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Constructing a Class of Symmetric Graphs

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We find a natural construction of a large class of symmetric graphs from point- and block-transitive 1-designs. The graphs in this class can be characterized as G-symmetric graphs whose vertex sets admit a G-invariant partition $\mathcal{B}$ of block size at least 3 such that, for any two blocks B, C of $\mathcal{B}$, either there is no edge between B and C, or there exists only one vertex in B not adjacent to any vertex in C. The special case where the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ is a complete graph occurs if and only if the 1-design needed in the construction is a G-doubly transitive and G-block-transitive 2-design, and in this case we give an explicit classification of Γ when G is a doubly transitive projective group or an affine group containing the affine general group. Examples of such graphs include cross ratio graphs studied recently by Gardiner, Praeger and Zhou and some other graphs with vertices the (point, line)-flags of the projective or affine geometry.

1. INTRODUCTION

For a finite graph Γ and an integer $s \geq 1$, an *s-arc* of Γ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices of Γ such that α_i, α_{i+1} are adjacent in Γ and $\alpha_{i-1} \neq \alpha_{i+1}$ for each i. If Γ admits a group G of automorphisms such that G is transitive on the vertex set $V(\Gamma)$ of Γ and, in its induced action, is transitive on the set $\mathrm{Arc}_s(\Gamma)$ of s-arcs of Γ, then Γ is said to be *(G, s)-arc transitive*. Often in the literature, a 1-arc is called an *arc* and a $(G, 1)$-arc transitive graph is called a *G-symmetric graph*. In this paper we will give a method of constructing a large class of G-symmetric graphs from G-point-transitive and G-block-transitive 1-designs. By using this we then classify all such graphs in the case where the 1-design involved is either a classical projective geometry, or a classical affine geometry, or a trivial doubly transitive linear space.

Let Γ be a finite G-symmetric graph. A partition $\mathcal{B}$ of $V(\Gamma)$ is said to be *G-invariant* if $B^g \in \mathcal{B}$ for any $B \in \mathcal{B}$ and $g \in G$, where $B^g := \{\alpha^g : \alpha \in B\}$; and $\mathcal{B}$ is *nontrivial* if $1 < |B| < |V(\Gamma)|$. If $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$, then Γ is said to be an *imprimitive G-symmetric graph*. In this case the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ is defined to be the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exists an edge of Γ joining a vertex of B to a vertex of C. In introducing a geometric approach to imprimitive symmetric graphs, Gardiner and Praeger [8] suggested an analysis of this quotient graph together with the 1-design $\mathcal{D}(B)$ with point set B and blocks $\Gamma(C) \cap B$ (with possible repetitions) for all $C \in \Gamma_{\mathcal{B}}(B)$, where $\Gamma(C) := \bigcup_{\alpha \in C} \Gamma(\alpha)$ with $\Gamma(\alpha)$ the *neighbourhood* of α in Γ (that is, the set of vertices adjacent to α in Γ), and $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$. Since Γ is G-symmetric, up to isomorphism, $\mathcal{D}(B)$ is independent of the choice of the block $B \in \mathcal{B}$. Thus the block size $k := |\Gamma(C) \cap B|$ of $\mathcal{D}(B)$ and the number of times each block of $\mathcal{D}(B)$ is repeated is independent of the choice of B. We will call this number the *multiplicity* of $\mathcal{D}(B)$ and denote it by m.

The graphs we are going to construct can be characterized as imprimitive G-symmetric graphs Γ satisfying $v = k + 1 \geq 3$, where $v := |B|$ is the block size of $\mathcal{B}$. And this paper forms part of our study on such graphs and is a sequel to [12, 16–18]. The construction shows that such a graph can be reconstructed from the quotient $\Gamma_{\mathcal{B}}$ and the induced action of G on $\mathcal{B}$. Moreover, it unveils a strong connection between such graphs and certain kinds of 1-designs. In fact, the construction requires a 1-design $\mathcal{D}$ with block size $m+1$ which admits G as a point- and block-transitive group of automorphisms, and a 'feasible' G-orbit Ω (see Definition 2.3

in Section 2.2) on the flags of $\mathcal{D}$, where a *flag* is an incident point-block pair. For such an Ω, denote by $\mathrm{F}(\mathcal{D}, \Omega)$ the set of ordered pairs $((\sigma, L), (\tau, N)) \subseteq \Omega \times \Omega$ such that $\sigma \notin N$, $\tau \notin L$ but $\sigma \in N'$, $\tau \in L'$ for some $(\sigma, L'), (\tau, N') \in \Omega$. The construction also requires a self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$. Given these we define the G-flag graph $\Gamma(\mathcal{D}, \Omega, \Psi)$ of $\mathcal{D}$ with respect to (Ω, Ψ) to be the graph with vertex set Ω and arc set Ψ. We prove that this graph is a G-symmetric graph admitting a certain G-invariant partition $\mathcal{B} := \mathcal{B}(\Omega)$ (see (3) in Section 2.2 for the definition) such that $v = k + 1 \geq 3$. Conversely, we show that any G-symmetric graph having this property is isomorphic to a G-flag graph $\Gamma(\mathcal{D}, \Omega, \Psi)$. The main result of this paper is the following theorem.

THEOREM 1.1. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $v = k + 1 \geq 3$. Then $\Gamma \cong \Gamma(\mathcal{D}, \Omega, \Psi)$ for a certain G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with point set $\mathcal{B}$ and block size $m + 1$, a certain feasible G-orbit Ω on the flags of $\mathcal{D}$, and a certain self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$, where m is the multiplicity of $\mathcal{D}(B)$.*

Conversely, for any G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with block size $m + 1$, any feasible G-orbit Ω on the flags of $\mathcal{D}$, and any self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$, the graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, group G and partition $\mathcal{B} = \mathcal{B}(\Omega)$ satisfy all the conditions above. Moreover, the multiplicity of the 1-design $\mathcal{D}(B)$ (for $B \in \mathcal{B}$) is equal to m.

We will show further that, in both parts of this theorem, G is faithful on the vertices of Γ if and only if it is faithful on the points of $\mathcal{D}$.

In particular, if $\mathcal{D}(B)$ contains no repeated blocks (that is, $m = 1$), then the construction above gives rise to the 3-arc graphs introduced in [12] (see Example 2.4 for details). In the case where $v = k + 1 \geq 3$ and $\Gamma_{\mathcal{B}}$ is a complete graph, the G-symmetry of $\Gamma_{\mathcal{B}}$ implies that G is doubly transitive on $\mathcal{B}$, and hence the design $\mathcal{D}$ in Theorem 1.1 is a G-doubly transitive and G-block-transitive 2-design. (As usual in the literature, when we say that a design is G-doubly transitive, we mean that G is doubly transitive on its points.) Since, as a result of the classification of finite simple groups, all the finite doubly transitive groups are known (see [3, 11]), Theorem 1.1 makes possible the classification of all such graphs Γ. As a moderate goal, we will classify the G-flag graphs of the classical projective and affine geometries for G a doubly transitive projective group or an affine group containing the affine general group, respectively. Examples of such graphs include the cross ratio graphs studied in [8, 10] and some other G-flag graphs in which the adjacency is defined naturally in terms of relative positions of lines involved. We prove that, for such groups G, these are the only G-symmetric graphs Γ such that $v = k + 1 \geq 3$, $\Gamma_{\mathcal{B}}$ is complete and G is faithful on $V(\Gamma)$. (In general, if a graph Γ is G-symmetric, then it is also G/K-symmetric under the induced action of G/K on $V(\Gamma)$, where K is the kernel of the action of G on $V(\Gamma)$. Since G/K is faithful on $V(\Gamma)$, this means that, in dealing with G-symmetric graphs Γ we may suppose without loss of generality that G is faithful on $V(\Gamma)$. In this paper we require the faithfulness of G only in several occasions for some technical reasons.)

THEOREM 1.2. *Let $d \geq 2$ be an integer and q a prime power. Then, for any group G with $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$ or $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$, all G-symmetric graphs Γ such that G is faithful on $V(\Gamma)$, and that $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$ with $v = k + 1 \geq 3$ and $\Gamma_{\mathcal{B}}$ complete are known explicitly.*

The reader is referred to Theorems 3.6 and 3.13 for the explicit lists of such graphs Γ and the corresponding v, m. We will also study G-flag graphs of the G-doubly transitive complete 2-$(v+1, 2, 1)$ designs $\mathcal{D}$. In this case $\mathcal{D}$ is G-flag transitive and we will show (see Example 2.4 and Corollary 2.6) that such G-flag graphs are precisely the G-symmetric graphs Γ such that

$v = k+1 \geq 3$, $\mathcal{D}(B)$ contains no repeated blocks and $\Gamma_{\mathcal{B}}$ is complete. A classification of such graphs Γ together with the corresponding groups G follows from the main theorem of [9]. With the contribution of [10, Theorem 5.1] and Theorem 1.2 above, we will see that this can be achieved via a (perhaps) more organic approach, and we will present such a classification explicitly in Theorem 3.19.

The construction introduced in this paper fits into a more general approach to constructing imprimitive symmetric graphs with the dual 1-design of $\mathcal{D}(B)$ containing no repeated blocks, see [19] for details.

2. Flag Graphs and the Proof of Theorem 1.1

2.1. Preliminaries. The reader is referred to [1, 6] and [15] for notation and terminology on designs, permutation groups and finite geometries, respectively. For a group acting on two sets Δ_1 and Δ_2, if there exists a bijection $\rho : \Delta_1 \to \Delta_2$ such that $\rho(\alpha^g) = (\rho(\alpha))^g$ for any $\alpha \in \Delta_1$ and $g \in G$, then the actions of G on Δ_1 and Δ_2 are said to be *permutationally equivalent* with respect to ρ. For $\alpha, \beta \in \Delta_1$, we use G_α to denote the stabilizer of α in G, and we set $G_{\alpha\beta} = (G_\alpha)_\beta$. For a positive integer n, we use K_n to denote the complete graph on n vertices, and $n \cdot \Gamma$ the union of n vertex disjoint copies of a given graph Γ.

Let Γ be a G-symmetric graph. Then Γ is regular; we denote by $\mathrm{val}(\Gamma)$ the valency of Γ. Instead of $\mathrm{Arc}_1(\Gamma)$, we use $\mathrm{Arc}(\Gamma)$ to denote the set of arcs of Γ. For a nontrivial G-invariant partition $\mathcal{B}$ of $V(\Gamma)$, we use $B(\alpha)$ to denote the block of $\mathcal{B}$ containing α. Thus, since $\mathcal{B}$ is G-invariant, we have $B(\alpha^g) = (B(\alpha))^g$ for any $\alpha \in V(\Gamma)$ and $g \in G$. We will assume without mentioning explicitly that the quotient graph $\Gamma_{\mathcal{B}}$ has at least one edge, so each block of $\mathcal{B}$ is an independent set of Γ (see e.g., [2, Proposition 22.1] and [14, Lemma 1.1(c)]). In the following we suppose the block size $v = |B|$ of $\mathcal{B}$ and the block size $k = |\Gamma(C) \cap B|$ (where $C \in \Gamma_{\mathcal{B}}(B)$) of the 1-design $\mathcal{D}(B)$ satisfy $v = k + 1 \geq 3$. (The case where $v = k + 1 = 2$ was studied in [12, Section 3].) Then, for each $\alpha \in B$, $B \setminus \{\alpha\}$ appears m times as a block of $\mathcal{D}(B)$, where m is the multiplicity of $\mathcal{D}(B)$. Set

$$\mathcal{B}(\alpha) := \{C \in \mathcal{B} : \Gamma(C) \cap B = B \setminus \{\alpha\}\}$$

so that $|\mathcal{B}(\alpha)| = m$. If $B(\alpha) \in \mathcal{B}(\beta)$ and $B(\beta) \in \mathcal{B}(\alpha)$, then we say that α and β are *mates*, and that α is the *mate* of β in $B(\alpha)$ (so β is the mate of α in $B(\beta)$ as well). Define Γ' to be the graph with the same vertices as Γ in which two vertices are adjacent if and only if they are mates. It was proved in [12, Proposition 3] that Γ' is a G-symmetric graph. For $B, C \in \mathcal{B}$, we denote by G_B the setwise stabilizer of B in G, and set $G_{B,C} = (G_B)_C$. Then one can check that $\mathbf{B}(B) := \{\mathcal{B}(\alpha) : \alpha \in B\}$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$, and hence G_B induces an action on $\mathbf{B}(B)$. As in [12], for adjacent blocks B, C of $\mathcal{B}$, we use $\Gamma[B, C]$ to denote the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. In particular, if $\Gamma[B, C] = (v - 1) \cdot K_2$, then following [18] Γ is called an *almost cover* of $\Gamma_{\mathcal{B}}$. We illustrate the notation introduced so far by the following diagram (see Figure 1), where the dashed lines represent edges of Γ'.

We will introduce a natural 1-design associated with $(\Gamma, \mathcal{B})$. For this purpose, we set

$$\mathcal{L}(\alpha) := \{B(\alpha)\} \cup \mathcal{B}(\alpha)$$

for each $\alpha \in V(\Gamma)$. Then $(\mathcal{L}(\alpha))^g = \mathcal{L}(\alpha^g)$ for any $g \in G$. In the particular case where $m = 1$ (that is, $\mathcal{D}(B)$ contains no repeated blocks), we have $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ whenever α and β are mates of each other. In general, part (d) of the following lemma tells us when $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ happens for distinct vertices α and β. Parts (a) and (b) of this lemma were proved in [12, Theorem 5(a) and (d)].

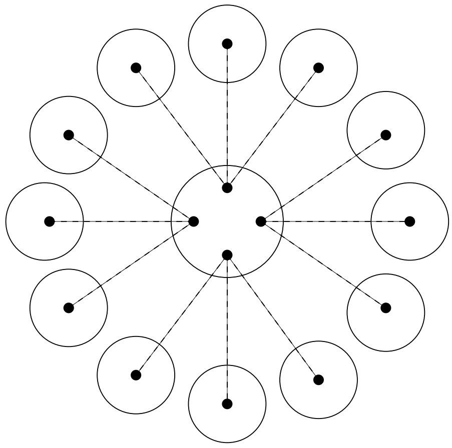

FIGURE 1. The sets $\mathcal{B}(\alpha)$ and the adjacency of Γ' in the case where $v = k + 1 = 4$ and $m = 3$. The edges of Γ and the three other vertices in each of the outskirt blocks are not shown in the diagram.

LEMMA 2.1. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $v = k + 1 \geq 3$. Then the following statements hold.*

(a) *$\Gamma_{\mathcal{B}}$ has valency* $\mathrm{val}(\Gamma_{\mathcal{B}}) = mv$.
(b) *If G is faithful on $V(\Gamma)$, then the induced action of G on $\mathcal{B}$ is faithful.*
(c) *For distinct vertices α, β in the same block B of $\mathcal{B}$, we have $\mathcal{L}(\alpha) \cap \mathcal{L}(\beta) = \{B\}$; in particular, $\mathcal{L}(\alpha) \neq \mathcal{L}(\beta)$.*
(d) *There exist distinct vertices $\alpha, \beta \in V(\Gamma)$ such that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ if and only if the graph Γ' is a union of vertex disjoint copies of K_{m+1} (hence $m + 1$ divides $|V(\Gamma)|$). In this case $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ holds for any two vertices α, β in the same component of Γ', and hence each $\mathcal{L}(\alpha)$ is repeated exactly $m + 1$ times; moreover, $\mathcal{L}(\alpha)$ induces a complete subgraph K_{m+1} of $\Gamma_{\mathcal{B}}$ and the components of Γ' constitute a G-invariant partition of $V(\Gamma)$ with block size $m + 1$.*

PROOF. The truth of (c) follows from the definition of $\mathcal{L}(\alpha)$. So we need to prove (d) only. Suppose $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ for some vertices $\alpha \neq \beta$, say $\alpha \in B$ and $\beta \in C$. Then $B \neq C$ by (c). Also, $C \in \mathcal{B}(\alpha)$ and $B \in \mathcal{B}(\beta)$, and in particular B, C are adjacent blocks. Moreover, by definition α, β must be mates of each other. Since Γ' is G-symmetric, as mentioned above, G_α is transitive on $\Gamma'(\alpha)$. Thus, for any $\gamma \in \Gamma'(\alpha)$, there exists $g \in G_\alpha$ such that $\beta^g = \gamma$. From $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ we then have $\mathcal{L}(\alpha) = \mathcal{L}(\alpha^g) = (\mathcal{L}(\alpha))^g = (\mathcal{L}(\beta))^g = \mathcal{L}(\beta^g) = \mathcal{L}(\gamma)$. In particular, this implies that each block $B(\delta) \in \mathcal{L}(\alpha) \setminus \{B(\gamma)\}$ contains a mate δ' of γ, where $\delta \in \Gamma'(\alpha) \setminus \{\gamma\}$; and thus any two blocks in $\mathcal{L}(\alpha)$ are adjacent. Again, by the G-symmetry of Γ' there exists $h \in G$ such that $(\alpha, \delta)^h = (\gamma, \delta')$. Hence $(\mathcal{L}(\alpha))^h = \mathcal{L}(\gamma)$ and $(\mathcal{L}(\delta))^h = \mathcal{L}(\delta')$. But $\mathcal{L}(\alpha) = \mathcal{L}(\delta)$ as $\delta \in \Gamma'(\alpha)$, so we have $\mathcal{L}(\delta') = \mathcal{L}(\gamma) = \mathcal{L}(\alpha) = \mathcal{L}(\delta)$, which implies $\delta' = \delta$. Thus, γ and δ are mates of each other and any two vertices in $\Gamma'(\alpha)$ are adjacent in Γ'. Hence $\{\alpha\} \cup \Gamma'(\alpha)$ induces the complete graph K_{m+1}, which must be a connected component of Γ' since Γ' has valency m. Therefore, Γ' is a union of vertex disjoint copies of K_{m+1}. From the proof above, in this case $\mathcal{L}(\gamma) = \mathcal{L}(\delta)$ holds for any vertices γ, δ in the same component of Γ'. Conversely, if Γ' is a union of vertex disjoint copies of K_{m+1}, then it is clear that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ for any vertices α, β in the same component of Γ'. Thus, $\mathcal{L}(\alpha)$ is repeated exactly $m + 1$ times and $\mathcal{L}(\alpha)$ induces a complete subgraph K_{m+1} of $\Gamma_{\mathcal{B}}$. From [12, Proposition 6] it follows that the components of Γ' constitute a G-invariant partition of $V(\Gamma)$. □

Denote by $\mathbf{L}$ the set of all $\mathcal{L}(\alpha)$, $\alpha \in V(\Gamma)$, with repeated ones identified. Then the action of G on $\mathcal{B}$ induces a natural action on $\mathbf{L}$ defined by $(\mathcal{L}(\alpha))^g = \mathcal{L}(\alpha^g)$, for $\alpha \in V(\Gamma)$ and $g \in G$. The subset $\mathbf{L}(B) := \{\mathcal{L}(\alpha) : \alpha \in B\}$ (for $B \in \mathcal{B}$) of $\mathbf{L}$ is G_B-invariant under this action, and thus G_B induces an action on $\mathbf{L}(B)$. It is easily checked that the action of G_B on B is permutationally equivalent to the actions of G_B on $\mathbf{B}(B)$ and $\mathbf{L}(B)$ with respect to the bijections defined by $\alpha \mapsto \mathcal{B}(\alpha)$, $\alpha \mapsto \mathcal{L}(\alpha)$, for $\alpha \in B$, respectively. Thus, we have $G_{B,\mathcal{B}(\alpha)} = G_{B,\mathcal{L}(\alpha)} = G_\alpha$, where $G_{B,\mathcal{B}(\alpha)}$, $G_{B,\mathcal{L}(\alpha)}$ are the setwise stabilizers of $\mathcal{B}(\alpha)$, $\mathcal{L}(\alpha)$ in G_B, respectively. We define

$$\mathcal{D}(\Gamma, \mathcal{B}) := (\mathcal{B}, \mathbf{L}) \tag{1}$$

to be the incidence structure with point set $\mathcal{B}$ and block set $\mathbf{L}$ in which a 'point' B is incident with a 'block' $\mathcal{L}(\alpha)$ if and only if $B \in \mathcal{L}(\alpha)$. Note that the flags of $\mathcal{D}(\Gamma, \mathcal{B})$ of the form $(B(\alpha), \mathcal{L}(\alpha))$ are pairwise distinct. We define

$$\Omega(\Gamma, \mathcal{B}) := \{(B(\alpha), \mathcal{L}(\alpha)) : \alpha \in V(\Gamma)\} \tag{2}$$

to be the set of all such flags.

LEMMA 2.2. *Under the same assumptions as in Lemma* 2.1*, the following statements hold* (*where, in* (c) *and* (d)*, $B \in \mathcal{B}$, $\alpha \in B$ and $C \in \mathcal{B}(\alpha)$*)*.*

- (a) *$\mathcal{D}(\Gamma, \mathcal{B})$ is a* 1*-design of block size $m+1$ which admits G as a point- and block-transitive group of automorphisms.*
- (b) *$\Omega(\Gamma, \mathcal{B})$ is a G-orbit on the set of flags of $\mathcal{D}(\Gamma, \mathcal{B})$, and the actions of G on $V(\Gamma)$ and $\Omega(\Gamma, \mathcal{B})$ are permutationally equivalent with respect to the bijection $\alpha \mapsto (B(\alpha), \mathcal{L}(\alpha))$, for $\alpha \in V(\Gamma)$.*
- (c) *$G_{B,\mathcal{L}(\alpha)} = G_\alpha$ is transitive on $\mathcal{B}(\alpha)$.*
- (d) *$G_{B,C}$ is transitive on $\mathbf{L}(B) \setminus \{\mathcal{L}(\alpha)\}$.*

PROOF. It is clear that G is transitive on $\mathcal{B}$ and on $\mathbf{L}$, and that G preserves the incidence relation of $\mathcal{D}(\Gamma, \mathcal{B})$. So G induces a group of automorphisms of $\mathcal{D}(\Gamma, \mathcal{B})$, and each $B \in \mathcal{B}$ is incident with the same number of elements of $\mathbf{L}$. Clearly, $\mathcal{D}(\Gamma, \mathcal{B})$ has block size $m + 1$, and thus (a) is proved. The assertions in (b) follow immediately from the definition of $\mathcal{D}(\Gamma, \mathcal{B})$ and the action of G on $\mathbf{L}$. To prove (c), let $B \in \mathcal{B}$ and $\alpha \in B$, and let $C, D \in \mathcal{B}(\alpha)$. Let β, γ be the unique mates of α in C and D, respectively. Since Γ' is G-symmetric [12, Proposition 3], there exists $g \in G_\alpha$ such that $\beta^g = \gamma$. This implies $C^g = D$, and hence $G_{B,\mathcal{L}(\alpha)} = G_\alpha$ is transitive on $\mathcal{B}(\alpha)$.

Finally, we prove (d). Let $B \in \mathcal{B}$, $\alpha \in B$ and $C \in \mathcal{B}(\alpha)$. Let δ be the mate of α in C. Since $v = k + 1 \geq 3$, for distinct vertices $\beta, \gamma \in B \setminus \{\alpha\}$ there exist $\varepsilon, \eta \in C \setminus \{\delta\}$ which are adjacent in Γ to β, γ respectively (ε, η are not needed to be distinct). By the G-symmetry of Γ, there exists $g \in G$ such that $(\beta, \varepsilon)^g = (\gamma, \eta)$. So we have $g \in G_{B,C}$ and $(\mathcal{B}(\beta))^g = \mathcal{B}(\gamma)$, and thus $G_{B,C}$ is transitive on $\mathbf{B}(B) \setminus \{\mathcal{B}(\alpha)\}$. Since the actions of G_B on $\mathbf{B}(B)$ and $\mathbf{L}(B)$ are transitive, and are permutationally equivalent with respect to the bijection $\mathcal{B}(\zeta) \mapsto \mathcal{L}(\zeta)$ for $\zeta \in B$, this implies that $G_{B,C}$ is transitive on $\mathbf{L}(B) \setminus \{\mathcal{L}(\alpha)\}$. $\square$

Using the notation in the proof above, since $C \in \mathcal{B}(\alpha)$ and $\mathbf{B}(B)$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$, we have $G_{B,C} \leq G_{B,\mathcal{B}(\alpha)}$. So Lemma 2.2(d) implies that $G_{B,\mathcal{B}(\alpha)}$ is transitive on $\mathbf{B}(B) \setminus \{\mathcal{B}(\alpha)\}$. Since G_B is transitive on $\mathbf{B}(B)$, it follows that G_B is doubly transitive on $\mathbf{B}(B)$, and hence doubly transitive on B and $\mathbf{L}(B)$. This is a restatement of [12, Theorem 5(b)].

2.2. Flag graph construction. For simplicity we assume that the 1-designs used in our construction have no repeated blocks. Let $\mathcal{D}$ be such a 1-design with point set V. As usual we may identify each block L of $\mathcal{D}$ with the subset of V consisting of the points incident with L. Let Ω be a subset of flags of $\mathcal{D}$, and let $\Psi \subseteq \Omega \times \Omega$. We say that Ψ is *self-paired* if $((\sigma, L), (\tau, N)) \in \Psi$ implies $((\tau, N), (\sigma, L)) \in \Psi$. If Ψ is self-paired, then we define the *flag graph* of $\mathcal{D}$ with respect to (Ω, Ψ), denoted by $\Gamma(\mathcal{D}, \Omega, \Psi)$, to be the graph with vertex set Ω in which two 'vertices' $(\sigma, L), (\tau, N) \in \Omega$ are adjacent if and only if $((\sigma, L), (\tau, N)) \in \Psi$. The self-parity of Ψ guarantees that this defines an undirected graph. For a given point σ of $\mathcal{D}$, we denote by $\Omega(\sigma)$ the set of flags of Ω with point entry σ. If Ω is a G-orbit on the flags of $\mathcal{D}$, for some group G of automorphisms of $\mathcal{D}$, then $\Omega(\sigma)$ is a G_σ-orbit on the flags of $\mathcal{D}$ with point entry σ. In this case, $\Gamma(\mathcal{D}, \Omega, \Psi)$ is G-vertex-transitive and its vertex set Ω admits a natural G-invariant partition, namely,

$$\mathcal{B}(\Omega) := \{\Omega(\sigma) : \sigma \in V\}. \tag{3}$$

If furthermore Ψ is a G-orbit on $\Omega \times \Omega$ (under the induced action), then $\Gamma(\mathcal{D}, \Omega, \Psi)$ is G-symmetric. For a flag (σ, L) of $\mathcal{D}$, we use $G_{\sigma,L}$ to denote the subgroup of G fixing (σ, L), that is, the subgroup of G fixing σ and L setwise. For our construction, we require some additional properties to be met by Ω.

DEFINITION 2.3. Let $\mathcal{D}$ be a 1-design which admits a point- and block-transitive group G of automorphisms. Let σ be a point of $\mathcal{D}$. A G-orbit Ω on the flags of $\mathcal{D}$ is said to be *feasible* if the following conditions are satisfied:

(a) $|\Omega(\sigma)| \geq 3$;
(b) $L \cap N = \{\sigma\}$, for distinct $(\sigma, L), (\sigma, N) \in \Omega(\sigma)$;
(c) $G_{\sigma,L}$ is transitive on $L \setminus \{\sigma\}$, for $(\sigma, L) \in \Omega$; and
(d) $G_{\sigma\tau}$ is transitive on $\Omega(\sigma) \setminus \{(\sigma, L)\}$, for $(\sigma, L) \in \Omega$ and $\tau \in L \setminus \{\sigma\}$.

For such a feasible Ω, we say that $((\sigma, L), (\tau, N)) \in \Omega \times \Omega$ is *compatible* with Ω if $\sigma \notin N$, $\tau \notin L$ but $\sigma \in N'$, $\tau \in L'$ for some $(\sigma, L'), (\tau, N') \in \Omega$.

Since G is transitive on the points of $\mathcal{D}$, the validity of (a)–(d) above does not depend on the choice of σ. Let Ω be a feasible G-orbit on the flags of $\mathcal{D}$, and let $((\sigma, L), (\tau, N))$ be compatible with Ω. Since $\sigma \in L$ but $\tau \notin L$, and $\sigma \in L$ but $\sigma \notin N$, we have $\sigma \neq \tau$ and $L \neq N$. Similarly, $L \neq L'$ and $N \neq N'$. (But it may happen that $L' = N'$, see Remark 2.7(a).) Since $\{\sigma, \tau\} \subseteq L'$ and $(\sigma, L') \in \Omega(\sigma)$, the requirement (b) in Definition 2.3 implies that (σ, L') is unique; and similarly (τ, N') is unique. Moreover, for any $(\sigma, L_1), (\tau, N_1) \in \Omega$ with $L_1 \neq L'$ and $N_1 \neq N'$, the ordered pair $((\sigma, L_1), (\tau, N_1))$ is also compatible with Ω. We use $\mathrm{F}(\mathcal{D}, \Omega)$ to denote the set of all ordered pairs of flags of $\mathcal{D}$ which are compatible with Ω. Then $\mathrm{F}(\mathcal{D}, \Omega)$ is a G-invariant subset of $\Omega \times \Omega$. In the following we will consider only those flag graphs $\Gamma(\mathcal{D}, \Omega, \Psi)$ such that $\mathcal{D}$ and G are as in Definition 2.3, Ω is a feasible G-orbit on the flags of $\mathcal{D}$, and Ψ is a self-paired G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$; and to be precise we will call such graphs *G-flag graphs* of $\mathcal{D}$.

Before proceeding to the proof of Theorem 1.1, let us illustrate our construction by examining a simple but important special case. (This case is 'simple' only in the sense that the design involved is degenerate with block size 2.) A 1-design $\mathcal{D}$ with block size 2 can be viewed as a regular graph Σ, and vice versa, if we identify the blocks of $\mathcal{D}$ with the edges of Σ. The automorphism groups of the design $\mathcal{D}$ and the graph Σ are the same. Moreover, under this identification the flag (σ, L) of $\mathcal{D}$, say $L = \{\sigma, \tau\}$, is the arc (σ, τ) of Σ. Hence $\mathcal{D}$ is G-flag-transitive if and only if Σ is G-symmetric, and in this case $\mathcal{D}$ is also G-point-transitive and

G-block-transitive. The following example shows that the G-flag graphs of G-flag-transitive 1-designs $\mathcal{D}$ with block size 2 are precisely the 3-arc graphs of Σ with respect to self-paired G-orbits on $\mathrm{Arc}_3(\Sigma)$, the set of 3-arcs of Σ. In general, for a regular graph Σ and a self-paired subset Δ of $\mathrm{Arc}_3(\Sigma)$, the 3-*arc graph* $\Xi(\Sigma, \Delta)$ of Σ with respect to Δ, as defined in [12, Section 6], is the graph with vertex set $\mathrm{Arc}(\Sigma)$ in which (σ, σ_1), (τ, τ_1) are adjacent if and only if $(\sigma_1, \sigma, \tau, \tau_1) \in \Delta$.

EXAMPLE 2.4. *Three-arc graphs.* A G-flag-transitive 1-design $\mathcal{D}$ with block size 2 can be viewed as a G-symmetric graph Σ, and vice versa. The valency v of Σ is equal to the number of blocks of $\mathcal{D}$ incident with a given point. We assume $v \geq 3$ in the following. Since $\mathcal{D}$ is G-flag-transitive, the only G-orbit on the flags of $\mathcal{D}$ is the set Ω of all flags of $\mathcal{D}$, that is, the arc set $\mathrm{Arc}(\Sigma)$ of Σ. Clearly, Ω satisfies (a)–(c) in Definition 2.3, and the requirement (d) therein is equivalent to requiring that Σ is $(G, 2)$-arc transitive. Therefore, $\mathcal{D}$ has a feasible G-orbit on its flags if and only if Σ is $(G, 2)$-arc transitive, and in this case the only such feasible G-orbit is the flag set Ω of $\mathcal{D}$. The G-invariant partition $\mathcal{B}(\Omega)$ of Ω (defined in (3)) can be identified with the G-invariant partition

$$\mathcal{B}(\Sigma) := \{\mathrm{Arc}(\Sigma; \sigma) : \sigma \in V(\Sigma)\} \tag{4}$$

of $\mathrm{Arc}(\Sigma)$, where $\mathrm{Arc}(\Sigma; \sigma) := \{(\sigma, \eta) : \eta \in \Sigma(\sigma)\}$ is the set of arcs of Σ initiated at σ. Moreover, an ordered pair $((\sigma, L), (\tau, N))$ of flags of $\mathcal{D}$, say $L = \{\sigma, \sigma_1\}$, $N = \{\tau, \tau_1\}$, is compatible with Ω if and only if $(\sigma_1, \sigma, \tau, \tau_1)$ is a 3-arc of Σ. In this case we may identify $((\sigma, L), (\tau, N))$ with $(\sigma_1, \sigma, \tau, \tau_1)$, and thus identify $\mathrm{F}(\mathcal{D}, \Omega)$ with $\mathrm{Arc}_3(\Sigma)$. Hence a self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$ can be identified with a self-paired G-orbit Δ on $\mathrm{Arc}_3(\Sigma)$, and vice versa. Therefore, for a G-flag-transitive 1-design $\mathcal{D}$ with block size 2, a G-flag graph of $\mathcal{D}$ exists if and only if Σ is $(G, 2)$-arc transitive, and in this case $\Gamma(\mathcal{D}, \Omega, \Psi)$ is isomorphic to the 3-arc graph $\Xi(\Sigma, \Delta)$ of Σ with respect to Δ.

2.3. Proof of Theorem 1.1. Now we are ready to prove Theorem 1.1. Suppose that Γ, G and $\mathcal{B}$ are as in the first part of Theorem 1.1. By parts (a) and (b) of Lemma 2.2, $\mathcal{D} := \mathcal{D}(\Gamma, \mathcal{B})$ (as defined in (1)) is a G-point-transitive and G-block-transitive 1-design with block size $m + 1$, and $\Omega := \Omega(\Gamma, \mathcal{B})$ (as defined in (2)) is a G-orbit on the flags of $\mathcal{D}$, where m is the multiplicity of $\mathcal{D}(B)$. It follows from the definition that $\Omega(B) = \{(B, \mathcal{L}(\alpha)) : \alpha \in B\} = \{(B, \mathcal{L}) : \mathcal{L} \in \mathbf{L}(B)\}$, for $B \in \mathcal{B}$. So $|\Omega(B)| = v \geq 3$ and $\mathcal{L} \cap \mathcal{N} = \{B\}$ for distinct $(B, \mathcal{L}), (B, \mathcal{N}) \in \Omega(B)$. Thus Ω satisfies (a) and (b) in Definition 2.3. For $(B, \mathcal{L}) \in \Omega(B)$, say $\mathcal{L} = \mathcal{L}(\alpha)$ for some $\alpha \in B$, we have $\mathcal{L} \setminus \{B\} = \mathcal{B}(\alpha)$ and $\Omega(B) \setminus \{(B, \mathcal{L})\} = \{(B, \mathcal{N}) : \mathcal{N} \in \mathbf{L}(B) \setminus \{\mathcal{L}\}\}$. So it follows from parts (c) and (d) of Lemma 2.2 that Ω satisfies (c) and (d) in Definition 2.3. Therefore, Ω is a feasible G-orbit on the flags of $\mathcal{D}$.

For an arc (α, β) of Γ, the blocks $B := B(\alpha)$ and $C := B(\beta)$ are adjacent in $\Gamma_{\mathcal{B}}$. So there exist $\alpha' \in B$ and $\beta' \in C$ such that α', β' are mates, that is, $B \in \mathcal{B}(\beta')$ and $C \in \mathcal{B}(\alpha')$. Thus, we have $B \in \mathcal{L}(\beta')$, $C \in \mathcal{L}(\alpha')$; and moreover $(B, \mathcal{L}(\alpha')) = (B(\alpha'), \mathcal{L}(\alpha')) \in \Omega$, $(C, \mathcal{L}(\beta')) = (B(\beta'), \mathcal{L}(\beta')) \in \Omega$. It follows from the definition that $B \notin \mathcal{L}(\beta)$ and $C \notin \mathcal{L}(\alpha)$, and therefore we have $((B, \mathcal{L}(\alpha)), (C, \mathcal{L}(\beta))) \in \mathrm{F}(\mathcal{D}, \Omega)$. Set

$$\Psi := \{((B(\alpha), \mathcal{L}(\alpha)), (B(\beta), \mathcal{L}(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\}.$$

Then clearly Ψ is self-paired, and the argument above shows that $\Psi \subseteq \mathrm{F}(\mathcal{D}, \Omega)$. By Lemma 2.2(b), the actions of G on $V(\Gamma)$ and Ω are permutationally equivalent with respect to the bijection $\rho : \gamma \mapsto (B(\gamma), \mathcal{L}(\gamma))$, for $\gamma \in V(\Gamma)$. Since Γ is G-symmetric, this implies that Ψ is a (self-paired) G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$. It is easily checked that ρ defines an isomorphism from Γ to the G-flag graph $\Gamma(\mathcal{D}, \Omega, \Psi)$, and hence the first part of Theorem 1.1 is proved. In addition, from Lemma 2.1(b), if G is faithful on the vertices of Γ, then it is also faithful on the points of $\mathcal{D}$.

Suppose conversely that $\mathcal{D}, G, \Omega, \Psi$ and m are as in the second part of Theorem 1.1. Let $\Gamma := \Gamma(\mathcal{D}, \Omega, \Psi)$, and let $\mathcal{B} := \mathcal{B}(\Omega)$ be as defined in (3). Then by the discussion before Definition 2.3, Γ is a G-symmetric graph with vertex set Ω, and $\mathcal{B}$ is a nontrivial G-invariant partition of Ω with block size $v := |\Omega(\sigma)| \geq 3$, where σ is a point of $\mathcal{D}$. To complete the proof, we need to show that the block size k of the 1-design $\mathcal{D}(\Omega(\sigma))$ induced on the block $\Omega(\sigma)$ of $\mathcal{B}$ satisfies $v = k + 1$, and that the multiplicity of $\mathcal{D}(\Omega(\sigma))$ is equal to m.

Let $\Omega(\sigma), \Omega(\tau)$ be adjacent blocks of $\mathcal{B}$. Then there exist $(\sigma, L) \in \Omega(\sigma)$ and $(\tau, N) \in \Omega(\tau)$ such that $(\sigma, L), (\tau, N)$ are adjacent in Γ, that is, $((\sigma, L), (\tau, N)) \in \Psi$. Since Ψ is a G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, it follows that $\sigma \notin N, \tau \notin L$ and $\sigma \in N', \tau \in L'$ for some $(\sigma, L'), (\tau, N') \in \Omega$. For any $(\sigma, L_1) \in \Omega(\sigma)$, since $\sigma \in N'$, we have $((\sigma, L_1), (\tau, N')) \notin \mathrm{F}(\mathcal{D}, \Omega)$, and hence (τ, N') is not adjacent in Γ to any 'vertex' of $\Omega(\sigma)$. Similarly, (σ, L') is not adjacent in Γ to any 'vertex' of $\Omega(\tau)$. On the other hand, since Ω is feasible and $\tau \in L' \setminus \{\sigma\}$, it follows from Definition 2.3(d) that $G_{\sigma\tau}$ is transitive on $\Omega(\sigma) \setminus \{(\sigma, L')\}$. Thus, for any $(\sigma, L_1) \in \Omega(\sigma) \setminus \{(\sigma, L')\}$, there exists $g \in G_{\sigma\tau}$ such that $(\sigma, L)^g = (\sigma, L_1)$. Since g fixes σ and $\sigma \notin N$, we have $\sigma \notin N^g$. But $\sigma \in N'$, so we have $(\tau, N_1) := (\tau, N)^g \in \Omega(\tau) \setminus \{(\tau, N')\}$, and $(\sigma, L_1), (\tau, N_1)$ are adjacent in Γ. Thus each 'vertex' of $\Omega(\sigma) \setminus \{(\sigma, L')\}$ is adjacent in Γ to at least one 'vertex' of $\Omega(\tau) \setminus \{(\tau, N')\}$. That is, $\Gamma(\Omega(\tau)) \cap \Omega(\sigma) = \Omega(\sigma) \setminus \{(\sigma, L')\}$ and hence $v = k + 1$.

Finally, we prove that the multiplicity of $\mathcal{D}(\Omega(\sigma))$ is equal to m. From Definition 2.3(c), $G_{\sigma,L'}$ is transitive on $L' \setminus \{\sigma\}$. So, for any $\varepsilon \in L' \setminus \{\sigma\}$ there exists $h \in G_{\sigma,L'}$ such that $\tau^h = \varepsilon$. Set $M' := (N')^h$. Then we have $(\varepsilon, M') = (\tau, N')^h \in \Omega$. Since h fixes σ, it fixes $\Omega(\sigma)$ setwise; and moreover $\sigma \in N'$ implies $\sigma \in M'$. Since h also fixes the flag (σ, L'), it must fix $\Omega(\sigma) \setminus \{(\sigma, L')\}$ setwise. Set $L_1 := L^h$ and $N_1 := N^h$. Then $(\sigma, L_1) = (\sigma, L)^h \in \Omega(\sigma) \setminus \{(\sigma, L')\}$, $(\varepsilon, N_1) = (\tau, N)^h \in \Omega(\varepsilon) \setminus \{(\varepsilon, M')\}$, and $(\sigma, L_1), (\varepsilon, N_1)$ are adjacent in Γ. By a similar discussion as in the previous paragraph, one can prove that $\Gamma(\Omega(\varepsilon)) \cap \Omega(\sigma) = \Omega(\sigma) \setminus \{(\sigma, L')\}$. Conversely, if $\Gamma(\Omega(\varepsilon)) \cap \Omega(\sigma) = \Omega(\sigma) \setminus \{(\sigma, L')\}$ for some block $\Omega(\varepsilon)$ of $\mathcal{B}$, then $\Omega(\sigma)$ is adjacent to $\Omega(\varepsilon)$ in $\Gamma_{\mathcal{B}}$, and (σ, L') is the unique 'vertex' of $\Omega(\sigma)$ not adjacent to any 'vertex' of $\Omega(\varepsilon)$. Since $\Omega(\sigma)$ is adjacent to $\Omega(\tau)$, by the G-symmetry of $\Gamma_{\mathcal{B}}$ there exists $z \in G$ such that $(\Omega(\sigma), \Omega(\tau))^z = (\Omega(\sigma), \Omega(\varepsilon))$. This implies $\tau^z = \varepsilon$. Moreover, since (σ, L') is the unique 'vertex' of $\Omega(\sigma)$ not adjacent to any 'vertex' of $\Omega(\tau)$, as shown in the previous paragraph, we must have $z \in G_{\sigma,L'}$. This together with $\tau \in L' \setminus \{\sigma\}$ implies that $\varepsilon = \tau^z \in L' \setminus \{\sigma\}$. In summary, we have proved that $\Gamma(\Omega(\varepsilon)) \cap \Omega(\sigma) = \Omega(\sigma) \setminus \{(\sigma, L')\}$ if and only if $\varepsilon \in L' \setminus \{\sigma\}$. Therefore, the multiplicity of $\mathcal{D}(\Omega(\sigma))$ is equal to $|L' \setminus \{\sigma\}| = m$. In addition, if an element of G fixes each flag in Ω, then it must fix each point of $\mathcal{D}$. So, if G is faithful on the points of $\mathcal{D}$, then it is faithful on the vertices of Γ. This completes the proof of Theorem 1.1, as well as that of the statement immediately following it.

2.4. Corollaries and remarks. From the proof above, one can see that the graph Γ' defined in the second paragraph of Section 2.1 is isomorphic to the flag graph $\Gamma(\mathcal{D}, \Omega, \Psi')$, where $\mathcal{D} := \mathcal{D}(\Gamma, \mathcal{B})$, $\Omega := \Omega(\Gamma, \mathcal{B})$ and $\Psi' := \{((B(\alpha'), \mathcal{L}(\alpha')), (B(\beta'), \mathcal{L}(\beta'))) : (\alpha', \beta') \in \mathrm{Arc}(\Gamma')\}$. Note that $\mathcal{D}$ has block size one larger than the multiplicity of $\mathcal{D}(B)$ (Lemma 2.2(a)). So $\mathcal{D}(B)$ contains no repeated blocks if and only if $\mathcal{D}$ has block size 2. In this case we may identify $\mathcal{D}$ with the quotient graph $\Gamma_{\mathcal{B}}$ by identifying each block $\{B, C\}$ of $\mathcal{D}$ with the edge of $\Gamma_{\mathcal{B}}$ joining B and C. Thus Theorem 1.1 and the discussion in Example 2.4 imply the following result.

COROLLARY 2.5 ([12, THEOREM 1]). *Let Γ be a finite G-symmetric graph, and $\mathcal{B}$ a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v \geq 3$ such that $\mathcal{D}(B)$ has block size $v-1$. Then $\mathcal{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. In this case $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired G-orbit Δ of 3-arcs of $\Gamma_{\mathcal{B}}$. Conversely, for any self-paired G-orbit Δ of 3-arcs of a $(G, 2)$-arc transitive graph Σ of valency $v \geq 3$, the graph $\Gamma = \Xi(\Sigma, \Delta)$, group G, and partition $\mathcal{B}(\Sigma)$ (defined in Example 2.4) satisfy all the conditions above.*

The case where $v = k + 1 \geq 3$ and $\Gamma_{\mathcal{B}}$ is a complete graph is of particular interest. In this case we have $\Gamma_{\mathcal{B}} \cong K_{mv+1}$ as $\mathrm{val}(\Gamma_{\mathcal{B}}) = mv$ (Lemma 2.1(a)). Since $\Gamma_{\mathcal{B}}$ is G-symmetric, this occurs precisely when G is doubly transitive on $\mathcal{B}$. So in this case $\mathcal{D}(\Gamma, \mathcal{B})$ is a G-doubly transitive and G-block-transitive 2-$(mv+1, m+1, \lambda)$ design, for some integer $\lambda \geq 1$. Conversely, if $\mathcal{D}$ is a G-doubly transitive and G-block-transitive 2-$(mv+1, m+1, \lambda)$ design, then for any G-flag graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$ of $\mathcal{D}$, we have $\Gamma_{\mathcal{B}(\Omega)} \cong K_{mv+1}$. So Theorem 1.1 has the following consequence.

COROLLARY 2.6. *Let $v \geq 3$ and $m \geq 1$ be integers, and let G be a group. Then the following statements are equivalent.*

(a) *Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size v such that $\mathcal{D}(B)$ has block size $v-1$ and $\Gamma_{\mathcal{B}} \cong K_{mv+1}$.*
(b) *$\Gamma \cong \Gamma(\mathcal{D}, \Omega, \Psi)$, for a G-doubly transitive and G-block-transitive 2-$(mv+1, m+1, \lambda)$ design $\mathcal{D}$, a feasible G-orbit Ω on the flags of $\mathcal{D}$, and a self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$.*

Moreover, the integer m above is equal to the multiplicity of $\mathcal{D}(B)$, and G is faithful on $V(\Gamma)$ if and only if it is faithful on the points of $\mathcal{D}$.

A *linear space* [1] is an incidence structure of points and blocks, called *lines*, in which any two distinct points are incident with exactly one line, any point is incident with at least two lines, and any line with at least two points. We conclude this section by making the following remarks.

REMARK 2.7. (a) A G-doubly transitive linear space $\mathcal{D}$ must be G-block-transitive and G-flag-transitive, and hence the only G-orbit on the flags of $\mathcal{D}$ is the flag set Ω of $\mathcal{D}$. In this case Ω satisfies (b) and (c) in Definition 2.3 automatically. Hence Ω is feasible if and only if any point is incident with at least three lines and, for distinct points σ, τ, $G_{\sigma\tau}$ is transitive on the lines incident with σ but not τ. Note that in this case we have $\mathrm{F}(\mathcal{D}, \Omega) = \{((\sigma, L), (\tau, N)) : (\sigma, L), (\tau, N) \in \Omega, \sigma \notin N, \tau \notin L\}$. Also, for the members $(\sigma, L'), (\tau, N')$ of Ω such that $\sigma \in N'$, $\tau \in L'$, we have $L' = N' = L_{\sigma\tau}$, where $L_{\sigma\tau}$ is the unique line incident with both σ and τ.

(b) Conversely, if the flag set of a G-flag-transitive 2-design $\mathcal{D}$ is feasible, then $\mathcal{D}$ is forced to be a linear space.

3. PROJECTIVE AND AFFINE FLAG GRAPHS

As a result of the classification of finite simple groups, all doubly transitive linear spaces are known [11, Theorem 1]. Thus, by using our flag graph construction, it seems possible to classify the G-flag graphs $\Gamma(\mathcal{D}, \Omega, \Psi)$ appearing in Corollary 2.6 for G-doubly transitive linear spaces $\mathcal{D}$, and this will contribute to the classification of all the graphs Γ therein. As an effort towards achieving this goal, we will classify in this section such graphs for the following typical doubly transitive linear spaces:

(i) the projective geometry $\mathrm{PG}(d-1,q)$ $(d \geq 2)$;
(ii) the affine geometry $\mathrm{AG}(d,q)$ $(d \geq 2)$; and
(iii) the *trivial* doubly transitive linear space, namely the complete 2-$(v+1,2,1)$ design.

For G a doubly transitive subgroup of $\mathrm{P\Gamma L}(d,q)$, we will characterize the G-flag graphs of $\mathrm{PG}(d-1,q)$ as the only G-flag graphs arising from any 2-design admitting G as a faithful, doubly transitive and block-transitive group of automorphisms. For the G-flag graphs of $\mathrm{AG}(d,q)$, where $\mathrm{AGL}(d,q) \leq G \leq \mathrm{A\Gamma L}(d,q)$, we have a similar characterization. From these and Corollary 2.6, the result in Theorem 1.2 then follows.

From Example 2.4, the G-flag graphs arising from (iii) above are precisely the 3-arc graphs $\Xi(\Sigma,\Delta)$ of complete $(G,2)$-arc transitive graphs $\Sigma := K_{v+1}$. In this case G is 3-transitive on $V(\Sigma)$, and is one of the groups that we will list at the beginning of Section 3.3. In particular, if Δ contains a 3-cycle of Σ, then Δ is the set of all 3-cycles of Σ. Hence $\Xi(\Sigma,\Delta) = (v+1)\cdot K_v$ ([12, Theorem 8(b)]) and $\Xi(\Sigma,\Delta)$ is an almost cover of Σ. So in the following discussion for this case, we may suppose that each 3-arc in Δ is *proper* in the sense that it is not a 3-cycle.

3.1. Projective flag graphs. Let $d \geq 2$ and $q = p^e$ with p a prime and $e \geq 1$. As usual we use the same notation for the projective geometry $\mathrm{PG}(d-1,q)$ and the (point, line)-incidence structure of $\mathrm{PG}(d-1,q)$. Let $V(d,q)$ be the d-dimensional linear space of row vectors over $\mathrm{GF}(q)$, and V the set of points of $\mathrm{PG}(d-1,q)$. Then $V = \{[\mathbf{x}] : \mathbf{x} \in V(d,q) \setminus \{\mathbf{0}\}\}$, where $[\mathbf{x}]$ is the point of $\mathrm{PG}(d-1,q)$ representing nonzero multiples of the vector $\mathbf{x}$. Moreover, $|V| = (q^d-1)/(q-1)$, and any group G with $\mathrm{PSL}(d,q) \leq G \leq \mathrm{P\Gamma L}(d,q)$ acts doubly transitively on V (see e.g., [6, p. 245]). By [11, Theorem 1], any G-doubly transitive linear space $\mathcal{D}$ with point set V (under the natural action) is either $\mathrm{PG}(d-1,q)$ for $d \geq 3$, or the trivial G-doubly transitive linear space. As mentioned above, in the latter case G is required to be 3-transitive on V and thus $d = 2$ (see e.g., [3, p. 8]), and the G-flag graphs of $\mathcal{D}$ are the 3-arc graphs $\Xi(\Sigma,\Delta)$ of $\Sigma := K_{q+1}$, where Σ has vertex set V and Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. These graphs $\Xi(\Sigma,\Delta)$ were classified in [10], and for completeness we describe them in the following.

Let $d = 2$ and let $\Sigma = K_{q+1}$ be as above. Then we may identify V with $\mathrm{GF}(q) \cup \{\infty\}$ and thus we have

$$\mathrm{Arc}(\Sigma) = \{yz : y, z \in \mathrm{GF}(q) \cup \{\infty\}, y \neq z\}$$

where ∞ satisfies the usual arithmetic rules such as $1/\infty = 0$, $\infty + y = \infty$, $\infty^p = \infty$, etc. The projective group $\mathrm{PGL}(2,q)$ consists of all Möbius transformations

$$t_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d} \quad (a,b,c,d \in \mathrm{GF}(q), ad-bc \neq 0)$$

of $\mathrm{GF}(q) \cup \{\infty\}$ (see e.g., [13, p. 20–21]), and $\mathrm{P\Gamma L}(2,q)$ is equal to the semidirect product $\mathrm{PGL}(2,q)\cdot\langle\psi\rangle$, where ψ is the Frobenius mapping defined by $\psi : x \mapsto x^p$, for $x \in \mathrm{GF}(q) \cup \{\infty\}$. From [10, Theorem 2.1], the 3-transitive subgroups of $\mathrm{P\Gamma L}(2,q)$ are the groups $\mathrm{PGL}(2,q)\cdot\langle\psi^s\rangle$, for some divisor s of e; and $\mathrm{M}(s,q) := \langle \mathrm{PSL}(2,q), \psi^s t_{a,0,0,1}\rangle$, where p is odd, e is even, s is a divisor of $e/2$, and a is a primitive element of $\mathrm{GF}(q)$.

For distinct elements $u, w, y, z \in \mathrm{GF}(q) \cup \{\infty\}$, the *cross ratio* (see e.g., [13, p. 59]) is defined as $c(u,w;y,z) := (u-y)(w-z)/(u-z)(w-y)$. The cross ratio can take all values in $\mathrm{GF}(q)$ except 0 and 1, and is invariant under the action of $\mathrm{PGL}(2,q)$ on quadruples of distinct elements of $\mathrm{GF}(q) \cup \{\infty\}$. Moreover, $\mathrm{PGL}(2,q)$ is transitive on such quadruples with a fixed cross ratio (see e.g., [13, p. 59]). Under the action of ψ, we have $c(u^\psi, w^\psi; y^\psi, z^\psi) = (c(u,w;y,z))^\psi$.

EXAMPLE 3.1. *Cross ratio graphs.* For each $x \in \mathrm{GF}(q) \setminus \{0, 1\}$, the subfield of $\mathrm{GF}(q)$ generated by x has the form $\mathrm{GF}(p^{s(x)})$, for some divisor $s(x)$ of e. For each divisor s of $s(x)$, the $\langle\psi^s\rangle$-orbit on $\mathrm{GF}(p^{s(x)})$ containing x is $B(x, s) := \{x^{\psi^{si}} : 0 \le i < s(x)/s\}$. The *untwisted cross ratio graph* $\mathrm{CR}(q; x, s)$, as defined in [10, Definition 3.2], is the graph with vertex set $\mathrm{Arc}(\Sigma)$ in which (u, w) and (y, z) are adjacent if and only if u, w, y, z are distinct elements of $\mathrm{GF}(q) \cup \{\infty\}$ and $c(u, w; y, z) \in B(x, s)$. Since $c(\infty, 0; 1, x) = x \in B(x, s)$, by the above-mentioned properties of cross ratio, (u, w) and (y, z) are adjacent in $\mathrm{CR}(q; x, s)$ if and only if $(w, u, y, z) \in (0, \infty, 1, x)^G$ for $G := \mathrm{PGL}(2, q) \cdot \langle\psi^s\rangle$. Thus $\mathrm{CR}(q; x, s)$ can be defined equivalently as the 3-arc graph $\Xi(\Sigma, \Delta)$ of Σ with respect to the self-paired G-orbit $\Delta := (0, \infty, 1, x)^G$ on $\mathrm{Arc}_3(\Sigma)$. (The G-orbit Δ is self-paired since $t_{1,-x,1,-1} \in G$ interchanges $\infty 0$ and $1x$.)

Now let p be an odd prime and e an even integer, and let $x \in \mathrm{GF}(q) \setminus \{0, 1\}$ be such that $s(x)$ is even and $x - 1$ is a square of $\mathrm{GF}(q)$. Let s be an even divisor of $s(x)$. The *twisted cross ratio graph* $\mathrm{TCR}(q; x, s)$, as defined in [10, Definition 3.4], is the graph with vertex set $\mathrm{Arc}(\Sigma)$ and arc set $(\infty 0, 1x)^{\mathrm{M}(s/2,q)}$. In other words, $\mathrm{TCR}(q; x, s)$ is the 3-arc graph $\Xi(\Sigma, \Delta)$ of Σ with respect to the self-paired $\mathrm{M}(s/2, q)$-orbit $\Delta := (0, \infty, 1, x)^{\mathrm{M}(s/2,q)}$ on $\mathrm{Arc}_3(\Sigma)$. (Note that $x - 1$ is a square implies that $t_{1,-x,1,-1} \in \mathrm{PSL}(2, q) \le \mathrm{M}(s/2, q)$. So Δ is self-paired by the same reason as in the last paragraph.) From the properties of cross ratio mentioned before this example, one can see that (y, z) is adjacent to $(\infty, 0)$ in $\mathrm{TCR}(q; x, s)$ precisely when $y \in \mathrm{GF}(q) \setminus \{0\}$ and $z/y \in B(x, s)$ or $z/y \in B(x, s)^{\psi^{s/2}}$, depending on whether y is a square or not.

From the discussion in Example 2.4 we know that the untwisted and twisted cross ratio graphs above admit the G-invariant partition $\mathcal{B}(\Sigma) := \{B(y) : y \in \mathrm{GF}(q) \cup \{\infty\}\}$, for suitable 3-transitive subgroups G of $\mathrm{P\Gamma L}(2, q)$, where $B(y) := \{(y, z) : z \in \mathrm{GF}(q) \cup \{\infty\}, y \ne z\}$. It was proved in [10, Theorem 5.1] that they are the only G-symmetric graphs with vertex set $\mathrm{Arc}(\Sigma)$ such that the block size of $\mathcal{D}(B(y))$ is $q - 1$. Therefore, they are the only 3-arc graphs of Σ with respect to some self-paired G-orbits on $\mathrm{Arc}_3(\Sigma)$. Moreover, for $(\Gamma, G) = (\mathrm{CR}(q; x, s), \mathrm{PGL}(2, q) \cdot \langle\psi^s\rangle)$ or $(\mathrm{TCR}(q; x, s), \mathrm{M}(s/2, q))$, the only 3-transitive subgroups H of $\mathrm{P\Gamma L}(2, q)$ such that Γ is H-symmetric are subgroups of G of the form $\mathrm{PGL}(2, q) \cdot \langle\psi^t\rangle$ or $\mathrm{M}(t/2, q)$ respectively, for some divisor t of e such that the greatest common divisor $\gcd(s(x), t)$ is equal to s. (See the comment immediately following [10, Theorems 5.1].) From the adjacency of Γ, one can see that in both cases Γ has valency $(q - 1)s(x)/s$ and, for distinct blocks $B(u)$, $B(y)$ of $\mathcal{B}(\Sigma)$, the bipartite subgraph $\Gamma[B(u), B(y)]$ has valency $s(x)/s$. In particular, Γ is an almost cover of Σ if and only if $s = s(x)$; and if this occurs then the integer t in H is a multiple of $s(x)$ as $\gcd(s(x), t) = s(x)$.

The reader is referred to [7] for two other interesting graphs, also relating to $\mathrm{GF}(q) \cup \{\infty\}$, which are connected 2-arc transitive 4-fold covers of $\Sigma = K_{q+1}$. They were discovered by Du, Marusic and Waller in their classification of a family of 2-arc transitive covers of complete graphs.

Now let us turn to the case where $d \ge 3$. In this case $\mathrm{PG}(d - 1, q)$ is a linear space with $mv + 1 := (q^d - 1)/(q - 1)$ points such that each line contains $m + 1 := q + 1$ points. So we have $v = (q^{d-1} - 1)/(q - 1)$ and $m = q$. For $1 \le s \le d - 1$, any $s + 1$ points of $\mathrm{PG}(d - 1, q)$ are said to be *independent* [15, p. 72] if they do not lie on any $(s - 1)$-flat of $\mathrm{PG}(d - 1, q)$. In particular, three points of $\mathrm{PG}(d - 1, q)$ are *noncollinear* if they are independent, and *collinear* otherwise. We will exploit the following basic result in projective geometry, a proof of which can be derived from [5, 1.4.24].

LEMMA 3.2. *Suppose* $\mathrm{PSL}(d, q) \le G \le \mathrm{P\Gamma L}(d, q)$, *where* $d \ge 3$ *and* q *is a prime power. Then, for any integer* s *with* $1 \le s \le d - 1$, G *is transitive on the set of ordered* $(s + 1)$*-tuples of independent points of* $\mathrm{PG}(d - 1, q)$.

Let $\Omega(P; d, q)$ denote the set of flags (that is, (point, line)-flags) of $\mathrm{PG}(d-1, q)$. In the following lemma we will show that $\Omega(P; d, q)$ is feasible. Thus, setting $\mathrm{F}(P; d, q) := \mathrm{F}(\mathrm{PG}(d-1, q), \Omega(P; d, q))$, from Remark 2.7(a) we have

$$\mathrm{F}(P; d, q) = \{((\sigma, L), (\tau, N)) : (\sigma, L), (\tau, N) \in \Omega(P; d, q), \sigma \notin N, \tau \notin L\}.$$

We call two distinct lines L, N of $\mathrm{PG}(d-1, q)$ *intersecting* if there exists a unique point incident with both L and N (that is, L, N lie on the same plane of $\mathrm{PG}(d-1, q)$), and *skew* otherwise. We use $\Psi^{+}(P; d, q)$ (respectively, $\Psi^{\simeq}(P; d, q)$) to denote the set of ordered pairs $((\sigma, L), (\tau, N)) \in \mathrm{F}(P; d, q)$ such that L, N are intersecting (respectively, skew). Here we use '$+$' and '$\simeq$' to symbolise relative positions of L and N. Clearly, $\Psi^{+}(P; d, q)$ and $\Psi^{\simeq}(P; d, q)$ consist of a partition of $\mathrm{F}(P; d, q)$. Note that $\Psi^{\simeq}(P; d, q) \neq \emptyset$ if and only if $d \geq 4$ (see e.g., [15, p.71]). So we have $\mathrm{F}(P; 3, q) = \Psi^{+}(P; 3, q)$.

LEMMA 3.3. *Suppose* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, *where* $d \geq 3$ *and* q *is a prime power.*

(a) *There exists a unique feasible* G*-orbit on the flags of* $\mathrm{PG}(d-1, q)$, *namely* $\Omega(P; d, q)$.
(b) *If* $d = 3$, *then* G *is transitive on* $\mathrm{F}(P; 3, q)$; *if* $d \geq 4$, *then* G *has two orbits on* $\mathrm{F}(P; d, q)$, *namely* $\Psi^{+}(P; d, q)$ *and* $\Psi^{\simeq}(P; d, q)$.

PROOF. (a) Since $\mathrm{PG}(d-1, q)$ is a G-doubly transitive linear space, it is G-flag-transitive, and hence $\Omega(P; d, q)$ is the only candidate for a feasible G-orbit on the flags of $\mathrm{PG}(d-1, q)$. In $\mathrm{PG}(d-1, q)$ each point is incident with $(q^{d-1}-1)/(q-1) \geq 3$ lines ([15, Theorem 2.5(iii)]). For distinct points σ, τ, let N_1, N_2 be two lines incident with σ but not τ, and let $\delta_i \in N_i \setminus \{\sigma\}$, $i = 1, 2$. Then (σ, τ, δ_1), (σ, τ, δ_2) are triples of noncollinear points. So by Lemma 3.2 there exists $g \in G$ such that $(\sigma, \tau, \delta_1)^g = (\sigma, \tau, \delta_2)$, and hence $g \in G_{\sigma\tau}$. Since N_i is the unique line incident with σ and δ_i, this implies $N_1^g = N_2$, and hence $\Omega(P; d, q)$ is feasible by Remark 2.7(a).

(b) Let $((\sigma_1, L_1), (\tau_1, N_1)), ((\sigma_2, L_2), (\tau_2, N_2)) \in \Psi^{+}(P; d, q)$. Let δ_i be the common point of L_i and N_i, for $i = 1, 2$. Then $(\sigma_1, \tau_1, \delta_1)$, $(\sigma_2, \tau_2, \delta_2)$ are triples of noncollinear points. By Lemma 3.2 we have $(\sigma_1, \tau_1, \delta_1)^g = (\sigma_2, \tau_2, \delta_2)$ for some $g \in G$. This implies $((\sigma_1, L_1), (\tau_1, N_1))^g = ((\sigma_2, L_2), (\tau_2, N_2))$, and hence G is transitive on $\Psi^{+}(P; d, q)$. Similarly, for $((\sigma_1, L_1), (\tau_1, N_1)), ((\sigma_2, L_2), (\tau_2, N_2)) \in \Psi^{\simeq}(P; d, q)$, we can choose $\sigma_i' \in L_i \setminus \{\sigma_i\}$ and $\tau_i' \in N_i \setminus \{\tau_i\}$, for $i = 1, 2$. So $(\sigma_1', \sigma_1, \tau_1, \tau_1')$, $(\sigma_2', \sigma_2, \tau_2, \tau_2')$ are quadruples of independent points of $\mathrm{PG}(d-1, q)$. Again by Lemma 3.2 we have $(\sigma_1', \sigma_1, \tau_1, \tau_1')^g = (\sigma_2', \sigma_2, \tau_2, \tau_2')$ for some $g \in G$. This implies $((\sigma_1, L_1), (\tau_1, N_1))^g = ((\sigma_2, L_2), (\tau_2, N_2))$, and hence G is transitive on $\Psi^{\simeq}(P; d, q)$. Since G preserves relative positions between lines and since $\Psi^{+}(P; d, q)$ and $\Psi^{\simeq}(P; d, q)$ consist of a partition of $\mathrm{F}(P; d, q)$, the assertions in (b) follow immediately. $\square$

By definition both $\Psi^{+}(P; d, q)$ and $\Psi^{\simeq}(P; d, q)$ are self-paired. Hence the following graphs are well-defined.

DEFINITION 3.4. The flag graphs of $\mathrm{PG}(d-1, q)$ with respect to $(\Omega(P; d, q), \Psi^{+}(P; d, q))$ and $(\Omega(P; d, q), \Psi^{\simeq}(P; d, q))$ are called *projective flag graphs*, denoted by $\Gamma^{+}(P; d, q)$ and $\Gamma^{\simeq}(P; d, q)$, respectively.

Note that we require $d \geq 4$ in defining $\Gamma^{\simeq}(P; d, q)$. From Lemma 3.3, $\Gamma^{+}(P; d, q)$ and $\Gamma^{\simeq}(P; d, q)$ are the only G-flag graphs of $\mathrm{PG}(d-1, q)$. Moreover, we have the following characterization of them.

LEMMA 3.5. *Suppose* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, *where* $d \geq 3$ *and* q *is a prime power. Suppose further that* $\mathcal{D}$ *is a* 2*-design, other than the trivial linear space, which admits* G *as a faithful, doubly transitive, and block-transitive group of automorphisms. Then any* G*-flag graph of* $\mathcal{D}$ *is isomorphic to* $\Gamma^{+}(P; d, q)$ *or* $\Gamma^{\simeq}(P; d, q)$.

PROOF. The group G has only two faithful permutation representations, namely the natural actions on the points and the hyperplanes of $\mathrm{PG}(d-1, q)$. Such representations are interchangable by an outer automorphism of $\mathrm{P\Gamma L}(d, q)$. So in the following it suffices to consider the usual action of G on the point set V of $\mathrm{PG}(d-1, q)$. Thus we may suppose that $\mathcal{D}$ has point set V.

Since $\mathcal{D}$ is G-doubly transitive and is not the trivial linear space, its block size is at least three. Suppose Ω is a feasible G-orbit on the flags of $\mathcal{D}$, and let $(\sigma, L) \in \Omega$. The double transitivity of G on V implies that, for any $\varepsilon \in V \setminus \{\sigma\}$, there exists a flag $(\sigma, N) \in \Omega$ such that $\varepsilon \in N$. This, together with the requirement (b) in Definition 2.3 and the fact that $\Omega(\sigma)$ is a G_σ-orbit on the flags of $\mathcal{D}$ with point entry σ, implies the following claim:

(i) $\{N \setminus \{\sigma\} : (\sigma, N) \in \Omega\}$ is a G_σ-invariant partition of $V \setminus \{\sigma\}$.

We claim further that:

(ii) For any $\tau, \delta \in L \setminus \{\sigma\}$, the points σ, τ, δ must be collinear in $\mathrm{PG}(d-1, q)$.

Suppose otherwise, and let ε be a point in a block N of $\mathcal{D}$ with $(\sigma, N) \in \Omega(\sigma)$ and $N \neq L$. Then in $\mathrm{PG}(d-1, q)$ either $\sigma, \tau, \varepsilon$ are noncollinear, or $\sigma, \delta, \varepsilon$ are noncollinear, since otherwise σ, τ, δ would be collinear, which contradicts our assumption. Without loss of generality we may suppose that $\sigma, \tau, \varepsilon$ are noncollinear in $\mathrm{PG}(d-1, q)$. Then by Lemma 3.2 there exists $g \in G$ such that $(\sigma, \tau, \delta)^g = (\sigma, \tau, \varepsilon)$. So we have $g \in G_{\sigma\tau}$. Since g fixes τ, by (i) it must fix L setwise. On the other hand, since g maps δ to ε, again from (i), g must map L to N. This is a contradiction and hence (ii) is proved. From this it follows that, for each $(\sigma, L) \in \Omega$, the block L of $\mathcal{D}$ consists of some collinear points of $\mathrm{PG}(d-1, q)$. Moreover, we have:

(iii) For each $(\sigma, L) \in \Omega$, the block L of $\mathcal{D}$ is a line of $\mathrm{PG}(d-1, q)$.

Suppose otherwise, then from (i), (ii) there exists $(\sigma, N_1) \in \Omega$ such that the points of L and N_1 lie on the same line, say L^*, of $\mathrm{PG}(d-1, q)$. Since $d \geq 3$, we can take $(\sigma, N_2) \in \Omega$ such that the points in L and those in N_2 do not lie on the same line of $\mathrm{PG}(d-1, q)$. Take a point $\tau \in L \setminus \{\sigma\}$. Since Ω is feasible, by the requirement (d) in Definition 2.3, there exists $g \in G_{\sigma\tau}$ such that $N_1^g = N_2$. Since g fixes σ and τ, it must fix the line L^* of $\mathrm{PG}(d-1, q)$. Hence the points in N_1 are mapped by g to some points on L^*. That is, the points in N_2 must lie on L^*. This is a contradiction and hence (iii) is proved.

The claims (i) and (iii) together imply that $\Omega(\sigma) = \Omega(P; d, q)(\sigma)$ for each $\sigma \in V$. So we have $\Omega = \Omega(P; d, q)$. In particular each line of $\mathrm{PG}(d-1, q)$ is a block of $\mathcal{D}$. Thus it follows from the definition that $\mathrm{F}(\mathcal{D}, \Omega) = \mathrm{F}(P; d, q)$. From Lemma 3.3(b), the result in Lemma 3.5 follows. $\square$

Applying Corollary 2.6, the discussion above leads to the following classification theorem, which is the main result in this subsection.

THEOREM 3.6. *Suppose* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, *where* $d \geq 2$ *and* $q = p^e$ *with* p *a prime and* $e \geq 1$. *Then, if and only if either* $d \geq 3$, *or* $d = 2$ *and* G *is* 3*-transitive, there exists a* G*-symmetric graph* Γ *with* G *faithful on* $V(\Gamma)$ *which admits a nontrivial* G*-invariant partition* $\mathcal{B}$ *such that* $v = k + 1 \geq 3$ *and* $\Gamma_{\mathcal{B}} \cong K_{mv+1}$, *where* m *is the multiplicity of* $\mathcal{D}(B)$. *Moreover, all the possibilities of such* Γ, G *and the corresponding* m, v *can be classified as follows.*

(a) $\Gamma = (q+1) \cdot K_q$, G *is* $\mathrm{PGL}(2, q) \cdot \langle \psi^s \rangle$ (*where* s *is a divisor of* e) *or* $\mathrm{M}(s, q)$ (*where* q *is odd,* e *is even and* s *is a divisor of* $e/2$), *and* $(m, v) = (1, q)$.
(b) $(\Gamma, G) = (\mathrm{CR}(q; x, s), \mathrm{PGL}(2, q) \cdot \langle \psi^t \rangle)$ *and* $(m, v) = (1, q)$, *where* $x \in \mathrm{GF}(q) \setminus \{0, 1\}$, s *is a divisor of* $s(x)$, *and* t *is a divisor of* e *with* $\gcd(s(x), t) = s$.

(c) $(\Gamma, G) = (\mathrm{TCR}(q; x, s), \mathrm{M}(t/2, q))$ *and* $(m, v) = (1, q)$*, where* p *is odd,* $e \geq 2$ *is even,* $x \in \mathrm{GF}(q) \setminus \{0, 1\}$ *with* $s(x)$ *even and* $x - 1$ *a square of* $\mathrm{GF}(q)$*,* s *is an even divisor of* $s(x)$*, and* t *is a divisor of* e *with* $\gcd(s(x), t) = s$*.*

(d) Γ *is either* $\Gamma^{+}(P; d, q)$ *or* $\Gamma^{\simeq}(P; d, q)$*, where* $d \geq 3$*,* G *is any doubly transitive subgroup of* $\mathrm{P\Gamma L}(d, q)$*, and* $(m, v) = (q, (q^{d-1} - 1)/(q - 1))$*.*

Note that the graph $\Gamma^{\simeq}(P; d, q)$ in (d) above appears only when $d \geq 4$. We conclude this subsection by proving the following properties of the projective flag graphs. As before, we denote by $L_{\sigma\tau}$ the unique line of $\mathrm{PG}(d - 1, q)$ through two distinct points σ and τ.

THEOREM 3.7. *Let* $d \geq 3$ *and* q *a prime power. Let* $\Omega = \Omega(P; d, q)$ *and* $\mathcal{B}(\Omega) = \{\Omega(\sigma) : \sigma$ a point of $\mathrm{PG}(d - 1, q)\}$ *as in* (3)*. Then the following statements hold.*

(a) $\Gamma^{+}(P; d, q)$ *and* $\Gamma^{\simeq}(P; d, q)$ *are connected graphs with diameter two and girth three, and with valencies* $(q^{d+1} - q^3)/(q - 1)$ *and* $(q^{d-1} - q^2)(q^d - q^2)/(q - 1)^2$*, respectively.*

(b) *For distinct blocks* $\Omega(\sigma), \Omega(\tau)$ *of* $\mathcal{B}(\Omega)$*, each vertex of* $\Omega(\sigma)$ *other than* $(\sigma, L_{\sigma\tau})$ *is adjacent to exactly* q *vertices of* $\Omega(\tau)$ *in* $\Gamma^{+}(P; d, q)$*, and adjacent to exactly* $(q^{d-1} - q^2)/(q - 1)$ *vertices of* $\Omega(\tau)$ *in* $\Gamma^{\simeq}(P; d, q)$*. In particular, for* $\Gamma := \Gamma^{+}(P; 3, q)$ *we have* $\Gamma[\Omega(\sigma), \Omega(\tau)] \cong K_{q,q}$*.*

(c) *For* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$*, any* G*-symmetric graph with vertex set* Ω (*under the induced action of* G *on* Ω) *is isomorphic to either* $\Gamma^{+}(P; d, q)$*, or* $\Gamma^{\simeq}(P; d, q)$*, or* $((q^d - 1)/(q - 1)) \cdot K_{(q^{d-1}-1)/(q-1)}$ *with connected components the sets of flags incident with a common point, or* $((q^{d-1} - 1)(q^d - 1)/(q - 1)(q^2 - 1)) \cdot K_{q+1}$ *with connected components the sets of flags incident with a common line.*

PROOF. Let $(\sigma, L), (\tau, N) \in \Omega$ be distinct flags of $\mathrm{PG}(d-1, q)$. If $L \neq N$ then, since each line of $\mathrm{PG}(d-1, q)$ contains $q+1 \geq 3$ points, we can take $\delta \in L \setminus \{\sigma, \tau\}$, $\varepsilon \in N \setminus \{\sigma, \tau\}$ and $\eta \in L_{\delta\varepsilon} \setminus \{\delta, \varepsilon\}$. One can check that the sequence $(\sigma, L), (\eta, L_{\delta\varepsilon}), (\tau, N)$ is a path of $\Gamma^{+}(P; d, q)$ with length two. In particular, if $(\sigma, L), (\tau, N)$ are adjacent in $\Gamma^{+}(P; d, q)$, then the sequence $(\sigma, L), (\eta, L_{\delta\varepsilon}), (\tau, N), (\sigma, L)$ is a triangle. Similarly, if $\sigma \neq \tau$ but $L = N$, then we can take $\delta \in L \setminus \{\sigma, \tau\}$ and a point ε not incident with L. Thus the sequence $(\sigma, L), (\varepsilon, L_{\delta\varepsilon}), (\tau, L)$ is a path of $\Gamma^{+}(P; d, q)$ with length two. Hence $\Gamma^{+}(P; d, q)$ is connected with diameter two and girth three. The definition of $\Gamma^{\simeq}(P; d, q)$ requires that $d \geq 4$. So for any distinct $(\sigma, L), (\tau, N) \in \Omega$, we can choose a line M which is skew with both L and N. For any $\delta \in M$, the sequence $(\sigma, L), (\delta, M), (\tau, N)$ is a path of $\Gamma^{\simeq}(P; d, q)$ with length two. Moreover, if $(\sigma, L), (\tau, N)$ are adjacent in $\Gamma^{\simeq}(P; d, q)$, then the sequence $(\sigma, L), (\delta, M), (\tau, N), (\sigma, L)$ is a triangle. Hence $\Gamma^{\simeq}(P; d, q)$ is connected with diameter two and girth three as well.

For any flag (σ, L) and any point τ not incident with L, there are exactly q lines which are incident with τ and intersect with L at a point other than σ, namely those lines joining τ and one of the points in $L \setminus \{\sigma\}$. Hence there are exactly $v - q - 1$ lines which are incident with τ and skew with L (note that $L_{\sigma\tau}$ is not skew with L), where $v = (q^{d-1} - 1)/(q - 1)$ as before. From these the assertions in (b) follow immediately. Note that, for a point τ incident with L, (σ, L) is not adjacent to any vertex of $\Omega(\tau)$ in either $\Gamma^{+}(P; d, q)$ or $\Gamma^{\simeq}(P; d, q)$. Since L contains $q + 1$ points and $\mathrm{PG}(d - 1, q)$ has $(q^d - 1)/(q - 1)$ points in total, from (b) the assertion in (a) concerning the valencies of $\Gamma^{+}(P; d, q)$ and $\Gamma^{\simeq}(P; d, q)$ follows.

Now let us prove (c). Suppose Γ is a graph with vertex set Ω which is G-symmetric under the induced action of G on Ω. Let $((\sigma, L), (\tau, N))$ be an arc of Γ. If $\sigma = \tau$, then $L \neq N$, and two flags $(\sigma_1, L_1), (\tau_1, N_1)$ are adjacent in Γ if and only if $\sigma_1 = \tau_1$ and $L_1 \neq N_1$. Since $\mathrm{PG}(d-1, q)$ has $(q^d - 1)/(q - 1)$ points, and since each point is incident with exactly $(q^{d-1} - 1)/(q - 1)$ lines, in this case we have $\Gamma \cong ((q^d - 1)/(q - 1)) \cdot K_{(q^{d-1}-1)/(q-1)}$. Similarly, if $L = N$, then we have $\Gamma \cong ((q^{d-1} - 1)(q^d - 1)/(q - 1)(q^2 - 1)) \cdot K_{q+1}$. In the following we suppose that $\sigma \neq \tau$ and $L \neq N$. Then the G-symmetry of Γ implies that there exists $g \in G$ which interchanges (σ, L) and (τ, N). So we have $\sigma \notin N$ for otherwise we would have $\sigma \in L \cap N$ and thus $\tau = \sigma^g \in (L \cap N)^g = L \cap N$, which implies $\sigma = \tau$ and so contradicts our assumption. Similarly, we have $\tau \notin L$ and hence $((\sigma, L), (\tau, N)) \in \mathrm{F}(P; d, q)$. Thus, since Γ is G-symmetric, its arc set $\mathrm{Arc}(\Gamma)$ is a self-paired G-orbit on $\mathrm{F}(P; d, q)$. Therefore, from Lemma 3.3, Γ is isomorphic to either $\Gamma^+(P; d, q)$ or $\Gamma^{\simeq}(P; d, q)$. □

3.2. Affine flag graphs. For an integer $d \geq 2$ and a prime power q, we use the same notation $\mathrm{AG}(d, q)$ for the (point, line)-incidence structure of the affine geometry $\mathrm{AG}(d, q)$. Thus, for any group G with $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$, $\mathrm{AG}(d, q)$ is a G-doubly transitive linear space. The aim of this subsection is to classify and characterize the G-flag graphs of $\mathrm{AG}(d, q)$. For this purpose we need the following basic result in affine geometry.

LEMMA 3.8. *Suppose* $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$*, where* $d \geq 2$ *and* q *is a prime power. Then, for* $1 \leq s \leq d$*,* G *is transitive on ordered* $(s+1)$*-tuples of points of* $\mathrm{AG}(d, q)$ *not lying on any* $(s-1)$*-flat of* $\mathrm{AG}(d, q)$.

From this and Remark 2.7(a), it is easily verified that the flag set $\Omega(A; d, q)$ of $\mathrm{AG}(d, q)$ is feasible. Thus, setting $\mathrm{F}(A; d, q) := \mathrm{F}(\mathrm{AG}(d, q), \Omega(A; d, q))$, we have

$$\mathrm{F}(A; d, q) = \{((\sigma, L), (\tau, N)) : (\sigma, L), (\tau, N) \in \Omega(A; d, q), \sigma \notin N, \tau \notin L\}.$$

We call two distinct lines of $\mathrm{AG}(d, q)$ *intersecting* if they share a unique common point, *parallel* if they lie on the same plane but have no point in common, and *skew* in the remaining case. We use $\Psi^+(A; d, q)$ ($\Psi^=(A; d, q)$, $\Psi^{\simeq}(A; d, q)$, respectively) to denote the set of ordered pairs $((\sigma, L), (\tau, N))$ in $\mathrm{F}(A; d, q)$ such that L, N are intersecting (parallel, skew, respectively). Then $\Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ and $\Psi^{\simeq}(A; d, q)$ consist of a partition of $\mathrm{F}(A; d, q)$. (Note that $\Psi^{\simeq}(A; d, q) \neq \emptyset$ if and only if $d \geq 3$, see [15, Theorem 1.15(i)].) Using Lemma 3.8 and by a similar argument as in the proof of Lemma 3.3, one can prove the following lemma.

LEMMA 3.9. *Suppose* $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$*, where* $d \geq 2$ *and* q *is a prime power.*

(a) *There exists a unique feasible* G*-orbit on the flags of* $\mathrm{AG}(d, q)$*, namely* $\Omega(A; d, q)$.
(b) *If* $d = 2$*, then* G *has two orbits on* $\mathrm{F}(A; d, q)$*, namely* $\Psi^+(A; 2, q)$ *and* $\Psi^=(A; 2, q)$*; if* $d \geq 3$*, then* G *has three orbits on* $\mathrm{F}(A; d, q)$*, namely* $\Psi^+(A; d, q)$*,* $\Psi^=(A; d, q)$ *and* $\Psi^{\simeq}(A; d, q)$.

Clearly, $\Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ and $\Psi^{\simeq}(A; d, q)$ are all self-paired. So the following graphs are well-defined.

DEFINITION 3.10. The flag graphs of $\mathrm{AG}(d, q)$ with respect to $(\Omega(A; d, q), \Psi)$, for $\Psi = \Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ and $\Psi^{\simeq}(A; d, q)$, are called *affine flag graphs*, and are denoted by $\Gamma^+(A; d, q)$, $\Gamma^=(A; d, q)$ and $\Gamma^{\simeq}(A; d, q)$, respectively.

From Lemma 3.9, these are the only G-flag graphs of $\mathrm{AG}(d, q)$, for $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$. Moreover, the following lemma shows that they are the only G-flag graphs of any G-doubly transitive and G-block-transitive 2-design. The proof of this result is similar to that of Lemma 3.5 and hence is omitted. In the proof we exploit the following fact: the only faithful permutation representation of G is its natural action on $V(d, q)$.

LEMMA 3.11. *Suppose* $\mathrm{AGL}(d,q) \le G \le \mathrm{A\Gamma L}(d,q)$, *where* $d \ge 2$ *and* q *is a prime power. Suppose further that* $\mathcal{D}$ *is a* 2*-design which admits* G *as a faithful, doubly transitive, and block-transitive group of automorphisms. Then any* G*-flag graph of* $\mathcal{D}$ *is isomorphic to* $\Gamma^{+}(A;d,q)$, *or* $\Gamma^{=}(A;d,q)$, *or* $\Gamma^{\simeq}(A;d,q)$.

REMARK 3.12. The affine geometry $\mathrm{AG}(d,q)$ has $mv+1 := q^d$ points, and each line of it contains $m+1 := q$ points. So we have $v = (q^d-1)/(q-1)$ and $m = q-1$. Thus, $\mathrm{AG}(d,q)$ is the trivial linear space if and only if $q = 2$, which in turn is true if and only if $\mathrm{AGL}(d,q)$ is 3-transitive on $V(d,q)$. Hence, from Example 2.4, $\Gamma^{+}(A;d,2)$, $\Gamma^{=}(A;d,2)$ and $\Gamma^{\simeq}(A;d,2)$ are all 3-arc graphs of the complete graph $\Sigma = K_{2^d}$ with vertex set $V(d,2)$. The vertices of these three graphs are ordered pairs $\mathbf{uw}$ of distinct vectors of $V(d,2)$. Since each plane of $\mathrm{AG}(d,2)$ contains exactly four points ([15, Theorem 1.17]), one can see that $\mathbf{uw}, \mathbf{yz}$ are adjacent in $\Gamma^{+}(A;d,2)$ if and only if $\mathbf{w} = \mathbf{z}$. So $\Gamma^{+}(A;d,2)$ is isomorphic to $2^d \cdot K_{2^d-1}$ and is the 3-arc graph of Σ with respect to the set of all 3-cycles of Σ. Similarly, $\mathbf{uw}, \mathbf{yz}$ are adjacent in $\Gamma^{=}(A;d,2)$ if and only if $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z}$ are distinct and $\mathbf{u}-\mathbf{w} = \mathbf{y}-\mathbf{z}$, and they are adjacent in $\Gamma^{\simeq}(A;d,2)$ if and only if $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z}$ do not lie on the same plane of $\mathrm{AG}(d,2)$. From this it follows that $\Gamma^{=}(A;d,2)$ is, and $\Gamma^{\simeq}(A;d,2)$ is not, an almost cover of Σ.

From Corollary 2.6 and the discussion above we get the following theorem, which together with Theorem 3.6 gives the proof of Theorem 1.2 in the introduction.

THEOREM 3.13. *Suppose* $\mathrm{AGL}(d,q) \le G \le \mathrm{A\Gamma L}(d,q)$, *where* $d \ge 2$ *and* q *is a prime power. Then there exists a* G*-symmetric graph* Γ *with* G *faithful on* $V(\Gamma)$ *which admits a nontrivial* G*-invariant partition* $\mathcal{B}$ *such that* $v = k+1 \ge 3$ *and* $\Gamma_{\mathcal{B}} \cong K_{mv+1}$. *Moreover, any such graph* Γ *is isomorphic to* $\Gamma^{+}(A;d,q)$, $\Gamma^{=}(A;d,q)$, *or* $\Gamma^{\simeq}(A;d,q)$. *In each case we have* $v = (q^d-1)/(q-1)$ *and the multiplicity* m *of* $\mathcal{D}(B)$ *(for* $B \in \mathcal{B}$*) is equal to* $q-1$.

In this theorem the graph $\Gamma^{\simeq}(A;d,q)$ appears only when $d \ge 3$. By a similar argument as in the proof of Theorem 3.7, one can prove the following properties of the affine flag graphs.

THEOREM 3.14. *Let* $d \ge 2$ *and* $q \ge 2$ *be a prime power. Let* $\Omega := \Omega(A;d,q)$ *and* $\mathcal{B}(\Omega) = \{\Omega(\sigma) : \sigma \text{ a point of } \mathrm{AG}(d,q)\}$. *Then the following statements hold.*

(a) $\Gamma^{+}(A;d,q)$ *and* $\Gamma^{\simeq}(A;d,q)$ *are connected graphs with diameter two and girth three, and with valencies* $(q-1)(q^d-q)$ *and* $(q^d-q^2)(q^d-q)/(q-1)$, *respectively.*

(b) $\Gamma^{=}(A;d,q)$ *has valency* q^d-q *and contains* $(q^d-1)/(q-1)$ *connected components, each of which is a complete* q^{d-1}*-partite graph with* q *vertices in each part. Moreover,* $\Gamma^{=}(A;d,q)$ *is an almost cover of* K_{q^d}.

(c) *For distinct blocks* $\Omega(\sigma), \Omega(\tau)$ *of* $\mathcal{B}(\Omega)$, *each vertex* (σ, L) *of* $\Omega(\sigma)$ *other than* $(\sigma, L_{\sigma\tau})$ *is adjacent to exactly* $q-1$ *vertices of* $\Omega(\tau)$ *in* $\Gamma^{+}(A;d,q)$, *and adjacent to exactly* $(q^d-q^2)/(q-1)$ *vertices of* $\Omega(\tau)$ *in* $\Gamma^{\simeq}(A;d,q)$. *In particular, for* $\Gamma := \Gamma^{+}(A;2,q)$, $\Gamma[\Omega(\sigma), \Omega(\tau)]$ *is isomorphic to* $K_{q,q}$ *minus a perfect matching.*

(d) *For* $\mathrm{AGL}(d,q) \le G \le \mathrm{A\Gamma L}(d,q)$, *any* G*-symmetric graph with vertex set* Ω *(under the induced action of* G *on* Ω*) is isomorphic to* $\Gamma^{+}(A;d,q)$, *or* $\Gamma^{\simeq}(A;d,q)$, *or* $\Gamma^{=}(A;d,q)$, *or* $q^d \cdot K_{(q^d-1)/(q-1)}$ *with connected components the sets of flags incident with a common point, or* $(q^{d-1}(q^d-1)/(q-1)) \cdot K_q$ *with connected components the sets of flags incident with a common line.*

3.3. A classification theorem. From the discussion at the beginning of this section, the G-flag graphs of a trivial G-doubly transitive linear space are precisely 3-arc graphs of the $(G,2)$-arc transitive graph $\Sigma := K_{v+1}$. In this case, G is 3-transitive on $V(\Sigma)$. In the following we suppose this is the case and furthermore G is faithful on $V(\Sigma)$. By the classification of highly transitive permutation groups (see [3, 11]), G is one of the following groups of degree $v+1$ with the natural 3-transitive permutation representation on $V(\Sigma)$:

(i) S_{v+1} $(v \geq 3)$;
(ii) A_{v+1} $(v \geq 4)$;
(iii) $\mathrm{AGL}(d, 2)$ $(v = 2^d - 1 \geq 3)$;
(iv) $\mathbb{Z}_2^4 \cdot A_7$ $(v = 15)$;
(v) Mathieu groups M_{v+1} $(v = 10, 11, 21, 22, 23)$ and M_{11} $(v = 11)$; and
(vi) 3-transitive groups G satisfying $\mathrm{PGL}(2, v) \leq G \leq \mathrm{P\Gamma L}(2, v)$ $(v \geq 3$ is a prime power, note that $\mathrm{PGL}(2, 4) \cong A_5)$.

By Corollary 2.5 the 3-arc graphs $\Gamma := \Xi(\Sigma, \Delta)$ of Σ, for self-paired G-orbits Δ on $\mathrm{Arc}_3(\Sigma)$, are G-symmetric graphs Γ such that $v = k + 1 \geq 3$ and $\Gamma_{\mathcal{B}} \cong K_{v+1}$ (and thus $\mathcal{D}(B)$ contains no repeated blocks). In this case the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent and doubly transitive ([12, Theorem 5(b)], see also the comments at the end of Section 2.1). So such 3-arc graphs Γ are precisely those graphs studied in [9] with the additional properties that $\mathrm{val}(\Gamma_{\mathcal{B}}) = v$ and $v = k + 1 \geq 3$. Thus, a classification of these 3-arc graphs follows from the main result of [9]. Mainly for the integrity and convenience of later reference we give such a classification explicitly in this subsection along a different route.

As mentioned earlier, we suppose in the following that each 3-arc in Δ is not a 3-cycle for otherwise we would have $\Gamma = (v+1) \cdot K_v$, Γ is an almost cover of Σ, and G can be any group listed above. The 3-arc graphs arising from the groups G in (vi) are (twisted and untwisted) cross ratio graphs, as shown in Example 3.1. The following example determines all the 3-arc graphs (other than $(v + 1) \cdot K_v$) of Σ arising from 4-transitive groups. For integers ℓ, n with $2 \leq 2\ell < n$, the *Kneser graph* $K(n, \ell)$ is the graph whose vertices are all ℓ-subsets of a given n-set and where two such ℓ-subsets are adjacent if and only if they have no element in common. For two graphs Γ_1, Γ_2, the lexicographic product $\Gamma_1[\Gamma_2]$ of Γ_1 by Γ_2 is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that $(\alpha_1, \alpha_2), (\beta_1, \beta_2)$ are adjacent if and only if either α_1, β_1 are adjacent in Γ_1, or $\alpha_1 = \beta_1$ and α_2, β_2 are adjacent in Γ_2.

EXAMPLE 3.15. *Lexicographic products.* If G is 4-transitive on $V(\Sigma)$, then either $G = S_{v+1}$ $(v \geq 3)$, or $G = A_{v+1}$ $(v \geq 5)$, or $G = \mathrm{M}_{v+1}$ $(v = 10, 11, 22, 23)$. In each case, G is transitive on the set Δ of proper 3-arcs of Σ, and hence Δ is the unique self-paired G-orbit on such 3-arcs. Clearly, $(\sigma, \tau), (\delta, \varepsilon)$ are adjacent in $\Xi(\Sigma, \Delta)$ if and only if $\{\sigma, \tau\} \cap \{\delta, \varepsilon\} = \emptyset$. Thus this 3-arc graph is isomorphic to $(K(v + 1, 2))[\overline{K}_2]$, the lexicographic product of the Kneser graph $K(v + 1, 2)$ and the empty graph $\overline{K}_2$ on two vertices. One can see that, for distinct blocks B, C of $\mathcal{B}(\Sigma)$ (defined in (4)), $\Gamma[B, C]$ is isomorphic to $K_{v-1,v-1}$ minus a perfect matching. This is the graph defined in [9, Proposition 5.1(a)].

EXAMPLE 3.16. *Special affine flag graphs.* The group $G := \mathbb{Z}_2^4 \cdot A_7$ is a subgroup of $\mathrm{AGL}(4, 2)$, where $\mathbb{Z}_2^4$ acts on $V(\Sigma) := V(4, 2)$ by translations and, for $\tau := \mathbf{0}$, $G_\tau \cong A_7$ is a subgroup of $\mathrm{GL}(4, 2) \cong A_8$ acting 2-transitively on $V(4, 2) \setminus \{\tau\}$ in its natural action. Let σ, σ' be distinct points of $V(4, 2) \setminus \{\tau\}$. Then from [4, p. 10] we have $G_{\sigma\tau} \cong \mathrm{PSL}(2, 7)$, which is transitive on $V(4, 2) \setminus \{\sigma, \tau\}$, and each involution in A_7 and also each element of order 3 in $\mathrm{PSL}(2, 7)$ fixes exactly three nonzero vectors in $V(4, 2)$. Hence in the action of $G_{\sigma\sigma'\tau} \cong A_4$ on $V(4, 2) \setminus \{\sigma, \sigma', \tau, \sigma + \sigma' + \tau\}$ the stabilizer of any vector is trivial, that is, $G_{\sigma\sigma'\tau}$ has an orbit of length 12. Apart from this orbit, $G_{\sigma\sigma'\tau}$ has another orbit on $V(4, 2) \setminus \{\sigma, \sigma', \tau\}$, namely $\{\sigma + \sigma' + \tau\}$. Since G is 3-transitive on $V(\Sigma)$, there are two G-orbits on proper 3-arcs of Σ. It is clear that these two G-orbits correspond to $\Psi^{=}(A; 4, 2)$ and $\Psi^{\simeq}(A; 4, 2)$ respectively. Therefore, we have exactly two 3-arc graphs of Σ, namely $\Gamma^{=}(A; 4, 2)$ and $\Gamma^{\simeq}(A; 4, 2)$. As mentioned in Remark 3.12, $\Gamma^{=}(A; 4, 2)$ is, and $\Gamma^{\simeq}(A; 4, 2)$ is not, an almost cover of Σ.

EXAMPLE 3.17. *Mathieu graphs* $\Xi_1(\mathrm{M}_{11})$ *and* $\Xi_2(\mathrm{M}_{11})$. The Mathieu group M_{11} with degree $v + 1 = 12$ is the automorphism group of the unique 3-$(12, 6, 2)$ design $\mathcal{D}$. We assume that the point set of $\mathcal{D}$ is the same as the vertex set of $\Sigma := K_{12}$. For a 2-arc (σ', σ, τ) of Σ, let $X(\sigma', \sigma, \tau)$ denote the union of the two blocks of $\mathcal{D}$ containing σ', σ, τ. Then $(\mathrm{M}_{11})_{\sigma'\sigma\tau} \cong S_3$ has two orbits on $V(\Sigma) \setminus \{\sigma', \sigma, \tau\}$ (see [6, pp. 231–232]), namely $V(\Sigma) \setminus X(\sigma', \sigma, \tau)$ and $X(\sigma', \sigma, \tau) \setminus \{\sigma', \sigma, \tau\}$. Let $\tau' \in V(\Sigma) \setminus \{\sigma', \sigma, \tau\}$. By the 3-transitivity of M_{11}, there exists $g \in \mathrm{M}_{11}$ such that $(\sigma, \tau, \tau')^g = (\tau, \sigma, \sigma')$. Set $(\sigma')^g = \delta$, so $(\sigma', \sigma, \tau, \tau')^g = (\delta, \tau, \sigma, \sigma')$. Since g is an automorphism of $\mathcal{D}$, the points $\sigma', \sigma, \tau, \tau'$ lie in the same block of $\mathcal{D}$ if and only if $\delta, \tau, \sigma, \sigma'$ lie in the same block of $\mathcal{D}$. This implies that, $\tau' \in V(\Sigma) \setminus X(\sigma', \sigma, \tau)$ ($\tau' \in X(\sigma', \sigma, \tau) \setminus \{\sigma', \sigma, \tau\}$, respectively) if and only if $\delta \in V(\Sigma) \setminus X(\sigma', \sigma, \tau)$ ($\delta \in X(\sigma', \sigma, \tau) \setminus \{\sigma', \sigma, \tau\}$, respectively). That is, δ and τ' are in the same $(\mathrm{M}_{11})_{\sigma'\sigma\tau}$-orbit on $V(\Sigma) \setminus \{\sigma', \sigma, \tau\}$. So there exists $h \in (\mathrm{M}_{11})_{\sigma'\sigma\tau}$ such that $\delta^h = \tau'$. This implies that gh reverses $(\sigma', \sigma, \tau, \tau')$ and hence Δ is self-paired. So there are exactly two self-paired M_{11}-orbits on proper 3-arcs of Σ, namely $\Delta_1 := (\sigma', \sigma, \tau, \tau')^{\mathrm{M}_{11}}$ for $\tau' \in V(\Sigma) \setminus X(\sigma', \sigma, \tau)$, and $\Delta_2 := (\sigma', \sigma, \tau, \tau')^{\mathrm{M}_{11}}$ for $\tau' \in X(\sigma', \sigma, \tau) \setminus \{\sigma', \sigma, \tau\}$. Thus we get two 3-arc graphs, namely $\Xi_i(\mathrm{M}_{11}) := \Xi(\Sigma, \Delta_i)$ for $i = 1, 2$. Note that $|V(\Sigma) \setminus X(\sigma', \sigma, \tau)| = 3$ and $|X(\sigma', \sigma, \tau) \setminus \{\sigma', \sigma, \tau\}| = 6$. From these it follows that, for blocks $B = \mathrm{Arc}(\Sigma; \sigma)$ and $C = \mathrm{Arc}(\Sigma; \tau)$ of $\mathcal{B}(\Sigma)$ (defined in (4)), each vertex of B other than (σ, τ) is adjacent to three vertices of C in $\Xi_1(\mathrm{M}_{11})$, and adjacent to six vertices of C in $\Xi_2(\mathrm{M}_{11})$. Hence $\Xi_1(\mathrm{M}_{11})$ and $\Xi_2(\mathrm{M}_{11})$ have valencies $3(v-1) = 30$ and $6(v-1) = 60$, respectively, and none of them is an almost cover of Σ. One can see that $(\alpha, \alpha'), (\beta, \beta')$ are adjacent in $\Xi_1(\mathrm{M}_{11})$ ($\Xi_2(\mathrm{M}_{11})$, respectively) if and only if $\alpha', \alpha, \beta, \beta'$ are distinct and $\beta' \in V(\Sigma) \setminus X(\alpha', \alpha, \beta)$ ($\beta' \in X(\alpha', \alpha, \beta) \setminus \{\alpha', \alpha, \beta\}$, respectively). Thus, $\Xi_1(\mathrm{M}_{11})$ and $\Xi_2(\mathrm{M}_{11})$ are the graphs defined in Proposition 5.1(e), (1) and (2) of [9], respectively.

EXAMPLE 3.18. *Mathieu graphs* $\Xi_1(\mathrm{M}_{22})$ *and* $\Xi_2(\mathrm{M}_{22})$. The Mathieu group M_{22} of degree $v + 1 = 22$ is the automorphism group of the 3-(22, 6, 1) Steiner system $\mathcal{D}$. We assume that the point set of $\mathcal{D}$ is the same as the vertex set of $\Sigma := K_{22}$. As in Example 3.17 above, we get two 3-arc graphs of Σ, namely the graph $\Xi_1(\mathrm{M}_{22})$ in which $(\alpha, \alpha'), (\beta, \beta')$ are adjacent if and only if $\alpha', \alpha, \beta, \beta'$ are distinct and $\beta' \in V(\Sigma) \setminus X(\alpha', \alpha, \beta)$, and the graph $\Xi_2(\mathrm{M}_{22})$ in which $(\alpha, \alpha'), (\beta, \beta')$ are adjacent if and only if $\alpha', \alpha, \beta, \beta'$ are distinct and $\beta' \in X(\alpha', \alpha, \beta) \setminus \{\alpha', \alpha, \beta\}$, where $X(\alpha', \alpha, \beta)$ denotes the unique block of $\mathcal{D}$ containing α', α, β. These two graphs are the graphs defined in Proposition 5.1(d), (1) and (2) of [9], respectively. Based on the same reason as in Example 3.17 one can see that, for blocks $B = \mathrm{Arc}(\Sigma; \alpha)$ and $C = \mathrm{Arc}(\Sigma; \beta)$ of $\mathcal{B}(\Sigma)$, each vertex of B other than (α, β) is adjacent to 16 vertices of C in $\Xi_1(\mathrm{M}_{22})$, and adjacent to three vertices of C in $\Xi_2(\mathrm{M}_{22})$. Thus, $\Xi_1(\mathrm{M}_{22})$ and $\Xi_2(\mathrm{M}_{22})$ have valencies $16(v-1) = 320$ and $3(v-1) = 60$, respectively. Moreover, none of them is an almost cover of Σ.

Combining the discussion in this subsection with Theorems 3.6(b) and (c), 3.13 and Remark 3.12, we get the following classification theorem, which is attributed to Gardiner and Praeger [9]. This theorem also gives an explicit list of all almost covers of the $(G, 2)$-arc transitive complete graph K_{v+1}. For a study of almost covers of 2-arc transitive noncomplete graphs, the reader is referred to [18].

THEOREM 3.19. *Suppose that* Γ *is a* G*-symmetric graph with* G *faithful on* $V(\Gamma)$ *which admits a nontrivial* G*-invariant partition* $\mathcal{B}$ *such that* $v = k + 1 \geq 3$, $\mathcal{D}(B)$ *contains no repeated blocks and* $\Gamma_{\mathcal{B}}$ *is complete. Then* $\Gamma_{\mathcal{B}} \cong K_{v+1}$, G *is* 3*-transitive and faithful on* $\mathcal{B}$*, and either* $\Gamma = (v+1) \cdot K_v$ *with* G *an arbitrary* 3*-transitive permutation group of degree* $v + 1$*, or one of the following* (a)–(f) *holds.*

(a) $\Gamma = (K(v+1, 2))[\overline{K}_2]$*, and* G *is either* S_{v+1} *(*$v \geq 3$*), or* A_{v+1} *(*$v \geq 5$*), or* M_{v+1} *(*$v = 10, 11, 22, 23$*).*

(b) $(\Gamma, G) = (\mathrm{CR}(v; x, s), \mathrm{PGL}(2, v) \cdot \langle \psi^t \rangle)$, *where* $v = p^e$ *with* p *a prime and* $e \geq 1$, $x \in \mathrm{GF}(v) \setminus \{0, 1\}$, s *is a divisor of* $s(x)$, *and* t *is a divisor of* e *with* $\gcd(s(x), t) = s$.
(c) $(\Gamma, G) = (\mathrm{TCR}(v; x, s), \mathrm{M}(t/2, v))$, *where* $v = p^e$ *with* p *an odd prime and* $e \geq 2$ *an even integer*, $x \in \mathrm{GF}(v) \setminus \{0, 1\}$ *with* $s(x)$ *even and* $x - 1$ *a square of* $\mathrm{GF}(v)$, s *is an even divisor of* $s(x)$, *and* t *is a divisor of* e *with* $\gcd(s(x), t) = s$.
(d) $\Gamma = \Gamma^{=}(A; d, 2)$ *or* $\Gamma^{\simeq}(A; d, 2)$, $v = 2^d - 1$, *where* $d \geq 2$, *and either* $G = \mathrm{AGL}(d, 2)$ *or* $d = 4$ *and* $G = \mathbb{Z}_2^4 \cdot A_7$.
(e) $(\Gamma, G) = (\Xi_1(\mathrm{M}_{11}), \mathrm{M}_{11})$ *or* $(\Xi_2(\mathrm{M}_{11}), \mathrm{M}_{11})$, *and* $v = 11$.
(f) $(\Gamma, G) = (\Xi_1(\mathrm{M}_{22}), \mathrm{M}_{22})$ *or* $(\Xi_2(\mathrm{M}_{22}), \mathrm{M}_{22})$, *and* $v = 21$.

Moreover, if in addition Γ *is an almost cover of* $\Gamma_{\mathcal{B}}$, *then either* $\Gamma = (v + 1) \cdot K_v$ *with* G *an arbitrary* 3*-transitive permutation group of degree* $v + 1$, *or* (Γ, G) *is as in* (*b*) *or* (*c*) *with* $s = s(x)$ *and* t *a multiple of* s, *or* $\Gamma = \Gamma^{=}(A; d, 2)$ *and* $G = \mathrm{AGL}(d, 2)$ *with* $d \geq 2$, *or* $\Gamma = \Gamma^{=}(A; 4, 2)$ *and* $G = \mathbb{Z}_2^4 \cdot A_7$.

In possibility (b) above, if $v = 3$ then $\mathrm{PGL}(2, 3) \cong S_4$ and we get only one cross ratio graph $\mathrm{CR}(3; 2, 1) \cong 3 \cdot C_4$; if $v = 4$, then $\mathrm{PGL}(2, 4) \cong A_5$ and we also have a unique cross ratio graph $\mathrm{CR}(4; t, 2) \cong \mathrm{CR}(4; t^2, 2)$, which is isomorphic to the dodecahedron (see [8, Example 2.4(a)]), where we set $\mathrm{GF}(4) = \{0, 1, t, t^2 = 1 + t\}$.

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