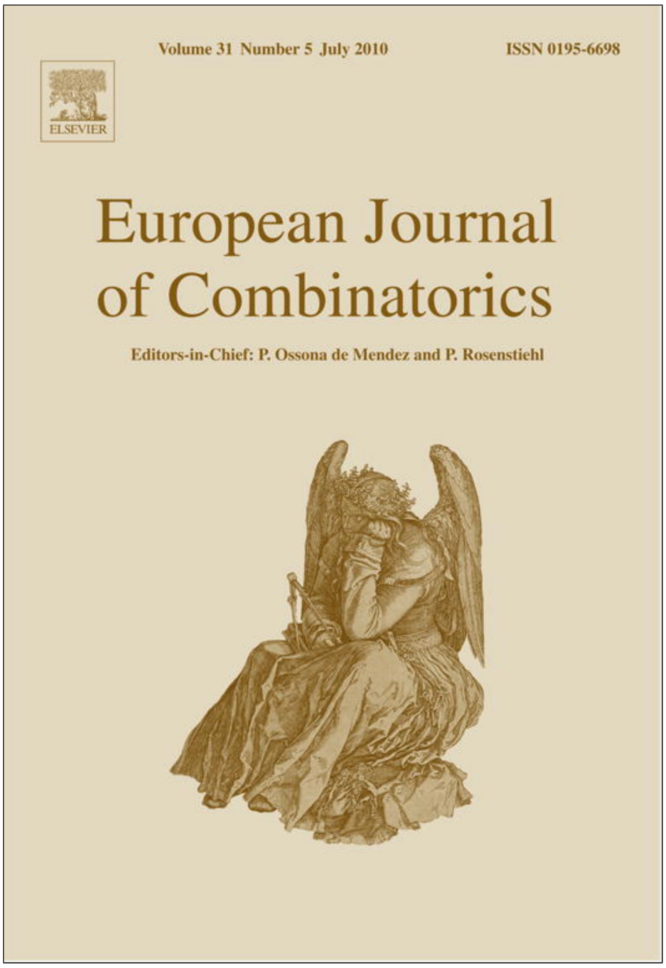
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The $L(h, 1, 1)$-labelling problem for trees

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A R T I C L E I N F O

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A B S T R A C T

Let $h \geq 1$ be an integer. An $L(h, 1, 1)$-labelling of a (finite or infinite) graph is an assignment of nonnegative integers (labels) to its vertices such that adjacent vertices receive labels with difference at least h, and vertices distance 2 or 3 apart receive distinct labels. The span of such a labelling is the difference between the maximum and minimum labels used, and the minimum span over all $L(h, 1, 1)$-labellings is called the $\lambda_{h,1,1}$-number of the graph. We prove that, for any integer $h \geq 1$ and any finite tree T of diameter at least 3 or infinite tree T of finite maximum degree, $\max\{\max_{uv \in E(T)} \min\{d(u), d(v)\} + h - 1, \Delta_2(T) - 1\} \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1$, and both lower and upper bounds are attainable, where $\Delta_2(T)$ is the maximum total degree of two adjacent vertices. Moreover, if h is less than or equal to the minimum degree of a non-pendant vertex of T, then $\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2$. In particular, $\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T)$. Furthermore, if T is a caterpillar and $h \geq 2$, then $\max\{\max_{uv \in E(T)} \min\{d(u), d(v)\} + h - 1, \Delta_2(T) - 1\} \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2$ with both lower and upper bounds achievable.

1. Introduction

Motivated by the problem [16] of assigning frequencies to transmitters in a radio communication network, various channel assignment problems have received extensive attention in recent years. Usually, such problems can be formulated as graph labelling problems, and a major concern is to minimize the span of a channel assignment subject to a set of constraints involving pairs of vertices within a given distance. Among others the following model has been studied widely, especially for the

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case when $d = 2$. Given a finite or infinite graph $G = (V(G), E(G))$ and a sequence $h_1, h_2, \ldots, h_d$ of nonnegative integers, an $L(h_1, h_2, \ldots, h_d)$-*labelling* of G is a mapping $\phi : V(G) \to \{0, 1, 2, \ldots\}$ such that, for $t = 1, 2, \ldots, d$ and any $u, v \in V(G)$ with $d(u, v) = t$,

$$|\phi(u) - \phi(v)| \geq h_t$$

where $d(u, v)$ is the distance in G between u and v. (In this paper an *infinite graph* means a graph with countably infinitely many vertices.) In practical terms, the *label* of u under ϕ, $\phi(u)$, is the channel assigned to the transmitter corresponding to u. Without loss of generality we will always assume $\min_{v \in V(G)} \phi(v) = 0$. Under this assumption the *span* of ϕ is defined as $\max_{v \in V(G)} \phi(v)$. Define [15,16]

$$\lambda_{h_1,h_2,\ldots,h_d}(G) := \min_{\phi} \max_{v \in V(G)} \phi(v)$$

to be the $\lambda_{h_1,h_2,\ldots,h_d}$-*number* of G, where the minimum is taken over all $L(h_1, h_2, \ldots, h_d)$-labellings of G. In practice [16] this parameter corresponds to the minimum bandwidth required by the radio communication network under the constraints above.

The $L(h_1, h_2, \ldots, h_d)$-labelling problem above is interesting in both theory and practical applications. For instance, when $d = 1$, it becomes the ordinary vertex-colouring problem since $\lambda_h(G) = h(\chi(G) - 1)$, where $\chi(G)$ is the chromatic number of G. In the case when $d = 2$, many interesting results (see e.g. [6–8,11,13,15,18,20,21]) have been obtained for various families of finite graphs, especially when $(h_1, h_2) = (2, 1)$. The reader is referred to [2] for an extensive bibliography on the $L(h_1, h_2)$-labelling problem and [22] for a short survey on Hamming graphs and hypercubes. In the following we just mention a few results for finite trees since they are more relevant to this article. In [15] it was proved that, for any finite tree T, $\lambda_{2,1}(T)$ is either $\Delta(T) + 1$ or $\Delta(T) + 2$, where $\Delta(T)$ is the maximum degree of T. A polynomial time algorithm for determining $\lambda_{2,1}(T)$ was given in [6], and a modification of it gave a polynomial algorithm for $\lambda_{h_1,h_2}(T)$ when h_2 divides h_1. It was conjectured [12] that the problem of determining λ_{h_1,h_2} for finite trees is NP-complete when h_2 does not divide h_1, and recently this was proved in [9]. In [10] it was proved that the $L(2, 1)$-labelling problem is NP-complete for graphs of treewidth 2. In [5] it was proved that $\Delta(T) + h - 1 \leq \lambda_{h,1}(T) \leq \min\{\Delta(T) + 2h - 2, 2\Delta(T) + h - 2\}$ with both lower and upper bounds attainable. In [14] the λ_{h_1,h_2}-number was derived for infinite regular trees when $h_1 \geq h_2$, and for $h_1 < h_2$ the authors of [4] studied the smallest integer λ such that every tree of maximum degree $\Delta \geq 2$ admits an $L(h_1, h_2)$-labelling of span at most λ.

More recently, researchers began to investigate the $L(h_1, h_2, h_3)$-labelling problem. For example, in [23] the third-named author studied the problem for hypercubes Q_n by using a group-theoretic approach, leading to upper bounds on $\lambda_{h_1,h_2,h_3}(Q_n)$ which are tight in certain cases. In [3] the $L(h, 1, 1)$-labelling problem (where $h \geq 1$) for outerplanar graphs was investigated. Nevertheless, in contrast to $L(h_1, h_2)$-labellings, we know only very little about $L(h_1, h_2, h_3)$-labellings even for some basic graphs such as trees.

In this paper we study the $L(h, 1, 1)$-labelling problem for finite and infinite trees, where $h \geq 1$. Define

$$\Delta_2(G) := \max_{uv \in E(G)} (d(u) + d(v))$$

for any graph G, where $d(u)$ is the degree of u in G. Note that, if G is infinite, then $\Delta_2(G) = \infty$ if and only if there exists no positive integer N such that $d(u) \leq N$ for all $u \in V(G)$, and in this case we have $\lambda_{h,1,1}(G) = \infty$. Thus, we consider only finite trees and infinite trees with finite maximum degree. We obtain the following bounds on $\lambda_{h,1,1}(T)$ in terms of $\Delta_2(T)$, which will be proved in Section 2 along with an algorithm for finding an $L(h, 1, 1)$-labelling of T with span $\Delta_2(T) + h - 1$. When T is finite, the running time of this algorithm is $O(|V(T)|^2)$.

Theorem 1. *Let $h \geq 1$ be an integer. Let T be a finite tree with diameter at least 3 or an infinite tree with finite maximum degree. Then*

$$\max\left\{\max_{uv \in E(T)} \min\{d(u), d(v)\} + h - 1, \Delta_2(T) - 1\right\} \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1. \quad (1)$$

Moreover, the lower bound is attainable for any $h \geq 1$ and the upper bound is attainable for any $h \geq 3$.

The lower bound $\Delta_2(T) - 1$ above is achieved by any tree T with diameter 3 and any h with $1 \le h \le \min\{d(u), d(v)\}$, where u, v are the two vertices of T with degree greater than 1. In fact, if we assign 0 to u, $\Delta_2(T)-1$ to v, $d(v), d(v)+1, \ldots, \Delta_2(T)-2$ to the neighbors of u other than v, and $1, 2, \ldots, d(v)-1$ to the neighbors of v other than u, then we get an $L(h, 1, 1)$-labelling of T with span $\Delta_2(T)-1$, and hence $\lambda_{h,1,1}(T) = \Delta_2(T)-1$. Let T' be the infinite tree obtained from T by attaching an infinite path (with one closed end) to a neighbor of u. It is easy to check that $\lambda_{h,1,1}(T') = \Delta_2(T') - 1$.

The lower bound $\max_{uv\in E(T)} \min\{d(u), d(v)\} + h - 1$ in (1) is achieved by any tree T with diameter 3 such that $d(u_1) = d(v_1)$ and any integer $h \ge d(u_1)$, where u_1, v_1 are the two vertices of T with degree greater than 1. In fact, if we assign 0 to u_1, $h + d(u_1) - 1$ to v_1, $h, h+1, \ldots, h + d(u_1) - 2$ to the neighbors of u_1 other than v_1, and $1, 2, \ldots, d(u_1) - 1$ to the neighbors of v_1 other than u_1, then we get an $L(h, 1, 1)$-labelling of T with span $h + d(u_1) - 1$. Hence $\lambda_{h,1,1}(T) = d(u_1) + h - 1 = \max_{uv\in E(T)} \min\{d(u), d(v)\} + h - 1$.

In the next section we will give for any $h \ge 3$ a family of trees which achieve the upper bound in (1). Our next result, to be proved in Section 3, implies that this upper bound can be improved when $h = 1, 2$. Define

$$\delta^*(T) := \min_{u\in V(T),\, d(u)\ge 2} d(u).$$

Theorem 2. *Let T be a finite tree with diameter at least 3 or an infinite tree with finite maximum degree. Then for any positive integer $h \le \delta^*(T)$ we have*

$$\lambda_{h,1,1}(T) \le \Delta_2(T) + h - 2. \tag{2}$$

A tree is called a *caterpillar* if the removal of all vertices of degree 1 results in a path, called the *spine*. Thus the spine of an infinite caterpillar is an infinite path with at least one open end. The next result, to be proved in Section 3, shows that for caterpillars the upper bound in (1) can also be reduced by 1 for any $h \ge 2$.

Theorem 3. *Let $h \ge 2$ be an integer. Let T be a finite caterpillar of diameter at least 3 or an infinite caterpillar of finite maximum degree. Then*

$$\max\left\{\max_{uv\in E(T)} \min\{d(u), d(v)\} + h - 1,\ \Delta_2(T) - 1\right\} \le \lambda_{h,1,1}(T) \le \Delta_2(T) + h - 2$$

and both lower and upper bounds are achievable. Moreover, if there exists no vertex on the spine with degree $\Delta_2(T) - 2$, then $\lambda_{h,1,1}(T) \le \Delta_2(T) + h - 3$; if there exist consecutive vertices u, v, w on the spine such that $d(u) = d(w) = \Delta_2 - 2$ and $d(v) = 2$, then $\lambda_{h,1,1}(T) = \Delta_2(T) + h - 2$.

Note that $\delta^*(T) \ge 2$ for any tree T with diameter at least 3. Thus, in the case when $h = 2$, Theorems 1 and 2 give the following corollary, which can be viewed as the counterpart of the result $\Delta(T) + 1 \le \lambda_{2,1}(T) \le \Delta(T) + 2$ mentioned above.

Corollary 4. *Let T be a finite tree with diameter at least 3 or an infinite tree with finite maximum degree. Then*

$$\Delta_2(T) - 1 \le \lambda_{2,1,1}(T) \le \Delta_2(T).$$

The nth *power* of a graph G, G^n, is the graph with the same vertices as G such that two vertices are adjacent if and only if the distance in G between them is at most n. From the definition of $\lambda_{1,1,1}$ it is evident that $\lambda_{1,1,1}(G) = \chi(G^3) - 1$. Thus, when $h = 1$, Theorems 1 and 2 together give the following corollary. (See, for example, [17,19,21] for related results on the chromatic number of power graphs.)

Corollary 5. *Let T be a finite tree with diameter at least 3 or an infinite tree with finite maximum degree. Then*

$$\chi(T^3) = \lambda_{1,1,1}(G) + 1 = \Delta_2(T).$$

In the case when T is finite, this result can also be deduced from the following facts: T^3 is chordal with clique number $\Delta_2(T)$ and the chromatic number of any chordal graph is equal to its clique number. (The nth power of a chordal graph is also chordal when n is odd. See [1] and also [24] for an independent and shorter proof. Since a finite tree T is chordal, T^3 is chordal.)

The λ_{h_1,h_2}-number of a graph is often bounded by its maximum degree Δ. For example, motivated by the conjecture [15] that $\lambda_{2,1}(G) \le \Delta(G)^2$ for any graph G with $\Delta(G) \ge 2$, a number of results in the literature relate $\lambda_{h_1,h_2}(G)$ to $\Delta(G)$ (see the survey paper [2]). Our results above suggest that labelling problems of distance 3 (not necessarily for trees) are more related to Δ_2.

We will use the following notation: for a vertex v of a tree T,

$$N(v) := \{u \in V(T) : uv \in E(T)\}$$
$$N[v] := N(v) \cup \{v\}$$
$$N_3(v) := \{u \in V(T) : 1 \le d(u, v) \le 3\}.$$

An edge uv of T is called *heavy* if it achieves $\Delta_2(T)$, that is, $d(u) + d(v) = \Delta_2(T)$.

2. Proof of Theorem 1

In this section we always assume that T is a finite tree with diameter at least 3 or an infinite tree with finite maximum degree. For integers $x < y$, let

$$[x, y] := \{x, x + 1, \ldots, y - 1, y\}.$$

The following lemma gives the lower bound in Theorem 1.

Lemma 6. *Let $h \ge 1$ be an integer. Then*

$$\lambda_{h,1,1}(T) \ge \max\left\{\max_{uv\in E(T)} \min\{d(u), d(v)\} + h - 1, \Delta_2(T) - 1\right\}.$$

Proof. Let $uv \in E(T)$ be a heavy edge. Then $N(u) \cup N(v)$ contains $\Delta_2(T)$ vertices with mutual distance at most 3. Since these $\Delta_2(T)$ vertices require $\Delta_2(T)$ distinct labels in any $L(h, 1, 1)$-labelling, we have $\lambda_{h,1,1}(T) \ge \Delta_2(T) - 1$ immediately.

To complete the proof it suffices to prove $\lambda_{h,1,1}(T) \ge \min\{d(u), d(v)\} + h - 1$ for every $uv \in E(T)$. Since this is clearly true when $\min\{d(u), d(v)\} = 1$, we consider edges uv with $\min\{d(u), d(v)\} \ge 2$ and denote $\lambda = \lambda_{h,1,1}(T)$. Let ϕ be an optimal $L(h, 1, 1)$-labelling of T, so all vertices of T receive labels from $[0, \lambda]$. Since u and v are adjacent, $\lambda \ge \max\{\phi(u), \phi(v)\} \ge \min\{\phi(u), \phi(v)\} + h$ and hence $\min\{\phi(u), \phi(v)\} \le \lambda - h$. Consider the case $\phi(u) \le \lambda - h$ first. If $\phi(u) \ge h$, then the available labels for the vertices in $N(u)$ are $[0, \phi(u) - h] \cup [\phi(u) + h, \lambda]$. Thus, $d(u) \le (\phi(u) - h + 1) + (\lambda - \phi(u) - h + 1)$; that is, $\lambda \ge d(u) + 2h - 2 \ge d(u) + h - 1$. If $0 \le \phi(u) \le h - 1$, then the available labels for the vertices in $N(u)$ are $[\phi(u) + h, \lambda]$. Hence $d(u) \le \lambda - (\phi(u) + h) + 1 \le \lambda - h + 1$. So we have proved $\lambda \ge d(u) + h - 1$ if $\phi(u) \le \lambda - h$. Similarly, if $\phi(v) \le \lambda - h$, then $\lambda \ge d(v) + h - 1$. Therefore $\lambda_{h,1,1}(T) \ge \min\{d(u), d(v)\} + h - 1$ for every $uv \in E(T)$ and the proof is complete. □

In the following we abbreviate $\Delta_2(T)$ to Δ_2 and fix a heavy edge uv of T. Let $T - uv$ be the graph obtained from T by deleting the edge uv. Denote by T_u, T_v the connected components of $T - uv$ containing u, v respectively. Let

$$l_u := \max_{w\in V(T_u)} d(u, w), \qquad l_v := \max_{w\in V(T_v)} d(v, w).$$

Note that, if T is infinite, then at least one of T_u, T_v must be infinite. Moreover, if T_u (resp., T_v) is infinite, then we define $l_u = \infty$ (resp., $l_v = \infty$). Define

$$L_i(u) := \{w \in V(T_u) : d(u, w) = i\}, \quad i = 0, 1, \ldots, l_u$$
$$L_i(v) := \{w \in V(T_v) : d(v, w) = i\}, \quad i = 0, 1, \ldots, l_v.$$

In particular, $L_0(u) = \{u\}$ and $L_0(v) = \{v\}$. To facilitate our labelling we index the vertices of $L_i(u)$ with sequences of positive integers of length i in the following way. First, we index the vertices in $L_1(u)$ $(= N(u) \setminus \{v\})$ with $1, 2, \ldots, d(u) - 1$ (sequences of length 1) respectively in an arbitrary but fixed order. Then for each vertex $a_1 \in \{1, 2, \ldots, d(u) - 1\}$ we index its neighbors other than u with a_1a_2 in an arbitrary order, where $a_2 = 1, 2, \ldots, d(a_1) - 1$. Inductively, for a vertex $a_1a_2 \cdots a_{i-1}a_i$ in level $L_i(u)$, if it is not a vertex of degree 1, then we index its neighbors other than $a_1a_2 \cdots a_{i-1}$ with $a_1a_2 \cdots a_{i-1}a_ia_{i+1}$ in an arbitrary order, where $a_{i+1} = 1, 2, \ldots, d(a_1a_2 \cdots a_{i-1}a_i) - 1$. In this way each vertex of T_u other than u is indexed with a unique sequence whose length is the distance between the vertex and u. Moreover, the unique path between u and a vertex $a_1a_2 \cdots a_{i-1}a_i \in L_i(u)$ is

$$u, a_1, a_1a_2, a_1a_2a_3, \ldots, a_1a_2 \cdots a_{i-1}a_i.$$

In the same fashion, we index the vertices of T_v other than v with sequences, and we use $b_1b_2 \cdots b_{i-1}b_i$ to denote a typical vertex in level $L_i(v)$ in order to avoid confusion with vertices of T_u. In the following, if $i = 1$ then $a_1 \cdots a_{i-1}, b_1 \cdots b_{i-1}$ are interpreted as u, v respectively, and $a_1 \cdots a_{i-2}, b_1 \cdots b_{i-2}$ are interpreted as v, u respectively. The following observations will be used without further explanation in the proof of Lemma 8.

Lemma 7. (a) *The following equalities (3)–(4) hold for $i = 1, 2, \ldots, l_u - 1$ and (5)–(6) for $i = 1, 2, \ldots, l_v - 1$:*

$$L_{i+1}(u) = \bigcup_{a_1 \cdots a_i \in L_i(u)} (N(a_1 \cdots a_i) \setminus \{a_1 \cdots a_{i-1}\}) \tag{3}$$

$$N_3(a_1 \cdots a_ia_{i+1}) \cap \left(\bigcup_{j=0}^{i} L_j(u) \right) = N[a_1 \cdots a_{i-1}], \quad \forall a_1 \cdots a_ia_{i+1} \in L_{i+1}(u) \tag{4}$$

$$L_{i+1}(v) = \bigcup_{b_1 \cdots b_i \in L_i(v)} (N(b_1 \cdots b_i) \setminus \{b_1 \cdots b_{i-1}\}) \tag{5}$$

$$N_3(b_1 \cdots b_ib_{i+1}) \cap \left(\bigcup_{j=0}^{i} L_j(v) \right) = N[b_1 \cdots b_{i-1}], \quad \forall b_1 \cdots b_ib_{i+1} \in L_{i+1}(v). \tag{6}$$

(b) *Any two vertices of T which are in the same level $L_i(u)$ (resp., $L_i(v)$) but not adjacent to the same vertex in level $L_{i-1}(u)$ (resp., $L_{i-1}(v)$) are distance 4 apart.*

The next lemma gives the upper bound in (1). For a labelling ϕ of T and a subset U of $V(T)$, define

$$\phi(U) := \{\phi(u) : u \in U\}.$$

Lemma 8. *Let $h \geq 1$ be an integer. Then*

$$\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1.$$

Proof. We construct recursively an $L(h, 1, 1)$-labelling ϕ of T with span $\Delta_2(T) + h - 1$. Recall that uv is a fixed heavy edge of T. If T is finite, then both l_u and l_v are finite; otherwise either l_u or l_v is ∞.

Part 1 (Initialization): Define

$$\phi(u) = 0, \qquad \phi(v) = \Delta_2 + h - 1; \tag{7}$$

$$\phi(a_1) = \Delta_2 + h - 1 - a_1, \quad a_1 = 1, 2, \ldots, d(u) - 1; \tag{8}$$

$$\phi(b_1) = b_1, \quad b_1 = 1, 2, \ldots, d(v) - 1. \tag{9}$$

Since $\Delta_2 + h - 1 - (d(u) - 1) = d(v) + h$, we have

$$\phi(N(u) \setminus \{v\}) = [d(v) + h, \Delta_2 + h - 2]$$
$$\phi(N(v) \setminus \{u\}) = [1, d(v) - 1].$$

Since $d(u) \geq 2$ and $d(v) \geq 2$, the labelling above satisfies the $L(h, 1, 1)$-conditions among the vertices of $N(u) \cup N(v)$.

Part 2 (Labelling T_u): If $l_u = 1$, then T_u has been labelled fully. Otherwise, for each $a_1 \in L_1(u)$ we label (in an arbitrary manner) the vertices of $N(a_1) \setminus \{u\} \subseteq L_2(u)$ with $1, 2, \ldots, d(a_1) - 1$ respectively, so $\phi(N(a_1) \setminus \{u\}) = [1, d(a_1) - 1]$. We do this for all vertices $a_1 \in L_1(u)$ independently, and in this way all vertices in $L_2(u)$ are labelled. Since $d(a_1) \leq \Delta_2 - d(u) = d(v)$ by the definition of Δ_2, in view of Lemma 7(b) the labelling so far satisfies the $L(h, 1, 1)$-conditions up to level $L_2(u)$.

If $l_u = 2$ then T_u has been labelled; otherwise we label $L_3(u)$ as follows: If $d(a_1a_2) \geq a_1 + 2$ (that is, $\phi(a_1) > \Delta_2 - d(a_1a_2) + 1$), then we label (in an arbitrary manner) the vertices of $N(a_1a_2) \setminus \{a_1\}$ with $\Delta_2 + h - d(a_1a_2), \ldots, \Delta_2 + h - 2 - a_1, \Delta_2 + h - a_1, \ldots, \Delta_2 + h - 1$ respectively, so $\phi(N(a_1a_2) \setminus \{a_1\}) = [\Delta_2 + h - d(a_1a_2), \Delta_2 + h - 1] \setminus \{\Delta_2 + h - 1 - a_1\}$. If $d(a_1a_2) \leq a_1 + 1$, then we label arbitrarily the vertices of $N(a_1a_2) \setminus \{a_1\}$ with $\Delta_2 + h + 1 - d(a_1a_2), \ldots, \Delta_2 + h - 1$ respectively, so $\phi(N(a_1a_2) \setminus \{a_1\}) = [\Delta_2 + h + 1 - d(a_1a_2), \Delta_2 + h - 1]$. Since $d(a_1) + d(a_1a_2) \leq \Delta_2$ by the definition of Δ_2 and $\phi(a_1a_2) \in [1, d(a_1) - 1]$ by the labelling above, in both cases these new labels satisfy the $L(h, 1, 1)$-conditions with existing labels up to level $L_2(u)$. Moreover, in view of Lemma 7(b), we can label $N(a_1a_2) \setminus \{a_1\}$ for all vertices $a_1a_2 \in L_2(u)$ independently, and thus label all vertices in $L_3(u)$ without violation of the $L(h, 1, 1)$-conditions.

If $l_u = 3$ then T_u has been labelled; otherwise we label $L_4(u)$ as follows. Note that $\phi(a_1) = \Delta_2 + h - 1 - a_1, \phi(a_1a_2) \in [1, d(a_1) - 1]$ and $\phi(a_1a_2a_3) \in [\Delta_2 + h - d(a_1a_2), \Delta_2 + h - 1] \setminus \{\Delta_2 + h - 1 - a_1\}$ or $[\Delta_2 + h + 1 - d(a_1a_2), \Delta_2 + h - 1]$ by the labelling above. We distinguish the following two cases for level $L_4(u)$.

We first consider the case where $d(a_1a_2a_3) \geq \phi(a_1a_2) + 1$. In this case, if $d(a_1a_2a_3) \leq \Delta_2 + h - 1 - a_1$, then we label arbitrarily the vertices of $N(a_1a_2a_3) \setminus \{a_1a_2\}$ with $0, 1, \ldots, \phi(a_1a_2) - 1, \phi(a_1a_2) + 1, \ldots, d(a_1a_2a_3) - 1$ (that is, $\phi(N(a_1a_2a_3) \setminus \{a_1a_2\}) = [0, d(a_1a_2a_3) - 1] \setminus \{\phi(a_1a_2)\}$); if $d(a_1a_2a_3) \geq \Delta_2 + h - a_1$, then we label these vertices arbitrarily with $0, 1, \ldots, \phi(a_1a_2) - 1, \phi(a_1a_2) + 1, \ldots, \Delta_2 + h - 2 - a_1, \Delta_2 + h - a_1, \ldots, d(a_1a_2a_3)$ (that is, $\phi(N(a_1a_2a_3) \setminus \{a_1a_2\}) = [0, d(a_1a_2a_3)] \setminus \{\phi(a_1a_2), \Delta_2 + h - 1 - a_1\}$). Since $d(a_1a_2) + d(a_1a_2a_3) \leq \Delta_2$, in each possibility these new labels satisfy the $L(h, 1, 1)$-conditions with existing labels up to level $L_3(u)$.

Next we assume $d(a_1a_2a_3) \leq \phi(a_1a_2)$. In this case, we have $d(a_1a_2a_3) \leq \phi(a_1a_2) \leq d(a_1) - 1 \leq d(v) - 1 < \phi(a_1) + 1 = \Delta_2 + h - a_1$. Thus, we label the vertices of $N(a_1a_2a_3) \setminus \{a_1a_2\}$ with $0, 1, \ldots, d(a_1a_2a_3) - 2$ (that is, $\phi(N(a_1a_2a_3) \setminus \{a_1a_2\}) = [0, d(a_1a_2a_3) - 2]$). Again, since $d(a_1a_2) + d(a_1a_2a_3) \leq \Delta_2$, these new labels satisfy the $L(h, 1, 1)$-conditions with the vertices up to level $L_3(u)$.

By Lemma 7(b) we can label $N(a_1a_2a_3) \setminus \{a_1a_2\}$ for all $a_1a_2a_3 \in L_3(u)$ independently in the above way, and thus label $L_4(u)$, without violating the $L(h, 1, 1)$-conditions.

In general, we prove by induction that the following hold for $i = 1, \ldots, l_u$ when T is finite and for all integers $i \geq 1$ when T is infinite:

(a) if i is odd, then for all $a_1 \cdots a_{i-1} \in L_{i-1}(u)$ we can label independently the vertices of $N(a_1 \cdots a_{i-1}) \setminus \{a_1 \cdots a_{i-2}\}$ with the $d(a_1 \cdots a_{i-1}) - 1$ *largest available integers* in $[\Delta_2 + h - 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]$ such that the $L(h, 1, 1)$-conditions are satisfied among vertices of T_u up to level $L_i(u)$;

(b) if i is even, then for all $a_1 \cdots a_{i-1} \in L_{i-1}(u)$ we can label independently the vertices of $N(a_1 \cdots a_{i-1}) \setminus \{a_1 \cdots a_{i-2}\}$ with the $d(a_1 \cdots a_{i-1}) - 1$ *smallest available integers* in $[0, d(a_1 \cdots a_{i-1})]$ such that the $L(h, 1, 1)$-conditions are satisfied among vertices of T_u up to level $L_i(u)$.

The discussion above established these statements for $i = 1, 2, 3, 4$. Suppose that (a) and (b) are true for all levels up to $i \leq l_u - 1$, implying that we have labelled all vertices of T_u up to level $L_i(u)$ without violating the $L(h, 1, 1)$-conditions. In the following we prove that they are true for level $i + 1$ as well. We will repeatedly use the property that $d(a_1 \cdots a_{i-1}) + d(a_1 \cdots a_i) \leq \Delta_2$ (by the definition of Δ_2) without mentioning it explicitly. Since there is no danger of confusion, we use the following simplified notation:

$$A_t := N(a_1 \cdots a_{t-1}) \setminus \{a_1 \cdots a_{t-2}\}, \qquad x_t := \phi(a_1 \cdots a_t), \quad i - 3 \leq t \leq i.$$

Case 1. *i is even.*

Since i is even, we have $\phi(A_i) \subset [0, d(a_1 \cdots a_{i-1})]$ by the induction hypothesis. Thus, $x_i \le d(a_1 \cdots a_{i-1}) \le \Delta_2 - d(a_1 \cdots a_{i-2}) \le x_{i-1}$, where the second inequality is from the definition of Δ_2 and the last one is from (a) applied to $i-1$. Similarly, $x_{i-1} \ge \Delta_2 - d(a_1 \cdots a_{i-2}) \ge d(a_1 \cdots a_{i-3}) \ge x_{i-2}$ and $x_{i-3} \ge \Delta_2 - d(a_1 \cdots a_{i-4}) \ge d(a_1 \cdots a_{i-3}) \ge x_{i-2}$. Thus, by the $L(h, 1, 1)$-conditions we have

$$x_{i-3} \ge x_{i-2} + h, \qquad x_{i-1} \ge x_{i-2} + h, \qquad x_{i-1} \ge x_i + h \tag{10}$$

and $x_{i-3}, x_{i-2}, x_{i-1}, x_i$ are pairwise distinct.

In the case where $x_{i-1}, x_{i-2} \le \Delta_2 + h - d(a_1 \cdots a_i)$, we can label the vertices of A_{i+1} with the integers in $[\Delta_2 + h + 1 - d(a_1 \cdots a_i), \Delta_2 + h - 1]$ without violating the $L(h, 1, 1)$-conditions. Henceforth we assume that at least one of x_{i-1} and x_{i-2} is at least $\Delta_2 + h + 1 - d(a_1 \cdots a_i)$. Since $x_{i-1} > x_{i-2}$ by (10), this implies that $x_{i-1} \ge \Delta_2 + h + 1 - d(a_1 \cdots a_i)$. If $x_{i-2} \ge \Delta_2 + h - 1 - d(a_1 \cdots a_i)$, then $x_{i-3} \ge \Delta_2 + 2h - 1 - d(a_1 \cdots a_i)$ by (10) and hence $\phi(A_i) = [0, d(a_1 \cdots a_{i-1}) - 2]$ by the induction hypothesis. In this case we can label the vertices of A_{i+1} with the integers in $[\Delta_2 + h - 1 - d(a_1 \cdots a_i), \Delta_2 + h - 1] \setminus \{x_{i-1}, x_{i-2}\}$ without violation of the $L(h, 1, 1)$-conditions. If $x_{i-2} \le \Delta_2 + h - 2 - d(a_1 \cdots a_i)$, then since $\phi(A_i) \subset [0, d(a_1 \cdots a_{i-1})]$ we can label the vertices of A_{i+1} with the integers in $[\Delta_2 + h - d(a_1 \cdots a_i), \Delta_2 + h - 1] \setminus \{x_{i-1}\}$ without violation of the $L(h, 1, 1)$-conditions.

Case 2. *i is odd.*

Since i is odd, by the induction hypothesis we have $\phi(A_i) \subset [\Delta_2 + h - 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]$. Applying the induction hypothesis to $L_{i-3}(u)$ and $L_{i-1}(u)$, we get $x_{i-3} \le d(a_1 \cdots a_{i-4}) \le \Delta_2 - d(a_1 \cdots a_{i-3}) \le x_{i-2}, x_{i-1} \le d(a_1 \cdots a_{i-2}) \le \Delta_2 - d(a_1 \cdots a_{i-3}) \le x_{i-2}$ and $x_{i-1} \le d(a_1 \cdots a_{i-2}) \le \Delta_2 - d(a_1 \cdots a_{i-1}) \le x_i$. Hence

$$x_{i-2} \ge x_{i-3} + h, \qquad x_{i-2} \ge x_{i-1} + h, \qquad x_i \ge x_{i-1} + h \tag{11}$$

and $x_{i-3}, x_{i-2}, x_{i-1}, x_i$ are pairwise distinct.

In the case where $x_{i-1}, x_{i-2} \ge d(a_1 \cdots a_i) - 1$, we label the vertices of A_{i+1} with the integers in $[0, d(a_1 \cdots a_i) - 2]$. So we may assume that at least one of x_{i-1} and x_{i-2} is smaller than $d(a_1 \cdots a_i) - 1$, which implies $x_{i-1} \le d(a_1 \cdots a_i) - 2$ in view of (11). If $x_{i-2} \le d(a_1 \cdots a_i)$, then $x_{i-3} \le d(a_1 \cdots a_i) - h$ by (11), and hence $\phi(A_i) = [\Delta_2 + h + 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]$ by the induction hypothesis. In this case we label A_{i+1} with $[0, d(a_1 \cdots a_i)] \setminus \{x_{i-1}, x_{i-2}\}$. If $x_{i-2} \ge d(a_1 \cdots a_i) + 1$, then since $\phi(A_i) \subset [\Delta_2 + h - 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]$ we can label A_{i+1} with $[0, d(a_1 \cdots a_i) - 1] \setminus \{x_{i-1}\}$. In each possibility the $L(h, 1, 1)$-conditions are satisfied by the labels for A_{i+1}.

Up to now we have proved (a) and (b) by induction and thus finished labelling T_u.

Part 3 (Labelling T_v): We label T_v by using techniques similar to those above. Note first that the vertices in $L_1(v)$ were labelled in the initialization. If $l_v \ge 2$, then for each $b_1 \in L_1(v)$, we label the vertices of $N(b_1) \setminus \{v\} \subseteq L_2(v)$ with $\Delta_2 + h - d(b_1), \ldots, \Delta_2 + h - 2$ respectively, so $\phi(N(b_1) \setminus \{v\}) = [\Delta_2 + h - d(b_1), \Delta_2 + h - 2]$. We do this for all $b_1 \in L_1(v)$ independently, and in this way all vertices in $L_2(v)$ are labelled. Since $\Delta_2 - d(b_1) \ge d(v)$ and $\phi(N(v) \setminus \{u\}) = [1, d(v) - 1]$, by Lemma 7 this labelling satisfies the $L(h, 1, 1)$-conditions with vertices in $\{u, v\} \cup L_1(v)$.

Note that for each vertex $w \in \bigcup_{i \ge 3} L_i(v)$ we have $N_3(w) \subseteq V(T_v)$ and hence we can label w without considering the labels used by T_u. Like for (a) and (b), by induction we can prove the following for $i = 1, 2, \ldots, l_v$ if T is finite and for all integers $i \ge 1$ if T is infinite:

(c) if i is odd, then for all $b_1 \cdots b_{i-1} \in L_{i-1}(v)$ we can label independently the vertices of $N(b_1 \cdots b_{i-1}) \setminus \{b_1 \cdots b_{i-2}\}$ with the $d(b_1 \cdots b_{i-1}) - 1$ *smallest available integers* in $[0, d(a_1 \cdots a_{i-1})]$ such that the $L(h, 1, 1)$-conditions are satisfied among vertices of T_v up to level $L_i(v)$;

(d) if i is even, then for all $b_1 \cdots b_{i-1} \in L_{i-1}(v)$ we can label independently the vertices of $N(b_1 \cdots b_{i-1}) \setminus \{b_1 \cdots b_{i-2}\}$ with the $d(b_1 \cdots b_{i-1}) - 1$ *largest available integers* in $[\Delta_2 + h - 1 - d(b_1 \cdots b_{i-1}), \Delta_2 + h - 1]$ such that the $L(h, 1, 1)$-conditions are satisfied among vertices of T_v up to level $L_i(v)$.

The proof of these statements is similar to that of (a) and (b) and hence is omitted.

In summary, we have proved that T admits an $L(h, 1, 1)$-labelling with span $\Delta_2 + h - 1$. Therefore, $\lambda_{h,1,1}(T) \le \Delta_2 + h - 1$. □

The proof of Lemma 8 is valid for both finite and infinite cases. Clearly, it gives an algorithm for constructing an $L(h, 1, 1)$-labelling of T with span Δ_2+h-1. In the case when T is finite, this algorithm takes $O(n)$ time (where $n = |V(T)|$) for vertex indexing and initialization, and $O(n)$ time for each of the $O(n)$ rounds of labelling as described in (a)–(d). Therefore, the algorithm runs in $O(n^2)$ time for finite trees.

The truth of (1) follows from Lemmas 6 and 8 immediately. In the introduction we have shown that the lower bound in (1) is achievable. To complete the proof of Theorem 1 we now prove that the upper bound in (1) is attainable as well when $h \geq 3$. Let T^* be the tree defined by

$$V(T^*) = \{w\} \cup \{w_i, u_i, v_i : 1 \leq i \leq h+2\} \cup \{v_{i,j}, z_{i,j} : 1 \leq i \leq h+2, 1 \leq j \leq h+1\}$$
$$E(T^*) = \{ww_i, w_iu_i, u_iv_i : 1 \leq i \leq h+2\} \cup \{v_iv_{i,j}, v_{i,j}z_{i,j} : 1 \leq i \leq h+2, 1 \leq j \leq h+1\}.$$

Lemma 9. *Let $h \geq 3$ be an integer. Let T be a finite tree or an infinite tree of finite maximum degree such that $\Delta(T) = \Delta(T^*)$ $(= h+2)$, $\Delta_2(T) = \Delta_2(T^*)$ $(= h+4)$ and T contains T^* as a subtree. Then*

$$\lambda_{h,1,1}(T) = \Delta_2(T) + h - 1 = 2h + 3.$$

Proof. Since $\lambda_{h,1,1}(T^*) \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1$ by Lemma 8, it suffices to prove $\lambda_{h,1,1}(T^*) \geq 2h + 3$.

Suppose to the contrary that $\lambda_{h,1,1}(T^*) \leq 2h + 2$ and let ϕ be an $L(h, 1, 1)$-labelling of T^* with span $2h + 2$. We first prove:

Claim. If v is a maximum degree vertex of T^*, then $\phi(v) \in \{0, 1, 2h + 1, 2h + 2\}$.

Suppose otherwise (that is, $\phi(v) \in [2, 2h]$) and let A be the set of available labels for the neighbours of v. If $2 \leq \phi(v) < h$, then $A \subseteq \{\phi(v) + h, \ldots, 2h + 2\}$ and so $|A| \leq h + 3 - \phi(v) \leq h + 1$. If $h \leq \phi(v) \leq h + 2$, then $A \subseteq \{0, \ldots, \phi(v) - h\} \cup \{\phi(v) + h, \ldots, 2h + 2\}$ and hence $|A| \leq 4 \leq h + 1$. If $h + 2 < \phi(v) \leq 2h$, then $A \subseteq \{0, \ldots, \phi(v) - h\}$ and hence $|A| \leq \phi(v) - h + 1 \leq h + 1$. Since $|N(v)| = h + 2$, in each of these cases there are not enough labels for the vertices in $N(v)$. This contradiction establishes the claim.

Since w is a maximum degree vertex, by using the dual labelling $\lambda_{h,1,1}(T^*) - \phi(z)$ instead of $\phi(z)$ $(z \in V(T^*))$ when necessary, by the claim above we may assume without loss of generality that $\phi(w) \in \{0, 1\}$. Assume first that $\phi(w) = 1$. Then $\phi(\{w_1, \ldots, w_{h+2}\}) = \{h + 1, \ldots, 2h + 2\}$. Suppose without loss of generality that $\phi(w_1) = h + 1$. Then the only label available for u_1 is 0, that is, $\phi(u_1) = 0$. Now v_1 is a maximum degree vertex so $\phi(v_1) \in \{2h + 1, 2h + 2\}$ by the claim above. If $\phi(v_1) = 2h + 1$, then $\phi(\{v_{1,1}, \ldots, v_{1,h+1}\}) \subseteq \{0, \ldots, h + 1\} \setminus \{0, h + 1\} = \{1, \ldots, h\}$. This is a contradiction because we need at least $h+1$ labels for $v_{1,1}, \ldots, v_{1,h+1}$. Therefore, $\phi(v_1) = 2h+2$ and so we must have $\phi(\{v_{1,1}, \ldots, v_{1,h+1}\}) = \{1, \ldots, h, h + 2\}$. Assume without loss of generality that $\phi(v_{1,1}) = h+2$. Then $\phi(z_{1,1}) \in \{0, 1, 2, 2h+2\}$. But this is a contradiction since $0, 2h+2, 1, 2$ have been used by u_1, v_1 and two vertices in $\{v_{1,1}, \ldots, v_{1,h+1}\}$ respectively.

Assume next that $\phi(w) = 0$. Then $\phi(\{w_1, \ldots, w_{h+2}\}) \subseteq \{h, \ldots, 2h + 2\}$ and exactly one label in this set is not used by these vertices. Suppose that $h + 1$ is not used and assume without loss of generality that $\phi(w_1) = h$. Then there is no label available for u_1, a contradiction. Therefore, $h + 1$ is used and without loss of generality we may assume $\phi(w_1) = h + 1$. The labels available for u_1 are 1, $2h + 1$ and $2h + 2$, except possibly at most one of these labels. We consider the case $\phi(u_1) = 1$ only since the other two cases are similar. Since v_1 is a maximum degree vertex, by our claim it must be labelled $2h + 1$ or $2h + 2$. If $\phi(v_1) = 2h + 1$, then the available label set for $v_{1,1}, \ldots, v_{1,h+1}$ is $\{0, \ldots, h + 1\} \setminus \{1, h + 1\}$, which contains less than $h + 1$ labels, a contradiction. If $\phi(v_1) = 2h + 2$, then the available label set for $v_{1,1}, \ldots, v_{1,h+1}$ is $\{0, \ldots, h + 2\} \setminus \{1, h + 1\}$, which has cardinality $h + 1$. So we may assume without loss of generality that $\phi(v_{1,1}) = h + 2$. However, there is no label available for $z_{1,1}$, again a contradiction.

So far we have completed the proof of Theorem 1. $\square$

3. Proofs of Theorems 2 and 3

As before we abbreviate $\Delta(T)$, $\Delta_2(T)$ to Δ, Δ_2 respectively. For a set X of integers, denote by $\max X$ ($\min X$) the maximum (minimum) integer in X.

Proof of Theorem 2. Let T be a finite tree with diameter at least 3 or an infinite tree of finite maximum degree. Let $h \leq \delta^*(T)$. Choose a maximum degree vertex w as the root of T and set

$$L_i := \{v \in V(T) : d(w, v) = i\}, \quad i = 0, 1, \ldots$$

For any $v \in V(T)$ we use $p(v)$ to denote the parent of v, and $c_1(v), \ldots, c_{d(v)-1}(v)$ the children of v.

Claim. There exists an $L(h, 1, 1)$-labelling ϕ of T such that, for any $k \geq 2$ and $v \in L_k$,

$$\begin{aligned} &\phi(\{p(p(v)), c_1(p(v)), \ldots, c_{d(p(v))-1}(p(v))\}) \\ &\quad = \{a \bmod (\Delta_2 + h - 1), \ldots, (a + d(p(v)) - 1) \bmod (\Delta_2 + h - 1)\} \end{aligned} \tag{12}$$

for some $a \in [0, \Delta_2 + h - 2]$.

We prove this claim by constructing ϕ inductively. To begin with we define

$$\begin{aligned} &\phi(w) = 0 \\ &\phi(c_i(w)) = \Delta_2 + h - i - 1, \quad i = 1, \ldots, \Delta. \end{aligned}$$

For each i such that $c_i(w)$ has at least one child, define

$$\phi(c_j(c_i(w))) = j, \quad j = 1, \ldots, d(c_i(w)) - 1.$$

Clearly, (12) holds for $k = 2$ and $v \in L_2$ (with $a = 0$). Observe that the smallest label used by a child of w is $\Delta_2 + h - \Delta - 1 \geq \Delta + 2 + h - \Delta - 1 = h + 1$. Note also that $\phi(c_i(w)) - \phi(c_j(c_i(w))) = (\Delta_2 + h - i - 1) - j \geq (\Delta_2 + h - \Delta - 1) - (d(c_i(w)) - 1) \geq (\Delta_2 + h - \Delta - 1) - (\Delta_2 - \Delta - 1) = h$. Thus ϕ satisfies the $L(h, 1, 1)$-conditions among vertices in $L_0 \cup L_1 \cup L_2$.

Assume that we have labelled the vertices of T up to some level $k \geq 2$ such that (12) holds for k and $v \in L_k$ and the $L(h, 1, 1)$-conditions are satisfied among vertices up to L_k. We extend ϕ to level L_{k+1} in the following way.

For any $u \in L_k$, let $C := \phi(\{p(p(u)), c_1(p(u)), \ldots, c_j(p(u))\})$, where $j = d(p(u)) - 1$. Then, by our induction hypothesis, $C = \{a_1 \bmod (\Delta_2+h-1), \ldots, (a_1+j) \bmod (\Delta_2+h-1)\}$ for some $a_1 \in [0, \Delta_2+h-2]$. Let $A := [0, \Delta_2+h-2] \setminus C$. Then $A = \{a_2 \bmod (\Delta_2+h-1), \ldots, (a_2+b_1) \bmod (\Delta_2+h-1)\}$, where $a_2 = (a_1 + j + 1) \bmod (\Delta_2 + h - 1)$ and $b_1 = \Delta_2 + h - j - 3$. All labels in $A \setminus \{\phi(p(u))\}$ are available for the children of u, except the ones in $B := [\phi(u)-h+1, \phi(u)-1] \cup [\phi(u)+1, \phi(u)+h-1]$. Since $\phi(u) \in C$ and $h \leq \delta^*(T) \leq d(p(u)) = j + 1 = |C|$, there are at least

$$(\phi(u) - \min C) + \min\{h - 1, \ \min C + j - \phi(u)\} \geq h - 1$$

labels in B which are either in C or not in $[0, \Delta_2 + h - 2]$ at all. So $|B \cap A| \leq h - 1$ as $|B| = 2(h - 1)$. Therefore, the label set available for the children of u is $(A \setminus \{\phi(p(u))\}) \setminus B$ which has cardinality at least $(\Delta_2 + h - j - 3) - (h - 1) = \Delta_2 - j - 2$. Since $d(u) + d(p(u)) \leq \Delta_2$, u has at most $\Delta_2 - j - 2$ children and so there are enough labels in $(A \setminus \{\phi(p(u))\}) \setminus B$ to label them without violating the $L(h, 1, 1)$-conditions. Note that $A \setminus B$ has the form $\{a_3 \bmod (\Delta_2 + h - 1), \ldots, (a_3 + b_2) \bmod (\Delta_2 + h - 1)\}$ for some $a_3 \in [0, \Delta_2 + h - 2]$ and $b_2 \geq \Delta_2 - j - 2$. Because of this we may select legal labels in $\{a_3 \bmod (\Delta_2 + h - 1), \ldots, (a_3 + b_2) \bmod (\Delta_2 + h - 1)\}$ around $\phi(p(u))$ to label the children of u such that (12) holds for each child v of u. (Note that $p(p(v)) = p(u)$.) In this way, we have extended ϕ to level L_{k+1} and hence completed the proof of the claim. (If T is finite then we stop in a finite number of inductive steps. If T is infinite then we continue the labelling process indefinitely.)

Since ϕ has span $\Delta_2 + h - 2$, we have $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 2$. □

Proof of Theorem 3. Because of Lemma 6 it suffices to prove the upper bounds.

We first consider the case where T is a finite caterpillar. Let $v_1, v_2, \ldots, v_n$ be the spine of T, where $n \geq 2$. Then $\Delta_2 \geq 4$ and $2 \leq d(v_i) \leq \Delta_2 - 2$ for $i = 1, 2, \ldots, n$. Let v_0 be a fixed neighbor of v_1 other than v_2, and v_{n+1} a fixed neighbor of v_n other than v_{n-1}. Let

$$V_i := N(v_i) \setminus \{v_{i-1}, v_{i+1}\}, \quad i = 1, 2, \ldots, n.$$

In the case when $n = 2$, assigning 0 to v_1, Δ_2+h-3 to v_2, $d(v_2)+h-2, d(v_2)+h-1, \ldots, \Delta_2+h-4$ to the neighbors of v_1 other than v_2, and $1, 2, \ldots, d(v_2) - 1$ to the neighbors of v_2 other than v_1, we get an $L(h, 1, 1)$-labelling of T with span $\Delta_2 + h - 3$. Hence $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 3$. Thus we assume $n \geq 3$ in the following.

Case 1. *There exists no v_i on the spine such that $d(v_i) = \Delta_2 - 2$.*

In this case $\Delta_2 - d(v_i) \geq 3$ for $i = 1, 2, \ldots, n$ and so $\Delta_2 \geq 5$. If $\Delta_2 = 5$, then there exists a heavy edge on the spine whose end-vertices have degrees 2 and 3 ($= \Delta_2 - 2$) respectively, a contradiction. Hence $\Delta_2 \geq 6$. We first define, for $i = 0, 1, \ldots, n + 1$,

$$\phi(v_i) = \begin{cases} 0, & i \equiv 0 \pmod 4 \\ \Delta_2 + h - 3, & i \equiv 1 \pmod 4 \\ 1, & i \equiv 2 \pmod 4 \\ \Delta_2 + h - 4, & i \equiv 3 \pmod 4. \end{cases} \tag{13}$$

Then for each $i = 1, \ldots, n$ with $d(v_i) > 2$ we assign $|V_i|$ distinct labels to the vertices in V_i, one label for each vertex but in an arbitrary manner, such that

$$\phi(V_i) = \begin{cases} [\Delta_2 + h - 2 - d(v_i), \Delta_2 + h - 5], & i \equiv 0 \pmod 2 \\ [2, d(v_i) - 1], & i \equiv 1 \pmod 2. \end{cases} \tag{14}$$

Since $\Delta_2 \geq 6$, it is clear that the vertices on the spine satisfy the $L(h, 1, 1)$-conditions. For $u_i \in V_i, u_j \in V_j$, $d(u_i, u_j)$ is 2 if $i = j$, 3 if $|i - j| = 1$, and at least 4 if $|i - j| \geq 2$. From the definition of ϕ it follows that $|\phi(u_i) - \phi(u_{i+1})| \geq \Delta_2 + h - 1 - (d(v_i) + d(v_{i+1})) \geq h - 1 \geq 1$ for $i = 1, 2, \ldots, n - 1$. For $i \equiv 1 \pmod 4$, $\phi(v_i) - \max \phi(V_i) = (\Delta_2 + h - 3) - (d(v_i) - 1) \geq h + 1$ since $\Delta_2 - d(v_i) \geq 3$. For $i \equiv 2 \pmod 4$, $\min \phi(V_i) - \phi(v_i) = \Delta_2 + h - 3 - d(v_i) \geq h$. Similarly, for $i \equiv 3 \pmod 4$, $\phi(v_i) - \max \phi(V_i) = \Delta_2 + h - 3 - d(v_i) \geq h$, and for $i \equiv 0 \pmod 4$, $\min \phi(V_i) - \phi(v_i) = \Delta_2 + h - 2 - d(v_i) \geq h + 1$. Since $h \geq 2$, by the definition of ϕ for any i between 0 and $n + 1$ and any vertex u not on the spine such that $d(u, v_i) = 2$ or 3, we have $\phi(u) \neq \phi(v_i)$. Therefore, ϕ is an $L(h, 1, 1)$-labelling of T with span $\Delta_2 + h - 3$, and hence $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 3$.

Case 2. *There exists v_{i^*} on the spine such that $d(v_{i^*}) = \Delta_2 - 2$, where $1 \leq i^* \leq n$.*

In this case we have $d(v_{i^*-1}) = 2$ (if $i^* > 1$) and $d(v_{i^*+1}) = 2$ (if $i^* < n$) by the definition of Δ_2. Define, for $i = 0, 1, \ldots, n + 1$,

$$\phi(v_i) = \begin{cases} 0, & i - i^* \equiv 0 \pmod 4 \\ \Delta_2 + h - 2, & i - i^* \equiv 1 \pmod 4 \\ 1, & i - i^* \equiv 2 \pmod 4 \\ \Delta_2 + h - 3, & i - i^* \equiv 3 \pmod 4. \end{cases} \tag{15}$$

Then for each $i = 1, \ldots, n$ with $d(v_i) > 2$ assign $|V_i|$ distinct labels to the vertices in V_i, one label per vertex, such that

$$\phi(V_i) = \begin{cases} [\Delta_2 + h - 1 - d(v_i), \Delta_2 + h - 4], & i - i^* \equiv 0 \pmod 2 \\ [2, d(v_i) - 1], & i - i^* \equiv 1 \pmod 2. \end{cases} \tag{16}$$

Like for Case 1, one can verify that ϕ is an $L(h, 1, 1)$-labelling of T with span $\Delta_2 + h - 2$. Hence $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 2$.

Under any $L(h, 1, 1)$-labelling ϕ of T, the vertices in $V_{i^*} \cup \{v_{i^*-1}, v_{i^*}, v_{i^*+1}\}$ receive distinct labels, and moreover the label of v_{i^*} must differ by at least h from the labels of the other $\Delta_2 - 2$ vertices in this set. This is possible only when the span is at least $\Delta_2 + h - 3$. Moreover, if the span of ϕ is $\Delta_2 + h - 3$, then we must have $\phi(v_{i^*}) = 0$ or $\Delta_2 + h - 3$, and both $\phi(v_{i^*-2})$ (if $i^* > 1$) and $\phi(v_{i^*+2})$

(if $i^* < n$) are at least 1 or at most $\Delta_2 + h - 4$, respectively. Thus, if the span of ϕ is $\Delta_2 + h - 3$, we must have $d(v_{i^*-2}) < \Delta_2 - 2$ and $d(v_{i^*+2}) < \Delta_2 - 2$, for otherwise $\phi(v_{i^*-2})$ and $\phi(v_{i^*+2})$ are 0 or Δ_2+h-3 by the previous sentence, a contradiction. In other words, if there exist consecutive vertices u, v, w on the spine such that $d(u) = d(w) = \Delta_2 - 2$ and $d(v) = 2$, then $\lambda_{h,1,1}(T) \geq \Delta_2 + h - 2$ and hence $\lambda_{h,1,1}(T) = \Delta_2 + h - 2$ by the upper bound in the previous paragraph.

Now we assume that T is an infinite caterpillar with finite maximum degree. Then either (i) the spine of T has one open end, or (ii) it has two open ends. In the former case let the spine be $v_1, v_2, \ldots$ and let v_0 be a neighbor of v_1 other than v_2, and in the latter case let the spine be $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$. In both cases we extend the rules (13) and (15) to all vertices v_i, where $i \geq 0$ in case (i) and $i = \cdots, -2, -1, 0, 1, 2, \ldots$ in case (ii). Then we apply (14) and (16) to all V_i, where $i \geq 1$ in (i) and $i = \cdots, -2, -1, 0, 1, 2, \ldots$ in (ii). The results follow from the same argument as in the finite case. □

4. Remarks and questions

If all vertices on the spine of a finite caterpillar T have maximum degree Δ, then $\lambda_{h,1,1}(T) = \max\{h, \Delta - 1\} + \Delta = \max\{h + \Delta_2(T)/2, \Delta_2(T) - 1\}$ as shown by Jinjiang Yuan. (We are grateful to Jinjiang for informing us of this result.) This indicates that the upper bound in Theorem 3 is far away from the actual value of $\lambda_{h,1,1}$ in certain cases, although it is attainable in some other cases.

The condition $h \leq \delta^*(T)$ is sufficient but not necessary to guarantee (2). In fact, if a finite tree T of diameter at least 3 has only one heavy edge, then we can prove $\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2$ by modifying the proof of Lemma 8. To achieve this we simply decrease the labels of the vertices in $L_i(u)$ ($i \geq 1$ is odd) and $L_i(v)$ ($i \geq 0$ is even) by 1. Since T has only one heavy edge, we have $d(a_1 \cdots a_i)+d(a_1 \cdots a_{i-1}) < \Delta_2$ and $d(b_1 \cdots b_i)+d(b_1 \cdots b_{i-1}) < \Delta_2$ for $i \geq 1$, and these inequalities ensure the validity of modified statements (a)–(d).

In view of Theorem 1 and Corollaries 4 and 5, we may ask the following questions naturally.

Question 10. (a) *Given $h \geq 3$, characterize those finite trees T with diameter at least 3 such that $\lambda_{h,1,1}(T) = \Delta_2(T) + h - 1$.*

(b) *Characterize finite trees T with diameter at least 3 such that $\lambda_{2,1,1}(T) = \Delta_2(T)$.*

Similar questions may be asked for infinite trees with finite maximum degree.

We speculate that 'most' finite trees of diameter at least 3 would have $\lambda_{2,1,1}$-number $\Delta_2 - 1$. To make this precise let $N(n)$ be the number of pairwise non-isomorphic trees with n vertices and diameter at least 3, and let $N_1(n)$ be the number of such trees with $\lambda_{2,1,1} = \Delta_2 - 1$.

Conjecture 11. $\lim_{n\to\infty} \frac{N_1(n)}{N(n)} = 1$.

We finish this article by asking the following question.

Question 12. *For a fixed integer $h \geq 2$, is the problem of determining $\lambda_{h,1,1}$ for finite trees solvable in polynomial time?*

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