal{A}) - |\mathcal{A}|}$$

which is the *dependence curvature* of $(\mathcal{B}, 2^{\mathcal{B}} \setminus \mathbf{J})$ in terms of [30, (5)]. Lemma 3.1 and [30, Theorem 6] together imply the following result.

Proposition 3.4 *Algorithm 3.3 is a $c(E, \mathcal{P})$-factor approximation algorithm for MPP.*

4 $\mathcal{P}$-colouring and domination

In this section we present a connection between $\chi(E, \mathcal{P})$ and the domination number of a graph associated with $(E, \mathcal{P})$, and then give upper bounds for $\chi(E, \mathcal{P})$ by using this connection.

Denote $E = \{x_1, \ldots, x_v\}$ and $\mathcal{B} = \{B_1, \ldots, B_b\}$, where $v = |E|$ and $b = |\mathcal{B}|$. Let $E^* = \{x_1^*, \ldots, x_b^*\}$ be a set without common elements with E. Define $G(E, \mathcal{P})$ to be the graph with vertex set $V(G(E, \mathcal{P})) = E \cup E^*$ and edge set $E(G(E, \mathcal{P})) = \{x_i x_j^* : x_i \in B_j, 1 \leq i \leq v, 1 \leq j \leq b\} \cup \{x_i^* x_j^* : 1 \leq i \neq j \leq b\}$. In other words, $G(E, \mathcal{P})$ is the graph obtained from the incidence graph of $(E, \mathcal{B})$ (as an incidence structure) by adding an edge between any two members of $\mathcal{B}$. Then $G(E, \mathcal{P})$ is a split graph since E is an independent set and E^* is a clique of $G(E, \mathcal{P})$. (A graph is called a *split graph* if its vertex set can be partitioned into an independent set and a clique.) Note that $G(E,\mathcal{P})$ is connected since each $x_i \in E$ is contained in at least one base of $(E,\mathcal{P})$.

The following definitions are standard in domination graph theory [29]. A *dominating set* of a graph $G = (V(G), E(G))$ is a subset D of $V(G)$ such that each vertex outside D is adjacent to at least one vertex in D. A *connected dominating set* is a dominating set D such that the subgraph $G[D]$ induced by D is connected. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and the *connected domination number* $\gamma_c(G)$ is defined similarly when G is connected. A dominating set with cardinality $\gamma(G)$ is called a *minimum dominating set* of G. A dominating set which is also a clique is called a *dominating clique* [14, 38]. The *clique domination number* [14, 38] of G, $\gamma_{cl}(G)$, is the minimum cardinality of a dominating clique of G if such a clique exists, and is defined to be ∞ otherwise. Clearly, we have

$$\gamma(G) \le \gamma_c(G) \le \gamma_{cl}(G). \tag{3}$$

Proposition 4.1 *Let $G(E,\mathcal{P})$ be the graph defined above. Then*

$$\chi(E,\mathcal{P}) = \gamma(G(E,\mathcal{P})) = \gamma_c(G(E,\mathcal{P})) = \gamma_{cl}(G(E,\mathcal{P})).$$

Proof There is a one-to-one correspondence between the set of covers of E by $\mathcal{B}$ and the set of dominating sets of $G(E,\mathcal{P})$ contained in E^*, namely a cover $\{B_{i_1}, \ldots, B_{i_n}\}$ of E by $\mathcal{B}$ gives rise to the dominating set $\{x^*_{i_1}, \ldots, x^*_{i_n}\} \subseteq E^*$ of $G(E,\mathcal{P})$, and vice versa. Taking a minimum $\mathcal{B}$-cover $\{B_{i_1}, \ldots, B_{i_n}\}$ of E, it follows that $\gamma(G(E,\mathcal{P})) \le \psi(E,\mathcal{B})$. Thus, since $\chi(E,\mathcal{P}) = \psi(E,\mathcal{B})$ by (2), to prove $\chi(E,\mathcal{P}) = \gamma(G(E,\mathcal{P}))$ it suffices to show that there exists a minimum dominating set of $G(E,\mathcal{P})$ which is contained entirely in E^*.

Let us first prove that there exists a minimum dominating set D of $G(E,\mathcal{P})$ such that $D \cap E^* \neq \emptyset$. To this end let $D = X \cup X^*$ be any minimum dominating set of $G(E,\mathcal{P})$, where $X \subseteq E$ and $X^* \subseteq E^*$. Since we are done in the case where $X^* \neq \emptyset$, let us assume in the following that $X^* = \emptyset$. Then $D = X = E$ since E is an independent set of $G(E,\mathcal{P})$. If $|B_j| \ge 2$ for each j, $1 \le j \le b$, then we take, say $x^*_1 \in E^*$ and $x \in B_1$. Since each x^*_i for $i \ge 2$ is adjacent to at least one vertex in $E \setminus \{x\}$, it follows that $(E \setminus \{x\}) \cup \{x^*_1\}$ is a dominating set of $G(E,\mathcal{P})$ which intersects E^* at x^*_1, and it is minimum since it has the same cardinality as D. On the other hand, if there exists j with $|B_j| = 1$, let, say, $|B_1| = \cdots = |B_t| = 1$ and $|B_j| > 1$ for $j \ge t+1$, where $1 \le t \le b$, and let $Y = \bigcup_{i=1}^{t} B_i$. Then $|Y| = t$ and, since B_i for $1 \le i \le t$ are bases of $(E,\mathcal{P})$, no vertex in Y can be contained in any B_j for $j \ge t+1$. Thus, $(E \setminus Y) \cup \{x^*_1, \ldots, x^*_t\}$ is a minimum dominating set of $G(E,\mathcal{P})$ which has nonempty intersection with E^*, as required.

From the result above there exists a minimum dominating set $D = X \cup X^*$ of $G(E,\mathcal{P})$ with $X \subseteq E$ and $\emptyset \neq X^* \subseteq E^*$. Note that each $x_i \in E \setminus X$ is adjacent to at least one $x^*_j \in X^*$. If there exists $x_i \in X$ which is adjacent to some $x^*_j \in X^*$, then we can delete x_i from D and thus obtain a smaller dominating set, a contradiction. So no vertex $x_i \in X$ is adjacent to any $x^*_j \in X^*$. Thus, each $x_i \in X$ must be contained in some $B_{i'}$ with $x^*_{i'} \in E^* \setminus X^*$. Replacing each $x_i \in X$ by its corresponding $x^*_{i'}$, we obtain a new minimum dominating set which is contained entirely in E^*. Therefore, we have proved that $\chi(E,\mathcal{P}) = \gamma(G(E,\mathcal{P}))$. Note that this new minimum domination set induces a clique and hence is a dominating clique of $G(E,\mathcal{P})$. Thus, $\gamma_{cl}(G(E,\mathcal{P})) \le \gamma(G(E,\mathcal{P}))$. This together with (3) gives $\gamma(G(E,\mathcal{P})) = \gamma_c(G(E,\mathcal{P})) = \gamma_{cl}(G(E,\mathcal{P}))$. □

It was observed in [14] that $\gamma(G) = \gamma_c(G) = \gamma_{cl}(G)$ for any connected split graph G. Thus, Proposition 4.1 says essentially that $\chi(E,\mathcal{P}) = \gamma(G(E,\mathcal{P}))$. This link enables us to obtain results for $\chi(E,\mathcal{P})$ by applying known results for the domination, connected domination or clique domination number of a split graph. As examples we now give several upper bounds for $\chi(E,\mathcal{P})$ by using Proposition 4.1 and known upper bounds for the domination number. For $B \in \mathcal{B}$, let

$$\bar{r}(B) = \sum_{x \in E \setminus B} r(x).$$

Note that if there exists a base with cardinality one, say, $B_1 = \{x_i\}$, then any minimum dominating set of $G(E,\mathcal{P})$ contains exactly one of x_i and x_1^*, and moreover $\chi(E,\mathcal{P}) = \chi(E \setminus \{x_i\}, \mathcal{P}) + 1$. *Therefore, without loss of generality we may assume $|B_j| \geq 2$, $1 \leq j \leq b$ in the following proposition.* Denote by $p(G), q(G), \delta(G)$ and $\Delta(G)$ respectively the number of vertices, the number of edges, the minimum degree and the maximum degree of a graph G.

Proposition 4.2 *Suppose that all bases of $(E,\mathcal{P})$ have cardinality at least two. Let $v\,(\geq 3), b, r(x)$ be as defined in (1), and let $\bar{r}(B)$ be as above. Let $\delta = \min_{x \in E} r(x)$, $\rho = \rho(E,\mathcal{P})$, and τ be the maximum cardinality of a subset of E which has at most one common element with each base of $(E,\mathcal{P})$. Then each of the following (i)-(vi) is an upper bound for $\chi(E,\mathcal{P})$.*

(i) $v + b + 1 - \max_{B \in \mathcal{B}} \left\{ \frac{|B|+b-1}{2} + \sqrt{\left(\frac{|B|+b-1}{2}\right)^2 + \bar{r}(B) + 1} \right\}$;

(ii) $(v - \rho)\left(1 - \frac{\delta+1}{v+b-1}\right) + 2$;

(iii) $\frac{1}{2}\left(v - \rho + \frac{\rho+b-1}{\delta}\right) + 1$;

(iv) $\frac{1}{2}\left(v + b - \tau(\delta - 2)\right)$;

(v) $\left(1 - \prod_{i=1}^{\delta+1} \frac{i\delta}{i\delta+1}\right)(v + b)$;

(vi) $\frac{1}{2}(v + b + 1 - \delta)$.

Proof Since, by our assumption, $|B| \geq 2$ for all $B \in \mathcal{B}$, we have $f = \sum_{B \in \mathcal{B}} |B| \geq 2b$. Let $E = \{x_1, \ldots, x_v\}$ and $\mathcal{B} = \{B_1, \ldots, B_b\}$ as before. By the definition of $G(E,\mathcal{P})$, the degree in $G(E,\mathcal{P})$ of $x_i \in E$ is

$$d(x_i) = r(x_i),\ 1 \leq i \leq v \tag{4}$$

and the degree in $G(E,\mathcal{P})$ of $x_j^* \in E^*$ is

$$d(x_j^*) = |B_j| + b - 1,\ 1 \leq j \leq b. \tag{5}$$

Since $r(x_i) \leq |B_j| + b - 1$ for all pairs i, j, we have $\delta(G(E,\mathcal{P})) = \delta$ and $\Delta(G(E,\mathcal{P})) = \rho + b - 1$. Note that $G(E,\mathcal{P})$ has $v + b$ vertices and $f + b(b-1)/2$ edges. Since $\chi(E,\mathcal{P}) = \gamma(G(E,\mathcal{P}))$ by Proposition 4.1, it suffices to prove the desired bounds for $\gamma(G(E,\mathcal{P}))$. This will be done by straightforward applications of certain upper bounds for the domination number of a graph.

(i) It was proved by Chen and Zhou [8, Corollary 6] that

$$\gamma(G) \leq p(G) + 1 - \frac{1}{2}\left\{ d(x) + \sqrt{(d(x))^2 + 8q(G) + 4 - 4d(\overline{N}(x))} \right\} \tag{6}$$

for any graph G and any vertex x of G, where $d(\overline{N}(x))$ is the sum of the degrees of x and the neighbours of x. For the vertex $x_j^* \in E^*$ of $G(E,\mathcal{P})$, we have $d(x_j^*) = |B_j| + b - 1$ by (5) and

$d(\overline{N}(x_j^*)) = \sum_{x:x\in B_j} r(x)+\sum_{B\in\mathcal{B}}(b+|B|-1) = (f-\bar{r}(B_j))+(b(b-1)+f) = 2f+b(b-1)-\bar{r}(B_j)$ by the definition of $G(E,\mathcal{P})$. Plugging these into (6) we obtain

$$\chi(E,\mathcal{P}) = \gamma(G(E,\mathcal{P})) \le v+b+1-\left\{\frac{|B_j|+b-1}{2}+\sqrt{\left(\frac{|B_j|+b-1}{2}\right)^2+\bar{r}(B_j)+1}\right\}.$$

Since this holds for all x_j^*, (i) follows immediately.

(ii) Payan [45] proved that $\gamma(G) \le (p(G)-1-\Delta(G))(p(G)-2-\delta(G))/(p(G)-1)+2$ for any graph G. Applying this to $G(E,\mathcal{P})$, the bound (ii) follows by using the data of $G(E,\mathcal{P})$ given at the beginning of the proof.

(iii) Flach and Volkmann [22] proved that $\gamma(G) \le \{p(G)+1-\Delta(G)(\delta(G)-1)/\delta(G)\}/2$ for any graph G. Applying this to $G(E,\mathcal{P})$ and using the data of $G(E,\mathcal{P})$, we have

$$\begin{aligned}\chi(E,\mathcal{P}) &= \gamma(G(E,\mathcal{P})) \\ &\le \frac{1}{2}\{v+b+1-(\rho+b-1)(1-(1/\delta))\} \\ &= \frac{1}{2}\{v-\rho+(\rho+b-1)/\delta\}+1.\end{aligned}$$

(iv) This follows from another inequality of Flach and Volkmann [22], which asserts that $\gamma(G) \le \{p(G)-(\delta(G)-2)\underline{\alpha}(G)\}/2$ for any graph G, where $\underline{\alpha}(G)$ is the maximum cardinality of an independent set of G such that each vertex of G is adjacent to at most one vertex in the set. One can check that $\underline{\alpha}(G(E,\mathcal{P})) \ge \tau$, and hence the result follows.

(v) This bound follows from Proposition 4.1 and a result in [10] which asserts that $\gamma(G) \le (1-S_\delta)p(G)$ for any graph G, where $\delta = \delta(G)$ and $S_\delta = \prod_{i=1}^{\delta+1}(i\delta/(i\delta+1))$.

(vi) It is known that $\gamma(G) \le (p(G)-\delta(G)+1)/2$ for any connected graph G which is not isomorphic to the cycle C_4 of length 4 (communicated in [45] and proved in [22]). Applying this to $G(E,\mathcal{P})$ and noting that $G(E,\mathcal{P}) \not\cong C_4$ as $v \ge 3$, we obtain (vi) directly. □

The bounds in Proposition 4.2 are valid for any independence system $(E,\mathcal{P})$. Because of this generality it is unrealistic to expect that they are good in all situations, although they do produce good upper bounds for some independence systems. Similar to any upper bound for the domination number of a graph, each bound in Proposition 4.2 has its advantages and drawbacks. As a benchmark let us consider the following simple bound:

$$\chi(E,\mathcal{P}) \le v-\rho+1, \tag{7}$$

which is due to the fact that $\{B\}\cup\{\{x\} : x \in E\setminus B\}$ is a $\mathcal{P}$-colouring of E for any base B of $(E,\mathcal{P})$ with maximum cardinality. One can easily see that (i) is always no worse than (7). The bound (ii) is better than (7) if and only if $\delta > (b+\rho-1)/(v-\rho)$. The same statement is true for (iii) as well. The bound (iv) is better than (7) if and only if $\tau(\delta-2) > b-v+2(\rho-1)$, and (v) is better than (7) if and only if $\prod_{i=1}^{\delta+1}(i\delta/(i\delta+1)) > (b+\rho-1)/(b+v)$. In general, (vi) is weak and it is better than (7) if and only if $\delta > b-v+2\rho-1$. In fact, (vi) is inferior to (iii) when $\delta \ge 2$, and is slightly better than (iii) when $\delta = 1$. Similar to (i)-(vi), one can derive other upper bounds for $\chi(E,\mathcal{P})$ from known bounds for domination number.

It can be easily verified that not every split graph is of the form $G(E,\mathcal{P})$. In fact, since no base of $(E,\mathcal{P})$ is contained in any other base of $(E,\mathcal{P})$, $G(E,\mathcal{P})$ has the property that the subsets $N(x_j^*)\cap E$ of E, for $j = 1,2,\ldots,b$, are mutually non-inclusive, where $N(x_j^*)$ is the neighbourhood of x_j^* in $G(E,\mathcal{P})$. Call a split graph with this property a *strong split graph.* More explicitly, a split graph G with vertex-set partitioned into an independent set V and a clique V^* is said to be a strong split graph if it satisfies the following condition (where $N_G(x^*)$ is the neighbourhood of x^* in G):

$$N_G(x^*) \cap V \not\subseteq N_G(y^*) \cap V, \text{ for distinct } x^*, y^* \in V^*. \tag{8}$$

Proposition 4.3 *A graph is a strong split graph if and only if it is isomorphic to $G(E,\mathcal{P})$ for some independence system $(E,\mathcal{P})$.*

Proof That $G(E,\mathcal{P})$ is a strong split graph has been justified above. Conversely, let G be a strong slit graph with partition $\{V, V^*\}$. For each $x^* \in V^*$, define $B(x^*) = \{x \in V : x \text{ is adjacent to } x^* \text{ in } G\}$. Since G is a strong split graph, by (8) we have $B(x^*) \not\subseteq B(y^*)$ for distinct $x^*, y^* \in V^*$. Thus, $\mathcal{B} = \{B(x^*) : x^* \in V^*\}$ defines an independence system, namely $(V,\mathcal{P})$ where $\mathcal{P} = \{X \subseteq V : X \subseteq B(x^*) \text{ for some } x^* \text{ in } V^*\}$, and $\mathcal{B}$ is the set of bases of this independence system. Clearly, we have $G \cong G(E,\mathcal{P})$ via the identification of x^* and $B(x^*)$ for each x^*. $\square$

5 Concluding remarks

In view of Propositions 4.1 and 4.3, investigations of the clique domination number of a strong split graph will benefit our understanding to the chromatic number of an independence system. From a computational point of view, Proposition 4.1 together with the construction of $G(E,\mathcal{P})$ can be taken as a transformation from MPP to the dominating clique problem for strong split graphs. (The *dominating clique problem* [38] is the problem of determining a dominating clique with minimum cardinality.) Unfortunately, even for split graphs this problem is **NP**-complete [14]. Moreover, the transformation itself is not necessarily polynomial since the construction of $G(E,\mathcal{P})$ involves all bases of $(E,\mathcal{P})$ and the problem of generating them is **NP**-hard [40]. Nevertheless, for some special types of independence systems [40] it is possible to generate all bases in polynomial time. (Here and in the following when we say an algorithm for $(E,\mathcal{P})$ is *polynomial*, we mean it is polynomial in v and b.) In general, if $\mathbf{F}$ is a class of independence systems $(E,\mathcal{P})$ such that generating all bases is achievable in polynomial time, then the transformation above is polynomial for $(E,\mathcal{P}) \in \mathbf{F}$ since the remaining time needed to construct $G(E,\mathcal{P})$ is $O(vb)$. Thus, Proposition 4.1 implies that MPP for $(E,\mathcal{P}) \in \mathbf{F}$ can be reduced in polynomial time to the dominating clique problem for $G(E,\mathcal{P})$, and so any polynomial-time approximation algorithm for the latter for strong split graphs would imply a polynomial-time approximation algorithm for the former for $(E,\mathcal{P}) \in \mathbf{F}$.

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