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On a class of finite symmetric graphs

Sanming Zhou [1]

Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

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Dedicated to Professor Yixun Lin with best wishes on the occasion of his 70th birthday

Abstract

Let Γ be a G-symmetric graph, and let $\mathcal{B}$ be a nontrivial G-invariant partition of the vertex set of Γ. This paper aims to characterize (Γ, G) under the conditions that the quotient graph $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and the induced subgraph between two adjacent blocks is $2 \cdot K_2$ or $K_{2,2}$. The results answer two questions about the relationship between Γ and $\Gamma_{\mathcal{B}}$ for this class of graphs.

1. Introduction

The purpose of this paper is to answer two questions [8] regarding 2-arc transitivity of quotient graphs for a class of finite symmetric graphs.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite graph. For an integer $s \geq 1$, an *s-arc* of Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices of Γ such that α_i, α_{i+1} are adjacent for $i = 0, \ldots, s-1$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. We will use $\mathtt{Arc}_s(\Gamma)$ to denote the set of s-arcs of Γ, and $\mathtt{Arc}(\Gamma)$ in place of $\mathtt{Arc}_1(\Gamma)$. Γ is said to admit a group G as a group of automorphisms if G acts on $V(\Gamma)$ and preserves the adjacency of Γ, that is, for any $\alpha, \beta \in V(\Gamma)$ and $g \in G$, α and β are adjacent in Γ if and only if α^g and β^g are adjacent in Γ. In the case where G is transitive on $V(\Gamma)$ and, under the induced action, transitive on $\mathtt{Arc}_s(\Gamma)$, Γ is said to be *(G, s)-arc transitive*. A (G, s)-arc transitive graph Γ is called *(G, s)-arc regular* if G is regular on $\mathtt{Arc}_s(\Gamma)$, that is, only the identity element of G can fix an s-arc of Γ. A 1-arc is usually called an *arc*, and a $(G, 1)$-arc transitive graph is called a *G-symmetric graph*. Since Tutte's seminal paper [16], symmetric graphs have been studied intensively; see [14,15] for a contemporary treatment of the subject.

E-mail address: smzhou@ms.unimelb.edu.au.

[1] Tel.: +61 3 8344 3453; fax: +61 3 8344 4599.

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Often a G-symmetric graph Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of $V(\Gamma)$ such that $B^g := \{\alpha^g : \alpha \in B\} \in \mathcal{B}$ and $1 < |B| < |V(\Gamma)|$ for any $B \in \mathcal{B}$ and $g \in G$. In this case Γ is called an *imprimitive G-symmetric graph*. The *quotient graph* of Γ with respect to $\mathcal{B}$, $\Gamma_{\mathcal{B}}$, is then defined to have vertex set $\mathcal{B}$ such that $B, C \in \mathcal{B}$ are adjacent if and only if there exists at least one edge of Γ between B and C. As usual we assume without mentioning explicitly that $\Gamma_{\mathcal{B}}$ contains at least one edge, so that each block of $\mathcal{B}$ is an independent set of Γ (e.g. [1, Proposition 22.1]). For blocks B, C of $\mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$, let $\Gamma[B, C]$ denote the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Here we define $\Gamma(D) := \bigcup_{\alpha \in D} \Gamma(\alpha)$ for each $D \in \mathcal{B}$, where $\Gamma(\alpha)$ is the neighbourhood of α in Γ. Γ is called [1] a *$|B|$-fold cover* of $\Gamma_{\mathcal{B}}$ if $\Gamma[B, C] \cong |B| \cdot K_2$ is a perfect matching between B and C. Similarly, if $\Gamma[B, C] \cong (|B|-1) \cdot K_2$, then Γ is called [20] an *almost cover* of $\Gamma_{\mathcal{B}}$. The reader is referred to [6,17–21] for recent results on imprimitive symmetric graphs.

In this paper we focus on the case where $|\Gamma(C) \cap B| = 2$ for adjacent $B, C \in \mathcal{B}$, that is, $\Gamma[B, C] \cong 2 \cdot K_2$ (two independent edges) or $K_{2,2}$ (complete bipartite graph with two vertices in each part). In this case we may associate a multigraph $[B]$ with each $B \in \mathcal{B}$, which is defined [6, Section 6] to have vertex set B and an edge joining the two vertices of $\Gamma(C) \cap B$ for all $C \in \Gamma_{\mathcal{B}}(B)$, where $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$. Denote by G_B the setwise stabilizer of B in G. A *near n-gonal graph* [13] is a connected graph Σ of girth at least 4 together with a set $\mathcal{E}$ of n-cycles of Σ such that each 2-arc of Σ is contained in a unique member of $\mathcal{E}$; we also say that Σ is a near n-gonal graph with respect to $\mathcal{E}$. The following theorem summarizes the main results of this paper.

Theorem 1.1. *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a G-symmetric graph. Suppose that $V(\Gamma)$ admits a G-invariant partition $\mathcal{B}$ of block size at least three such that $\Gamma_{\mathcal{B}}$ is connected, and for any two adjacent blocks $B, C \in \mathcal{B}$, $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if $[B] \cong K_3$ or $(|B|/2) \cdot K_2$, and G_B is 2-transitive on the edge set of $[B]$. Moreover, if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, then one of the following holds:*

(a) *$\Gamma \cong s \cdot C_t$ with $s, t \geq 3$, and $\Gamma_{\mathcal{B}} \cong K_4$ or $\Gamma_{\mathcal{B}}$ is a trivalent near n-gonal graph for some integer $n \geq 4$;*
(b) *$\Gamma[B, C] \cong K_{2,2}$, $\Gamma_{\mathcal{B}}$ is trivalent $(G, 3)$-arc transitive, Γ is 4-valent, connected and not $(G, 2)$-arc transitive;*
(c) *$\Gamma \cong 2q \cdot K_2$ or $q \cdot K_{2,2}$ for some integer $q \geq 3$.*

Thus $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive when $\mathtt{val}(\Gamma) \geq 5$.

The research in this paper was motivated by the following questions [8] for an imprimitive G-symmetric graph $(\Gamma, \mathcal{B})$.

(1) Under what circumstances is $\Gamma_{\mathcal{B}}$ $(G, 2)$-arc transitive, and what information can we obtain about Γ if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive?
(2) Assuming that Γ is $(G, 2)$-arc transitive, under what conditions is $\Gamma_{\mathcal{B}}$ also $(G, 2)$-arc transitive?

Theorem 1.1 answers Question (1) for the class of G-symmetric graphs $(\Gamma, \mathcal{B})$ such that $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$. We will also answer Question (2) for the same class (see Theorem 3.4). The full version of Theorem 1.1 with more technical details will be given in Theorem 3.1. A study of G-symmetric graphs $(\Gamma, \mathcal{B})$ with $|\Gamma(C) \cap B| = 2$ for adjacent $B, C \in \mathcal{B}$ was conducted in [6, Section 6] under the additional assumption that Γ is G-locally primitive. In the present paper we do not require Γ to be G-locally primitive. (A G-symmetric graph Γ is called *G-locally* *primitive* or G-*locally imprimitive* depending on whether G_α is primitive or imprimitive on $\Gamma(\alpha)$, where G_α is the stabilizer of α in G.)

The two questions above have been answered for the class [10] of imprimitive symmetric graphs with $|\Gamma(C) \cap B| = |B| - 1 \geq 2$, and the one [8] with $|\Gamma(C) \cap B| = |B| - 2 \geq 1$. In [11] symmetric graphs with 2-arc transitive quotients were studied and their connections with 2-point transitive block designs were explored. Relationships between a symmetric graph and a quotient graph of it in the context of Questions (1) and (2) often play an important role in studying 2-arc transitive graphs; see [9,12,14,15] for example.

2. Preliminaries

We follow the notation and terminology in [5] for permutation groups. Let G be a group acting on a set Ω, and let $X \subseteq \Omega$. As usual we use G_X and $G_{(X)}$ to denote the setwise and pointwise stabilizers of X in G, respectively. For a group G acting on two sets Ω_1 and Ω_2, if there exists a bijection $\psi : \Omega_1 \to \Omega_2$ such that $\psi(\alpha^g) = (\psi(\alpha))^g$ for all $\alpha \in \Omega_1$ and $g \in G$, then the actions of G on Ω_1 and Ω_2 are said to be *permutationally equivalent*. By a graph we mean a *simple* graph (i.e. without loops and multiple edges), whereas a *multigraph* means that multiple edges may exist. We use $\mathtt{val}(\Gamma)$ to denote the *valency* of a graph Γ. The union of n vertex-disjoint copies of Γ is denoted by $n \cdot \Gamma$. For two graphs Γ and Σ, the *lexicographic product* of Γ by Σ, $\Gamma[\Sigma]$, is the graph with vertex set $V(\Gamma) \times V(\Sigma)$ such that (α, β), (γ, δ) are adjacent if and only if either α, γ are adjacent in Γ, or $\alpha = \gamma$ and β, δ are adjacent in Σ.

Let $(\Gamma, \mathcal{B})$ be an imprimitive G-symmetric graph with $|\Gamma(C) \cap B| = 2$ for adjacent blocks $B, C \in \mathcal{B}$. Since Γ is G-symmetric, the multigraph $[B]$ defined in the introduction is independent of the choice of B up to isomorphism. For adjacent vertices α, β of $[B]$, define

$$\langle \alpha, \beta \rangle := \{C \in \Gamma_{\mathcal{B}}(B) : \Gamma(C) \cap B = \{\alpha, \beta\}\}.$$

The cardinality m of $\langle \alpha, \beta \rangle$ is independent of the choice of adjacent α and β, and is called the *multiplicity* of $[B]$. Let

$$\mathcal{M}(B) := \{\langle \alpha, \beta \rangle : \alpha, \beta \in B \text{ are adjacent in } [B]\}.$$

The following two lemmas are straightforward, and hence we omit their proofs.

Lemma 2.1. *Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent blocks $B, C \in \mathcal{B}$. Then*

(a) $\mathtt{val}(\Gamma) = \mathtt{val}([B])$ *or* $2\,\mathtt{val}([B])$*, accordingly;*
(b) $\mathtt{val}(\Gamma_{\mathcal{B}})$ *is equal to the number of edges of* $[B]$ *and thus is a multiple of* m*;*
(c) $\mathtt{val}([B]) = |\{C \in \mathcal{B} : \Gamma(\alpha) \cap C \neq \emptyset\}|$ *(where α is a fixed vertex of Γ), a multiple of m, and the valency of the underlying simple graph of* $[B]$ *is* $\mathtt{val}([B])/m$.

Lemma 2.2. *Let $(\Gamma, \mathcal{B}, G)$ be as in Lemma 2.1. Then $\mathcal{M}(B)$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$ with block size m, and the induced action of G_B on $\mathcal{M}(B)$ is permutationally equivalent to the action of G_B on the edge set of the underlying simple graph of $[B]$ via the bijection $\langle \alpha, \beta \rangle \leftrightarrow \{\alpha, \beta\}$. In particular, the following* (a) *and* (b) *hold.*

(a) *If $[B]$ is simple (that is, $m = 1$), then the actions of G_B on $\Gamma_{\mathcal{B}}(B)$ and on the edge set of $[B]$ are permutationally equivalent.*
(b) *If $[B]$ has multiple edges (that is, $m \geq 2$) and $|B| \geq 3$, then $\Gamma_{\mathcal{B}}$ is G-locally imprimitive and hence not $(G, 2)$-arc transitive.*

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Note that for $|B| = 2$ the statement in Lemma 2.2(b) is invalid. In fact, a 2-fold cover Γ of a $(G, 2)$-arc transitive graph Σ of valency at least 2 may be $(G, 2)$-arc transitive, and for the natural partition $\mathcal{B}$ of $V(\Gamma)$ we have $m = \mathtt{val}(\Sigma) \geq 2$, $\mathcal{M}(B)$ is a trivial partition, and $\Gamma_{\mathcal{B}} \cong \Sigma$ is $(G, 2)$-arc transitive.

The following theorem contains most information on $[B]$ that we will need to prove our main results. Let $G_{(B)}$ and $G_{[B]}$ denote the kernels of the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$, respectively.

Theorem 2.3. *Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent blocks $B, C \in \mathcal{B}$, where $G \leq \mathrm{Aut}(\Gamma)$. Then the underlying simple graph of $[B]$ is G_B-symmetric, and the components of $[B]$ for B running over $\mathcal{B}$ form a G-invariant partition $\mathcal{Q}$ of $V(\Gamma)$. This partition $\mathcal{Q}$ has block size $|B|/\omega$, is a refinement of $\mathcal{B}$, and is such that $G_{(\mathcal{B})} = G_{(\mathcal{Q})}$, $\mathtt{val}(\Gamma_{\mathcal{Q}}) = \mathtt{val}(\Gamma_{\mathcal{B}})/\omega$ and $\Gamma[P, Q] \cong \Gamma[B, C]$ for adjacent blocks $P, Q \in \mathcal{Q}$, where ω is the number of components of $[B]$. Moreover, the following* (a) *and* (b) *hold.*

(a) *In the case where the underlying simple graph of $[B]$ is a perfect matching (hence $|B|$ is even and the perfect matching is $(|B|/2) \cdot K_2$), we have $\mathcal{Q} = \{\Gamma(C) \cap B : (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}$ (ignoring the multiplicity of each $\Gamma(C) \cap B$), which has block size 2, and either $\Gamma \cong \Gamma_{\mathcal{Q}}[\overline{K}_2]$ or Γ is a 2-fold cover of $\Gamma_{\mathcal{Q}}$;*

(b) *In the case where the underlying simple graph of $[B]$ is not a perfect matching, G is faithful on both $\mathcal{B}$ and $\mathcal{Q}$, and $G_{[B]}$ is a subgroup of $G_{(B)}$; moreover, $G_{(B)} = G_{[B]}$ if in addition $[B]$ is simple, and $G_{(B)} = G_{[B]} = 1$ if $[B]$ is simple and $\Gamma_{\mathcal{B}}$ is a complete graph.*

Proof. It can be easily verified that the induced action of G_B on B preserves the adjacency of $[B]$ and hence the underlying simple graph of $[B]$ admits G_B as a group of automorphisms. Let $\alpha \in B$ and $\beta, \gamma \in [B](\alpha)$ (the neighbourhood of α in $[B]$). Then there exist $C, D \in \Gamma_{\mathcal{B}}(B)$ such that $\Gamma(C) \cap B = \{\alpha, \beta\}$ and $\Gamma(D) \cap B = \{\alpha, \gamma\}$. Hence α is adjacent to a vertex $\delta \in C$ and a vertex $\varepsilon \in D$. Since Γ is G-symmetric, there exists $g \in G$ such that $(\alpha, \delta)^g = (\alpha, \varepsilon)$. Thus, $g \in G_\alpha$ and $C^g = D$. Consequently, $(\Gamma(C) \cap B)^g = \Gamma(D) \cap B$, that is, $\{\alpha, \beta\}^g = \{\alpha, \gamma\}$ and hence $\beta^g = \gamma$. This means that G_α is transitive on $[B](\alpha)$. Since G_B is transitive on B, it follows that the underlying simple graph of $[B]$ is G_B-symmetric. Therefore, the connected components of $[B]$ form a G_B-invariant partition of B. From this it is straightforward to show that the set $\mathcal{Q}$ of such components, for B running over $\mathcal{B}$, is a G-invariant partition of $V(\Gamma)$. Clearly, $\mathcal{Q}$ is a refinement of $\mathcal{B}$ with block size $|B|/\omega$, $\mathtt{val}(\Gamma_{\mathcal{Q}}) = \mathtt{val}(\Gamma_{\mathcal{B}})/\omega$, and $\Gamma[P, Q] \cong \Gamma[B, C]$ for adjacent blocks $P, Q \in \mathcal{Q}$. Since $\mathcal{B}$ is G-invariant and $\mathcal{Q}$ refines $\mathcal{B}$, it follows that $G_{(\mathcal{Q})} \leq G_{(\mathcal{B})}$. On the other hand, if $g \in G_{(\mathcal{B})}$, then g fixes setwise each block of $\mathcal{B}$ and hence fixes $\Gamma(C) \cap B$, for all pairs B, C of adjacent blocks of $\mathcal{B}$. In other words, g fixes each edge of $[B]$, for all $B \in \mathcal{B}$. Thus, g fixes setwise each block of $\mathcal{Q}$ and so $g \in G_{(\mathcal{Q})}$. It follows that $G_{(\mathcal{B})} \leq G_{(\mathcal{Q})}$ and hence $G_{(\mathcal{B})} = G_{(\mathcal{Q})}$.

Assume that the underlying simple graph of $[B]$ is a perfect matching, namely $(|B|/2) \cdot K_2$. Then $\mathcal{Q} = \{\Gamma(C) \cap B : (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}$ and thus $\mathcal{Q}$ has block size 2. Since $\Gamma[P, Q] \cong \Gamma[B, C]$, either $\Gamma[P, Q] \cong K_{2,2}$ or $\Gamma[P, Q] \cong 2 \cdot K_2$. In the former case we have $\Gamma \cong \Gamma_{\mathcal{Q}}[\overline{K}_2]$, and in the latter case Γ is a 2-fold cover of $\Gamma_{\mathcal{Q}}$.

In the following we assume that the underlying simple graph of $[B]$ is not a perfect matching. Then $|B| \geq 3$ and this simple graph has valency at least two. Moreover, in this case distinct vertices of B are incident with distinct sets of edges of $[B]$; in other words, the vertices of B are distinguishable. Let $g \in G_{[B]}$. Then g fixes setwise each block of $\Gamma_{\mathcal{B}}(B)$ and hence fixes each edge of $[B]$. Since the vertices of B are distinguishable by different sets of edges of $[B]$, it follows that g fixes B pointwise. Therefore, $G_{[B]}$ is a subgroup of $G_{(B)}$. Similarly, any $g \in G_{(\mathcal{B})}$

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fixes each edge of $[B]$ for all $B \in \mathcal{B}$. Since the vertices of B are distinguishable, it follows that g fixes B pointwise for all $B \in \mathcal{B}$; in other words, g fixes each vertex of Γ. Since $G \leq \mathrm{Aut}(\Gamma)$ is faithful on $V(\Gamma)$, we conclude that $g = 1$, and hence $G_{(\mathcal{B})} = 1$. Thus, G is faithful on $\mathcal{B}$. Since $G_{(\mathcal{B})} = G_{(\mathcal{Q})}$, G is faithful on $\mathcal{Q}$ as well.

In the case where $[B]$ is simple, by Lemma 2.2(a) the actions of G_B on $\Gamma_{\mathcal{B}}(B)$ and on the edge set of $[B]$ are permutationally equivalent. Thus, since any $h \in G_{(B)}$ fixes each edge of $[B]$, it follows that h fixes setwise each block of $\Gamma_{\mathcal{B}}(B)$. That is, $h \in G_{[B]}$ and so $G_{(B)} \leq G_{[B]}$. This together with $G_{[B]} \leq G_{(B)}$ implies $G_{(B)} = G_{[B]}$. Finally, if $[B]$ is simple and $\Gamma_{\mathcal{B}}$ is a complete graph, then we have $\Gamma_{\mathcal{B}}(B) = \mathcal{B} \setminus \{B\}$ and hence $G_{(B)} = G_{[B]} = G_{(\mathcal{B})}$. However, G is faithful on $\mathcal{B}$, so we have $G_{(B)} = G_{[B]} = 1$ and the proof is complete. □

Note that the condition $G \leq \mathrm{Aut}(\Gamma)$ was required only in part (b) of Theorem 2.3.

Remark 2.4. (a) That the underlying simple graph of $[B]$ is G_B-symmetric was known in [6, Lemma 6.1] under the additional assumption that Γ is G-locally primitive. In this case, either $[B]$ is a simple graph, or the underlying simple graph of $[B]$ is a perfect matching.

(b) In the case where the underlying simple graph of $[B]$ is a perfect matching, the faithfulness of G ($\leq \mathrm{Aut}(\Gamma)$) on $\mathcal{B}$ is not guaranteed. For example, let $\Gamma = 2 \cdot C_n$ be a 2-fold cover of C_n (cycle of length n), and let $G = \mathbb{Z}_2 \mathrm{wr} D_{2n}$. Then Γ is G-symmetric, and it admits the natural G-invariant partition with quotient C_n such that the underlying simple graph of $[B]$ is isomorphic to K_2. Clearly, the induced action of G on $\mathcal{B}$ is unfaithful.

Let us end this section by the following observations, which will be used in the next section.

Lemma 2.5. *Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent blocks $B, C \in \mathcal{B}$.*

(a) *If $\Gamma[B, C] \cong 2 \cdot K_2$ and $[B]$ is simple, then for $\alpha \in B$ the actions of G_α on $\Gamma(\alpha)$ and $[B](\alpha)$ are permutationally equivalent, where $[B](\alpha)$ is the neighbourhood of α in $[B]$.*

(b) *If $\Gamma[B, C] \cong K_{2,2}$ and $\mathtt{val}([B]) \geq 2$, then the subsets $\Gamma(\alpha) \cap C$ of $\Gamma(\alpha)$, for C running over all $C \in \mathcal{B}$ such that $\Gamma(\alpha) \cap C \neq \emptyset$, form a G_α-invariant partition of $\Gamma(\alpha)$ of block size 2; in particular, Γ is G-locally imprimitive and hence not $(G, 2)$-arc transitive.*

Proof. (a) Since $[B]$ is simple and $\Gamma[B, C] \cong 2 \cdot K_2$, from (a) and (c) of Lemma 2.1 we have $|\Gamma(\alpha)| = |[B](\alpha)| = |\{C \in \mathcal{B} : \Gamma(\alpha) \cap C \neq \emptyset\}|$. For each $\beta \in \Gamma(\alpha)$, say, $\beta \in C$, the unique vertex γ of $(\Gamma(C) \cap B) \setminus \{\alpha\}$ is a neighbour of α in $[B](\alpha)$. It can be easily verified that $\beta \leftrightarrow \gamma$ defines a bijection between $\Gamma(\alpha)$ and $[B](\alpha)$, and the actions of G_α on $\Gamma(\alpha)$ and $[B](\alpha)$ are permutationally equivalent with respect to this bijection.

(b) The proof is straightforward and hence omitted. □

3. Main results and proofs

Let $(\Gamma, \mathcal{B})$ be an imprimitive G-symmetric graph such that $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent $B, C \in \mathcal{B}$. If $|B| = 2$, then either Γ is a 2-fold cover of $\Gamma_{\mathcal{B}}$, or $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_2]$. In the former case $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if Γ is $(G, 2)$-arc transitive, whilst in the latter case Γ is not $(G, 2)$-arc transitive unless $\Gamma \cong q \cdot K_{2,2}$ and $\Gamma_{\mathcal{B}} \cong q \cdot K_2$ for some $q \geq 1$. Thus, we may assume $|B| \geq 3$ in the following. In answering Question (1), the case $|B| = 3$ invokes 3-arc graphs of trivalent 2-arc transitive graphs, which were determined in [22]. For a regular graph Σ, a subset Δ of $\mathtt{Arc}_3(\Sigma)$ is called *self-paired* if $(\tau, \sigma, \sigma', \tau') \in \Delta$ implies $(\tau', \sigma', \sigma, \tau) \in \Delta$. For such a Δ, the 3-*arc graph* of Σ with respect to Δ, denoted by $\Xi(\Sigma, \Delta)$, is defined [10,18] to be the graph with vertex set $\mathtt{Arc}(\Sigma)$ in which $(\sigma, \tau), (\sigma', \tau')$ are adjacent if and only if

$(\tau, \sigma, \sigma', \tau') \in \Delta$. In the case where Σ is G-symmetric and G is transitive on Δ (under the induced action of G on $\mathtt{Arc}_3(\Sigma)$), $\Gamma := \Xi(\Sigma, \Delta)$ is a G-symmetric graph [10, Section 6] which admits

$$\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\} \tag{1}$$

as a G-invariant partition such that $\Gamma_{\mathcal{B}(\Sigma)} \cong \Sigma$ with respect to the bijection $\sigma \leftrightarrow B(\sigma)$, where $B(\sigma) := \{(\sigma, \tau) : \tau \in \Sigma(\sigma)\}$.

The following theorem is the full version of Theorem 1.1. In part (ii) of this theorem the graph $3 \cdot C_4$ in $(3 \cdot C_4, \mathrm{PGL}(2, 3))$ is the cross-ratio graph [7,17] CR(3; 2, 1), and in $(3 \cdot C_4, \mathrm{AGL}(2, 2))$ it should be interpreted as the affine flag graph [17] $\Gamma^{=}(A; 2, 2)$. It is well known that, for a connected trivalent G-symmetric graph Σ, G is a homomorphic image of one of seven finitely presented groups, G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5, with the subscript s indicating that Σ is (G, s)-arc regular. The reader is referred to [3,4] for this result and the presentations of these groups.

Theorem 3.1. *Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $|B| \geq 3$ and $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent $B, C \in \mathcal{B}$. Suppose that $\Gamma_{\mathcal{B}}$ is connected. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if $[B]$ is a simple graph and one of the following* (a) *and* (b) *occurs:*

(a) $|B| = 3$, *and $[B] \cong K_3$ is $(G_B, 2)$-arc transitive (that is, $G_B^B / G_{(B)} \cong S_3$);*
(b) $|B| \geq 4$ *is even, $[B] \cong (|B|/2) \cdot K_2$, and G_B is 2-transitive on the edges of $[B]$.*

Moreover, in case (a) *the following hold:*

(i) *$\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$, $\Gamma_{\mathcal{B}}$ is a trivalent $(G, 2)$-arc transitive graph of type other than G_2^2, and moreover any connected trivalent $(G, 2)$-arc transitive graph Σ of type other than G_2^2 can occur as $\Gamma_{\mathcal{B}}$;*
(ii) *one of the following* (1)–(2) *occurs:* (1) *$\Gamma \cong s \cdot C_t$ for some $s \geq 3, t \geq 3$, Γ is $(G, 2)$-arc transitive, $\Gamma[B, C] \cong 2 \cdot K_2$, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc regular of type G_2^1, and either $\Gamma_{\mathcal{B}} \cong K_4$ and $(\Gamma, G) \cong (4 \cdot C_3, S_4)$, $(3 \cdot C_4, \mathrm{PGL}(2, 3))$ or $(3 \cdot C_4, \mathrm{AGL}(2, 2))$, or $\Gamma_{\mathcal{B}} \not\cong K_4$ and $\Gamma_{\mathcal{B}}$ is a near n-gonal graph for some integer $n \geq 4$;* (2) *Γ is 4-valent, connected and not $(G, 2)$-arc transitive, $\Delta = \mathtt{Arc}_3(\Gamma_{\mathcal{B}})$, $\Gamma[B, C] \cong K_{2,2}$, and $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive.*

In case (b)*, we have:*

(iii) *$\mathtt{val}(\Gamma_{\mathcal{B}}) = |B|/2$, and $|V(\Gamma)| = 4q$ for some integer $q \geq 3$;*
(iv) *Γ is $(G, 2)$-arc transitive, and either $\Gamma \cong 2q \cdot K_2$ and $\Gamma[B, C] \cong 2 \cdot K_2$, or $\Gamma \cong q \cdot K_{2,2}$ and $\Gamma[B, C] \cong K_{2,2}$;*
(v) *$\Gamma_{\mathcal{Q}} \cong q \cdot K_2$, where $\mathcal{Q} = \{\Gamma(C) \cap B : (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}$ is as in Theorem 2.3*(a).

In the proof of Theorem 3.1 we will exploit the main results of [10,20] and a classification result in [17]. We will also use the following lemma, which is a restatement of a result in [22].

Lemma 3.2 (*[22]*). *A connected trivalent G-symmetric graph Σ has a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$ if and only if it is not of type G_2^2. Moreover, when Σ is $(G, 1)$-arc regular, there are exactly two self-paired G-orbits on $\mathtt{Arc}_3(\Sigma)$; when $\Sigma \neq K_4$ is $(G, 2)$-arc regular of type G_2^1, there are exactly two self-paired G-orbits on $\mathtt{Arc}_3(\Sigma)$, namely $\Delta_1 := (\tau, \sigma, \sigma', \tau')^G$ and $\Delta_2 := (\tau, \sigma, \sigma', \delta')^G$ (where σ, σ' are adjacent vertices, $\Sigma(\sigma) = \{\sigma', \tau, \delta\}$ and $\Sigma(\sigma') = \{\sigma, \tau', \delta'\}$), and $\Xi(\Sigma, \Delta_1)$, $\Xi(\Sigma, \Delta_2)$ are both almost covers of Σ with valency* 2; *when Σ is (G, s)-arc regular, where $3 \leq s \leq 5$, the only self-paired G-orbit is $\Delta := \mathtt{Arc}_3(\Sigma)$, and $\Xi(\Sigma, \Delta)$ is a connected G-symmetric but not $(G, 2)$-arc transitive graph of valency* 4.

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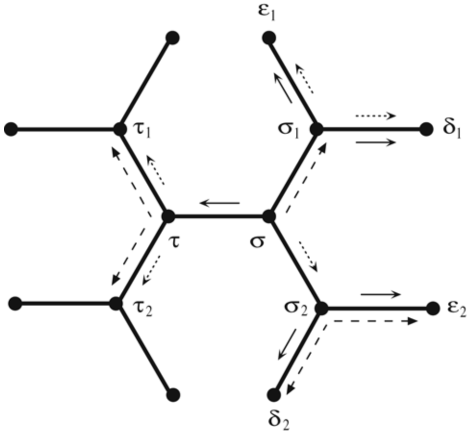

Fig. 1. Proof of part (ii)(2) of Theorem 3.1. In the 3-arc graph $\Xi(\Gamma_{\mathcal{B}}, \Delta)$, where $\Delta = \mathtt{Arc}_3(\Gamma_{\mathcal{B}})$, the vertex (σ, τ) is adjacent to $(\sigma_1, \varepsilon_1)$, (σ_1, δ_1), $(\sigma_2, \varepsilon_2)$ and (σ_2, δ_2). Similarly, (σ, σ_1) is adjacent to $(\sigma_2, \varepsilon_2)$, (σ_2, δ_2), (τ, τ_1) and (τ, τ_2), and (σ, σ_2) is adjacent to $(\sigma_1, \varepsilon_1)$, (σ_1, δ_1), (τ, τ_1) and (τ, τ_2).

Proof of Theorem 3.1 ($\Rightarrow$). Suppose $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Then $[B]$ is a simple graph by Lemma 2.2, and $\mathtt{val}([B]) = |\{C \in \mathcal{B} : \Gamma(\alpha) \cap C \neq \emptyset\}|$ by Lemma 2.1(c). Since $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, and hence 2-transitive on the set of edges of $[B]$ by Lemma 2.2(a). It follows that, whenever $[B]$ contains adjacent edges, any two edges of $[B]$ must be adjacent. Thus, one of the following possibilities occurs:

(A) $[B]$ contains at least two edges, and any two edges of $[B]$ are adjacent;
(B) $[B]$ consists of pairwise independent edges, that is, $[B]$ is a perfect matching.

Case (A) In this case we must have $|B| = 3$ and hence $[B] \cong K_3$. Thus, $\mathtt{val}([B]) = 2$ and hence $\mathtt{val}(\Gamma_{\mathcal{B}}) = 3$ by Lemma 2.1. Hence $\Gamma_{\mathcal{B}}$ is a trivalent $(G, 2)$-arc transitive graph. Let $B = \{\alpha, \beta, \gamma\}$, and let $C, D, E \in \Gamma_{\mathcal{B}}(B)$ be such that $\Gamma(C) \cap B = \{\alpha, \beta\}$, $\Gamma(D) \cap B = \{\beta, \gamma\}$ and $\Gamma(E) \cap B = \{\gamma, \alpha\}$. Since $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, there exists $g \in G_B$ such that $(C, E)^g = (E, C)$. Since g fixes B and interchanges C and E, it interchanges $\Gamma(C) \cap B$ and $\Gamma(E) \cap B$, that is, $\{\alpha, \beta\}^g = \{\gamma, \alpha\}$ and $\{\gamma, \alpha\}^g = \{\alpha, \beta\}$. Thus, we must have $\alpha^g = \alpha$, $\beta^g = \gamma$ and $\gamma^g = \beta$. Now that $g \in G_\alpha$ and $[B]$ is G_B-symmetric, it follows that $[B]$ is $(G_B, 2)$-arc transitive, or equivalently $G_B^B/G_{(B)} \cong S_3$. Since $|\Gamma(C) \cap B| = |B| - 1 = 2$ and $[B]$ is simple, we have $\Gamma(F) \cap B \neq \Gamma(F') \cap B$ for distinct $F, F' \in \Gamma_{\mathcal{B}}(B)$ and hence from [10, Theorem 1] there exists a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$ such that $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$. From Lemma 3.2 it follows that $\Gamma_{\mathcal{B}}$ is of type other than G_2^2. (It is also of type other than G_1 since it is $(G, 2)$-arc transitive.) From [10, Theorem 2] the case $\Gamma[B, C] \cong K_{2,2}$ occurs if and only if $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive, which in turn is true if and only if $\Delta = \mathtt{Arc}_3(\Gamma_{\mathcal{B}})$ in the 3-arc graph $\Xi(\Gamma_{\mathcal{B}}, \Delta)$ above. In this case $\Gamma_{\mathcal{B}}$ is of type G_3, G_4^1, G_4^2 or G_5, and it is clear that Γ is 4-valent. (See Fig. 1 for an illustration.) Moreover, since in this case $\Gamma_{\mathcal{B}}$ is connected and $\Gamma[B, C] \cong K_{2,2}$, Γ is connected and not $(G, 2)$-arc transitive.

In the case where $\Gamma[B, C] \cong 2 \cdot K_2$, which occurs if and only if $\Gamma_{\mathcal{B}}$ is of type G_2^1, we have $\mathtt{val}(\Gamma) = 2$ and hence Γ is a union of vertex-disjoint cycles of the same length. In this case the element g in the previous paragraph must interchange the two neighbours of α in Γ, and hence Γ is $(G, 2)$-arc transitive. If $\Gamma_{\mathcal{B}} \cong K_4$, then $(\Gamma, G) \cong (4 \cdot C_3, S_4)$, $(3 \cdot C_4, \mathrm{PGL}(2, 3))$ or $(3 \cdot C_4, \mathrm{AGL}(2, 2))$ by [17, Theorem 3.19]. In the general case where $\Gamma_{\mathcal{B}} \ncong K_4$, since Γ

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is an almost cover of $\Gamma_{\mathcal{B}}$, by [20, Theorem 3.1] there exists an integer $n \geq 4$ such that $\Gamma_{\mathcal{B}}$ is a near n-gonal graph with respect to a G-orbit $\mathcal{E}$ on n-cycles of $\Gamma_{\mathcal{B}}$. The cycles in $\mathcal{E}$ that contain the 2-arcs (C, B, D), (C, B, E), (D, B, E) respectively must be pairwise distinct, and so $|\mathcal{E}| \geq 3$. Moreover, since Δ is the set of 3-arcs contained in cycles in $\mathcal{E}$ [20, Theorem 3.1], by the definition of a 3-arc graph, each cycle in $\mathcal{E}$ gives rise to a cycle of $\Xi(\Gamma_{\mathcal{B}}, \Delta)$ and vice versa. Hence $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta) \cong s \cdot C_t$, where $s = |\mathcal{E}| \geq 3$, and $t \geq 3$ is the cycle length of $\mathcal{E}$.

To complete the proof for case (A), we now justify that any connected trivalent $(G, 2)$-arc transitive graph of type other than G_2^2 can occur as $\Gamma_{\mathcal{B}}$. In fact, by Lemma 3.2, for such a graph Σ there exists at least one self-paired G-orbit Δ on $\mathtt{Arc}_3(\Sigma)$. Thus, by [10, Theorem 1] the 3-arc graph $\Gamma := \Xi(\Sigma, \Delta)$ is a G-symmetric graph whose vertex set $\mathtt{Arc}(\Sigma)$ admits $\mathcal{B}(\Sigma)$ (defined in (1)) as a G-invariant partition such that $\Gamma_{\mathcal{B}(\Sigma)} \cong \Sigma$. Obviously, for such a graph $(\Gamma, \mathcal{B}(\Sigma))$ we have $|\Gamma(B(\tau)) \cap B(\sigma)| = |B(\sigma)| - 1 = 2$ for $(\sigma, \tau) \in \mathtt{Arc}(\Sigma)$ and $[B(\sigma)] \cong K_3$, where $B(\eta) := \{(\eta, \varepsilon) : \varepsilon \in \Sigma(\eta)\}$ for each $\eta \in V(\Sigma)$. Also, since Σ is $(G, 2)$-arc transitive, $G_{\sigma\tau}$ is 2-transitive on $\Sigma(\sigma) \setminus \{\tau\}$. This is equivalent to saying that $G_{\sigma\tau}$ is 2-transitive on $\{(\sigma, \varepsilon) : \varepsilon \in \Sigma(\sigma) \setminus \{\tau\}\}$, which is the neighbourhood of (σ, τ) in $[B(\sigma)]$. Since $G_{\sigma\tau} \leq G_{B(\sigma)}$, it follows that $[B(\sigma)]$ is $(G_{B(\sigma)}, 2)$-arc transitive. From [10, Theorem 2], $\Gamma[B(\sigma), B(\tau)] \cong 2 \cdot K_2$ if Σ is $(G, 2)$-arc regular, and $\Gamma[B(\sigma), B(\tau)] \cong K_{2,2}$ if Σ is $(G, 3)$-arc transitive.

Case (B) In this case we have $|B| \geq 4$, $|B|$ is even, and $[B] \cong (|B|/2) \cdot K_2$. Hence $\mathtt{val}(\Gamma_{\mathcal{B}}) = |B|/2$ and each vertex of Γ has neighbour in exactly one block of $\mathcal{B}$. Thus, $|\mathcal{B}| \geq \mathtt{val}(\Gamma_{\mathcal{B}}) + 1 \geq 3$, $|V(\Gamma)| = |B||\mathcal{B}| = 2\,\mathtt{val}(\Gamma_{\mathcal{B}})|\mathcal{B}| = 4q \geq 12$, where $q = |E(\Gamma_{\mathcal{B}})|$. Clearly, if $\Gamma[B, C] \cong 2 \cdot K_2$ then $\Gamma \cong 2q \cdot K_2$; whilst if $\Gamma[B, C] \cong K_{2,2}$ then $\Gamma \cong q \cdot K_{2,2}$. In the first case Γ has no 2-arc and hence is $(G, 2)$-arc transitive automatically. In the second possibility, since G_α is transitive on $\Gamma(\alpha)$ and $|\Gamma(\alpha)| = 2$, G_α is 2-transitive on $\Gamma(\alpha)$, and hence Γ is $(G, 2)$-arc transitive. Since G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, by Lemma 2.2(a), G_B is 2-transitive on the edges of $[B]$. Evidently, for the G-invariant partition $\mathcal{Q}$ of $V(\Gamma)$, we have $\Gamma_{\mathcal{Q}} \cong q \cdot K_2$.

($\Leftarrow$) We need to prove that if $[B]$ is simple and one of (a), (b) occurs then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Suppose first that (a) occurs. Since $[B] \cong K_3$ has three edges, $\Gamma_{\mathcal{B}}$ is trivalent by Lemma 2.1(b). Using the notation above, there exists $g \in G_\alpha$ such that g interchanges β and γ since $[B] \cong K_3$ is $(G_B, 2)$-arc transitive. Thus, g fixes D and interchanges C and E. Hence $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Now suppose (b) occurs. Then $[B]$ is simple and G_B is 2-transitive on the edges of $[B]$. From Lemma 2.2(a), this implies that G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, and hence $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. □

Remark 3.3. In the case where $|B| = 4$, we have $|\Gamma(C) \cap B| = |B| - 2 = 2$ for adjacent $B, C \in \mathcal{B}$, and hence the results in [8] apply. In fact, in this case $[B]$ agrees with the multigraph Γ^B introduced in [8], and moreover the underlying simple graph of $[B]$ is $2 \cdot K_2$, C_4 or K_4. (For a G-symmetric graph $(\Gamma, \mathcal{B})$ with $|\Gamma(C) \cap B| = |B| - 2 \geq 1$, Γ^B is defined [8] to be the multigraph with vertex set B and edges joining the two vertices of $B \setminus (\Gamma(C) \cap B)$ for $C \in \Gamma_{\mathcal{B}}(B)$.) From [8, Theorems 1.3], $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if $[B]$ is simple and $[B] \cong 2 \cdot K_2$, and in this case $\Gamma_{\mathcal{B}} \cong s \cdot C_t$ for some $s \geq 1$ and $t \geq 3$, and either $\Gamma \cong 2st \cdot K_2$ or $\Gamma \cong st \cdot C_4$, agreeing with (b) and (iv) of Theorem 3.1.

The next theorem tells us what happens when $(\Gamma, \mathcal{B})$ is a $(G, 2)$-arc transitive graph with $|\Gamma(C) \cap B| = 2$ for adjacent $B, C \in \mathcal{B}$. In particular, it answers Question (2) for such graphs.

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Theorem 3.4. *Let Γ be a $(G, 2)$-arc transitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent $B, C \in \mathcal{B}$. Then one of the following* (a)–(c) *holds.*

(a) *$\Gamma[B, C] \cong K_{2,2}$, $|B|$ is even, $[B] \cong (|B|/2) \cdot K_2$ is simple, and $\Gamma \cong q \cdot K_{2,2}$ for some $q \geq 1$;*
(b) *$\Gamma[B, C] \cong 2 \cdot K_2$, $[B]$ is simple and $(G_B, 2)$-arc transitive;*
(c) *$\Gamma[B, C] \cong 2 \cdot K_2$, $m = \mathtt{val}([B]) \geq 2$, $|B|$ is even, the underlying simple graph of $[B]$ is the perfect matching $(|B|/2) \cdot K_2$, $\Gamma_{\mathcal{Q}}$ is $(G, 2)$-arc transitive with valency m, and Γ is a 2-fold cover of $\Gamma_{\mathcal{Q}}$, where $\mathcal{Q} = \{\Gamma(C) \cap B : (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}$ is as in Theorem 2.3*(a).

Moreover, in case (a) *$\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if G_B is 2-transitive on the edges of $[B]$; in case* (b) *$\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if* (i) *$|B| = 3$, $[B] \cong K_3$, $G_B^B/G_{(B)} \cong S_3$ and $\Gamma \cong s \cdot C_t$ for some $s \geq 3$, $t \geq 3$, or* (ii) *$|B|$ is even, $[B] \cong (|B|/2) \cdot K_2$, $\Gamma \cong 2q \cdot K_2$ for some $q \geq 1$, and G_B is 2-transitive on the edges of $[B]$; in case* (c) *$\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if $|B| = 2$.*

Proof. We distinguish the following three cases.

Case (A) $\Gamma[B, C] \cong K_{2,2}$. In this case, since Γ is $(G, 2)$-arc transitive we have $\mathtt{val}([B]) = 1$ by Lemma 2.5(b). Thus, $m = 1$ and $|B| = 2\,\mathtt{val}(\Gamma_{\mathcal{B}})$ by Lemma 2.1. Therefore, $[B]$ is simple, $[B] \cong (|B|/2) \cdot K_2$, and $\Gamma \cong q \cdot K_{2,2}$ for some integer $q \geq 1$. Hence by Lemma 2.2(a) the actions of G_B on $\Gamma_{\mathcal{B}}(B)$ and on the edges of $[B]$ are permutationally equivalent. Thus, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if G_B is 2-transitive on the edges of $[B]$.

Case (B) $\Gamma[B, C] \cong 2 \cdot K_2$ and $[B]$ is simple. In this case the actions of G_α on $[B](\alpha)$ and $\Gamma(\alpha)$ are permutationally equivalent by Lemma 2.5(a). But G_α is 2-transitive on $\Gamma(\alpha)$ since Γ is $(G, 2)$-arc transitive by our assumption. Hence G_α is 2-transitive on $[B](\alpha)$ as well. Since $[B]$ is G_B-symmetric by Theorem 2.3, it follows that $[B]$ is $(G_B, 2)$-arc transitive. From Lemma 2.2(a), $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if G_B is 2-transitive on the set of edges of $[B]$. This occurs only if either (i) $|B| = 3$ and $[B] \cong K_3$; or (ii) $|B|$ is even, $[B] \cong (|B|/2) \cdot K_2$, and G_B is 2-transitive on the edges of $[B]$. In case (i), we have $G_B^B/G_{(B)} \cong S_3$ since $[B]$ is $(G_B, 2)$-arc transitive, and moreover Γ has valency 2 and thus is a union of vertex-disjoint cycles. Furthermore, in case (i), since $|\Gamma(C) \cap B| = |B| - 1 = 2$ for adjacent $B, C \in \mathcal{B}$, by [10, Theorem 1] Γ is a 3-arc graph of $\Gamma_{\mathcal{B}}$ with respect to a self-paired G-orbit on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$, and an argument similar to the third paragraph in the proof of Theorem 3.1 ensures that $\Gamma \cong s \cdot C_t$ for some $s \geq 3$, $t \geq 3$. In case (ii) we have $\Gamma \cong 2q \cdot K_2$ for some integer q. Clearly, if the conditions in (ii) are satisfied, then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. If the conditions in (i) are satisfied, then since $|B| = 3$ the $(G_B, 2)$-arc transitivity of $[B]$ implies that G_B is 2-transitive on the edges of $[B]$, and hence $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive by Lemma 2.2(a).

Case (C) $\Gamma[B, C] \cong 2 \cdot K_2$ and $m \geq 2$. In this case, for $\alpha \in B$ there exist distinct $C, D \in \Gamma_{\mathcal{B}}(B)$ such that $\alpha \in \Gamma(C) \cap B = \Gamma(D) \cap B$. (Hence C, D are in the same block of $\mathcal{M}(B)$.) Let $\beta \in C, \gamma \in D$ be adjacent to α in Γ. We first show that the underlying simple graph of $[B]$ is a perfect matching. Suppose otherwise, then there exists $E \in \Gamma_{\mathcal{B}}(B)$ such that $\alpha \in \Gamma(E) \cap B \neq \Gamma(D) \cap B$. Thus, α is adjacent to a vertex δ in E, and E, D belong to distinct blocks of $\mathcal{M}(B)$. Since Γ is $(G, 2)$-arc transitive, there exists $g \in G_{\alpha\beta}$ such that $\gamma^g = \delta$. Then $g \in G_{BC}$ and $D^g = E$. However, since $\mathcal{M}(B)$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$ by Lemma 2.2, $g \in G_{BC}$ implies that g fixes the block of $\mathcal{M}(B)$ containing C and D, and on the other hand $D^g = E$ implies that g permutes the block of $\mathcal{M}(B)$ containing D to the block of $\mathcal{M}(B)$ containing E. This contradiction shows that the underlying simple graph of $[B]$ must be the perfect matching $(|B|/2) \cdot K_2$ and hence $|B|$ is even. Thus, $m = \mathtt{val}([B]) \geq 2$. Since the underlying simple graph of $[B]$ is a perfect matching, from Theorem 2.3(a) it follows that $\mathcal{Q} = \{\Gamma(C) \cap B : (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}$ (ignoring the multiplicity of each $\Gamma(C) \cap B$) is a G-invariant partition of $V(\Gamma)$. It is readily seen that Γ is a 2-fold cover of $\Gamma_{\mathcal{Q}}$. Thus, both $\Gamma_{\mathcal{Q}}$ and Γ have valency m, and moreover $\Gamma_{\mathcal{Q}}$ is $(G, 2)$-arc transitive since Γ is $(G, 2)$-arc transitive. Hence, if $|B| = 2$, then $\mathcal{Q}$ coincides with $\mathcal{B}$ and hence $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. On the other hand, if $|B| \geq 4$, then since $[B]$ is not simple, $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive by Theorem 3.1. Therefore, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if $|B| = 2$. □

Remark 3.5. From Theorem 3.4, for a $(G, 2)$-arc transitive graph $(\Gamma, \mathcal{B})$ with $\Gamma[B, C] \cong 2 \cdot K_2$ or $K_{2,2}$ for adjacent $B, C \in \mathcal{B}$, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if one of the following (a)-(c) holds: (a) $|B|$ is even, $[B] \cong (|B|/2) \cdot K_2$ is simple, and G_B is 2-transitive on the edges of $[B]$; (b) $|B| = 3$, $[B] \cong K_3$ is simple, and $\Gamma[B, C] \cong 2 \cdot K_2$; (c) $|B| = 2$ and Γ is a 2-fold cover of $\Gamma_{\mathcal{B}}$. Moreover, in case (a) we have either $\Gamma[B, C] \cong K_{2,2}$ and $\Gamma \cong q \cdot K_{2,2}$, or $\Gamma[B, C] \cong 2 \cdot K_2$ and $\Gamma \cong 2q \cdot K_2$, and in case (b) we have $G_B^B/G_{(B)} \cong S_3$ and $\Gamma \cong s \cdot C_t$ for some $s \geq 3, t \geq 3$. In case (c), $[B]$ is not necessarily simple since it may happen that $m = \mathtt{val}([B]) \geq 2$. Note that cases (a) and (c) overlap when $|B| = 2$ and $\Gamma[B, C] \cong 2 \cdot K_2$.

The reader is referred to [8, Examples 4.7 and 4.8] for examples with $|\Gamma(C) \cap B| = |B| - 2 = 2$ for adjacent $B, C \in \mathcal{B}$ such that Γ is $(G, 2)$-arc transitive but $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive.

In part (c) of Theorem 3.4, $\Gamma_{\mathcal{Q}}$ is $(G, 2)$-arc transitive while $\Gamma_{\mathcal{B}}$ is not when $|B| \geq 4$. These two quotient graphs of Γ are connected by $(\Gamma_{\mathcal{Q}})_{\mathbf{B}} \cong \Gamma_{\mathcal{B}}$, where $\mathbf{B} := \{\{\Gamma(C) \cap B : C \in \Gamma_{\mathcal{B}}(B)\} : B \in \mathcal{B}\}$ (ignoring the multiplicity of $\Gamma(C) \cap B$), which is a G-invariant partition of $\mathcal{Q}$.

4. Concluding remarks

A scheme for constructing G-symmetric graphs with $\Gamma[B, C] \cong 2 \cdot K_2$ for adjacent $B, C \in \mathcal{B}$ was described in [6, Section 6]. In view of Theorem 3.4 and Lemma 2.5, to construct a 2-arc transitive graph $(\Gamma, \mathcal{B})$ with $\Gamma[B, C] \cong 2 \cdot K_2$ and $m = 1$ by using this scheme, we may start with a G-symmetric graph $\Gamma_{\mathcal{B}}$ and mutually isomorphic $(G_B, 2)$-arc transitive graphs $[B]$ (where $B \in V(\Gamma_{\mathcal{B}})$) on v vertices such that $v\,\mathtt{val}([B]) = 2\,\mathtt{val}(\Gamma_{\mathcal{B}})$. The action of G on $V(\Gamma_{\mathcal{B}})$ induces an action on such graphs $[B]$. To construct Γ we need to develop a rule [6] of labelling each edge of $[B]$ by an edge "BC" of $\Gamma_{\mathcal{B}}$, where $C \in \Gamma_{\mathcal{B}}(B)$, such that the actions of G_B on such labels and on the edges of $[B]$ are permutationally equivalent. We also need a "G-invariant joining rule" [6] to specify how to join the end-vertices of "BC" and the end-vertices of "CB" by two independent edges. If we can find such a rule such that, for each $\alpha \in B$, the actions of G_α on $\Gamma(\alpha)$ and $[B](\alpha)$ are permutationally equivalent, then by the $(G_B, 2)$-arc transitivity of $[B]$ the graph Γ thus constructed is $(G, 2)$-arc transitive. Theorem 3.4 suggests that we should choose $\Gamma_{\mathcal{B}}$ to be G-symmetric but not $(G, 2)$-arc transitive in order to obtain interesting $(G, 2)$-arc transitive graphs Γ by using this construction. The reader is referred to [6, Section 6] for a few examples of this construction. One of them is Conder's trivalent 5-arc transitive graph [2] on 75 600 vertices which can be obtained by taking $[B]$ as Tutte's 8-cage [1].

Finally, for a G-symmetric graph $(\Gamma, \mathcal{B})$ with $|\Gamma(C) \cap B| = 2$ for adjacent $B, C \in \mathcal{B}$, by Theorem 2.3 the underlying simple graph of $[B]$ is isomorphic to $K_{|B|}$ if and only if G_B is 2-transitive on B, and in this case we have $\mathtt{val}([B]) = m(|B| - 1)$ and $\mathtt{val}(\Gamma_{\mathcal{B}}) = m|B|(|B| - 1)/2$. Under the assumption that Γ is G-locally primitive, G_B is 3-transitive on B and $\Gamma_{\mathcal{B}}$ is a complete graph, it was shown in [6, Theorem 6.11] that $\Gamma[B, C] \cong 2 \cdot K_2$ and either Γ or the graph obtained from Γ by consistently swapping edges and non-edges of $\Gamma[B, C]$ is isomorphic to $(\mathtt{val}(\Gamma_{\mathcal{B}}) + 1) \cdot K_{|B|}$.

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References

[1] N.L. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1993. Cambridge Mathematical Library.
[2] M. Conder, A new 5-arc transitive cubic graph, J. Graph Theory 11 (1987) 303–307.
[3] M. Conder, P. Lorimer, Automorphism groups of symmetric graphs of valency 3, J. Combin. Theory Ser. B 47 (1989) 61–72.
[4] D.Z. Djokovic, G.L. Miller, Regular groups of automorphisms of cubic graphs, J. Combin. Theory Ser. B 29 (1980) 195–230.
[5] J.D. Dixon, B. Mortimer, Permutation Groups, Springer, New York, 1996.
[6] A. Gardiner, C.E. Praeger, A geometrical approach to imprimitive graphs, Proc. London Math. Soc. 3 (71) (1995) 524–546.
[7] A. Gardiner, C.E. Praeger, S. Zhou, Cross-ratio graphs, J. London Math. Soc. 2 (64) (2001) 257–272.
[8] M.A. Iranmanesh, C.E. Praeger, S. Zhou, Finite symmetric graphs with two-arc transitive quotients, J. Combin. Theory Ser. B 94 (2005) 79–99.
[9] C.H. Li, The finite primitive permutation groups containing an abelian regular subgroup, Proc. London Math. Soc. 87 (2003) 162–196.
[10] C.H. Li, C.E. Praeger, S. Zhou, A class of finite symmetric graphs with 2-arc transitive quotients, Math. Proc. Cambridge Philos. Soc. 129 (2000) 19–34.
[11] Z. Lu, S. Zhou, Finite symmetric graphs with 2-arc transitive quotients (II), J. Graph Theory (in press).
[12] D. Marušič, On 2-arc-transitivity of Cayley graphs, J. Combin. Theory Ser. B 87 (2003) 162–196.
[13] M. Perkel, Near-polygonal graphs, Ars Combin. 26(A) (1988) 149–170.
[14] C.E. Praeger, Finite transitive permutation groups and finite vertex transitive graphs, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry (Montreal, 1996), in: NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci., vol. 497, Kluwer Academic Publishing, Dordrecht, 1997, pp. 277–318.
[15] C.E. Praeger, Finite symmetric graphs, in: L.W. Beineke, R.J. Wilson (Eds.), Algebraic Graph Theory, in: Encyclopedia of Mathematics and its Applications, vol. 102, Cambridge University Press, Cambridge, pp. 179–202 (Chapter 7).
[16] W.T. Tutte, A family of cubical graphs, Proc. Cambridge Philos. Soc. 43 (1947) 459–474.
[17] S. Zhou, Constructing a class of symmetric graphs, Eur. J. Combin. 23 (2002) 741–760.
[18] S. Zhou, Imprimitive symmetric graphs, 3-arc graphs and 1-designs, Discrete Math. 244 (2002) 521–537.
[19] S. Zhou, Symmetric graphs and flag graphs, Monatsh. Math. 139 (2003) 69–81.
[20] S. Zhou, Almost covers of 2-arc transitive graphs, Combinatorica 24 (2004) 731–745.
[21] S. Zhou, A local analysis of imprimitive symmetric graphs, J. Algebraic Combin. 22 (2005) 435–449.
[22] S. Zhou, Trivalent 2-arc transitive graphs of type G_2^1 are near polygonal, Preprint.

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