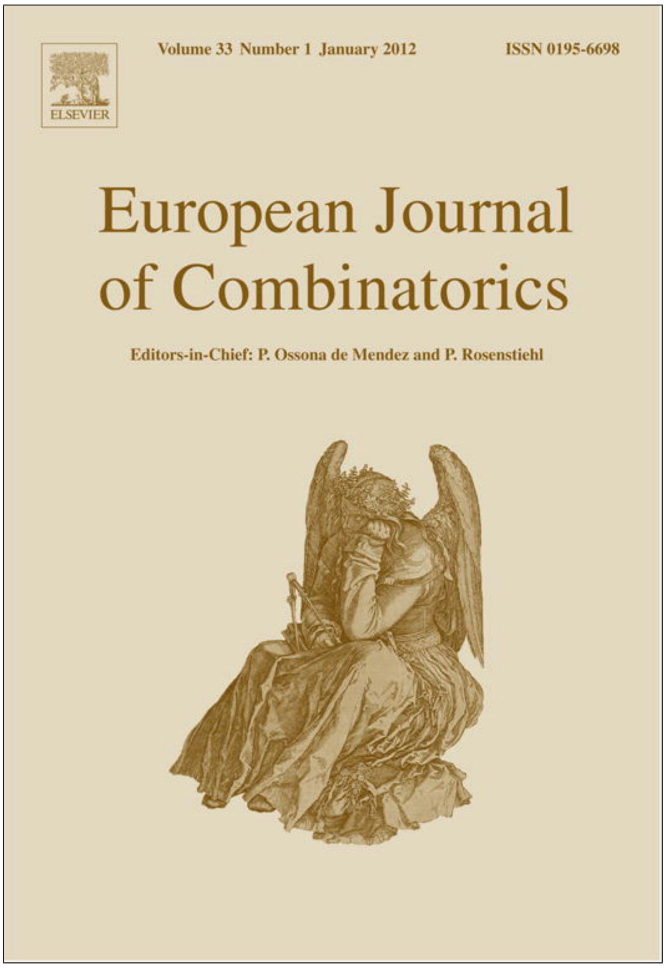
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Gossiping and routing in second-kind Frobenius graphs

Xin Gui Fang[a], Sanming Zhou[b]

[a] *LAMA and School of Mathematical Sciences, Peking University, Beijing 100871, PR China*

[b] *Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia*

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ABSTRACT

A Frobenius group is a permutation group which is transitive but not regular such that only the identity element can fix two points. It is well known that such a group is a semidirect product $G = K \rtimes H$, where K is a nilpotent normal subgroup of G. A second-kind G-Frobenius graph is a Cayley graph $\Gamma = \mathrm{Cay}(K, a^H \cup (a^{-1})^H)$ for some $a \in K$ with order $\neq 2$ and $\langle a^H \rangle = K$, where H is of odd order and x^H denotes the H-orbit containing $x \in K$. In the case when K is abelian of odd order, we give the exact value of the minimum gossiping time of Γ under the store-and-forward, all-port and full-duplex model and prove that Γ admits optimal gossiping schemes with the following properties: messages are always transmitted along shortest paths; each arc is used exactly once at each time step; at each step after the initial one the arcs carrying the message originated from a given vertex form a perfect matching. In the case when K is abelian of even order, we give an upper bound on the minimum gossiping time of Γ under the same model. When K is abelian, we give an algorithm for producing all-to-all routings which are optimal for both edge-forwarding and minimal edge-forwarding indices of Γ, and prove that such routings are also optimal for arc-forwarding and minimal arc-forwarding indices if in addition K is of odd order. We give a family of second-kind Frobenius graphs which contains all Paley graphs and connected generalized Paley graphs of odd order as a proper subfamily. Based on this and Dirichlet's prime number theorem we show that, for any even integer $r \geq 4$, there exist infinitely many second-kind Frobenius graphs with valency r and order greater than any given integer such that the kernels of the underlying Frobenius groups are abelian of odd order.

E-mail addresses: xgfang@math.pku.edu.cn (X.G. Fang), smzhou@ms.unimelb.edu.au (S. Zhou).

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1. Introduction

Cayley graphs play a significant role in the design of interconnection networks (see e.g. [1,2,8,15,20]). A number of important network topologies such as rings, hypercubes, cube-connected graphs, multi-loop networks, butterfly graphs, Knödel graphs, etc. are Cayley graphs [15,20]. In [10,27] a large class of Cayley graphs on the kernels of Frobenius groups was studied. It is shown [10] that such Frobenius graphs admit the best possible all-to-all routing and have the smallest possible edge-forwarding index. There are two kinds of Frobenius graphs depending on the nature of their Cayley sets. It is proved [31] that first-kind Frobenius graphs admit 'perfect' gossiping schemes in a sense under the store-and-forward, all-port and full-duplex model. Meanwhile, an algorithm for systematically producing such schemes (usually not unique) was given in [31]. The same paper also gave an algorithm for producing all-to-all routings in a first-kind Frobenius graph which are optimal for both edge- and arc-forwarding indices [16]. These results motivated studies of first-kind Frobenius circulant graphs, leading to classification [29,28] of such graphs with valency 4 or 6 and investigation of related combinatorial problems [30]. In contrast to first-kind Frobenius graphs, apart from the formulas [10,27] for the edge-forwarding index, no other result is known on gossiping and routing in second-kind Frobenius graphs. The purpose of this paper is to improve this situation. Since the Frobenius kernel of a finite Frobenius group is always abelian [7] except when the Frobenius complement is a group of odd order all of whose Sylow subgroups are cyclic, we will pay special attention to second-kind Frobenius graphs with abelian Frobenius kernels. The main results, Theorems 3.1 and 4.3, indicate that such graphs also exhibit appealing gossiping and routing properties. Moreover, some of them have small valency as we will see in Section 5.

Let us first introduce terminology and notation needed to present our results. Let K be a group whose identity element is denoted by 1. An *action* of K on a set V is a mapping $V \times K \to V$, $(v, x) \mapsto v^x$, such that $v^1 = v$ and $(v^x)^y = v^{xy}$ for $v \in V$ and $x, y \in K$. We use $v^K := \{v^x : x \in K\}$ to denote the *K-orbit* containing v and $K_v := \{x \in K : v^x = v\}$ the *stabilizer* of v in K. K is *semiregular* on V if $K_v = 1$ is the trivial subgroup of K for all $v \in V$, *transitive* on V if $v^K = V$ for some (and hence all) $v \in V$, and *regular* on V if it is both transitive and semiregular on V. If a group H acts on K such that $(xy)^h = x^h y^h$ for any $x, y \in K$ and $h \in H$, then H is said to act on K as a group. In this case we use $K \rtimes H$ to denote the semidirected product [9] of K by H with respect to the action.

An inverse-closed subset S of $K \setminus \{1\}$ gives rise to a *Cayley graph* $\Gamma = \mathrm{Cay}(K, S)$, which is defined to have vertex set K such that $x, y \in K$ are adjacent if and only if $xy^{-1} \in S$. Γ has valency (degree) $|S|$ and it is connected if and only if $\langle S \rangle = K$. It is well known (see e.g. [5]) that $(x, g) \mapsto xg$, $x, g \in K$, defines a regular action of K on K (as a set) which preserves the adjacency of Γ. So we may view K as a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. The permutation $x \mapsto xg$, $x \in K$, induced by g is called a *translation*. Let $\mathrm{Aut}(K, S) := \{\alpha \in \mathrm{Aut}(K) : S^\alpha = S\}$ be the setwise stabilizer of S in $\mathrm{Aut}(K)$ under the natural action of $\mathrm{Aut}(K)$ on K, and $\mathrm{Aut}(\Gamma)_1$ the stabilizer of the vertex 1 in $\mathrm{Aut}(\Gamma)$. It is readily seen (e.g. [5, Proposition 16.2]) that $\mathrm{Aut}(K, S) \le \mathrm{Aut}(\Gamma)_1$.

A *Frobenius group* G is a transitive group on a set V which is not regular on V, but has the property that the only element of G which fixes two points of V is the identity of G. It is well known (see e.g. [9, p. 86]) that a finite Frobenius group G has a nilpotent normal subgroup K, called the *Frobenius kernel*, which is regular on V. Hence $G = K \rtimes H$, where H is the stabilizer of a point of V; each such group H is called a *Frobenius complement* of K in G. Since K is regular on V, we may identify V with K in such a way that K acts on itself by right multiplication, and we may choose H to be the stabilizer of 1 so that H acts on K by conjugation. Obviously, H is semiregular on $K \setminus \{1\}$. A *G-Frobenius graph* [10] is a connected graph with vertex set V and edge set $\{\{x, y\} : (x, y) \in O\}$ for some G-orbit O on $\{(x, y) : x, y \in V, x \neq y\}$. It is proved [10, Theorem 1.4] that any G-Frobenius graph is a Cayley graph $\mathrm{Cay}(K, S)$ on its Frobenius kernel K, where for some $a \in K$ with $\langle a^H \rangle = K$ and order $|a|$,

(i) $S = a^H$ if $|H|$ is even or $|a| = 2$, or
(ii) $S = a^H \cup (a^{-1})^H$ if $|H|$ is odd and $|a| \neq 2$.

Hereinafter $x^H := \{h^{-1}xh : h \in H\}$ is the H-orbit containing $x \in K$ under the action of H on K (by conjugation). Conversely, for any $a \in K$ with $\langle a^H \rangle = K$, the Cayley graph $\mathrm{Cay}(K, S)$ with S as above is a G-Frobenius graph [10, Theorem 1.4]. Since G is a Frobenius group, H can be regarded as a subgroup of $\mathrm{Aut}(K)$. Hence $H \le \mathrm{Aut}(K, S) \le \mathrm{Aut}(\Gamma)_1$ and consequently $G \le \mathrm{Aut}(\Gamma)$. We call $\mathrm{Cay}(K, S)$ a *first-* or *second-kind* G-Frobenius graph according as whether S is given by (i) or (ii).

A process of disseminating a distinct message at every vertex in a network to all other vertices is called *gossiping*. Motivated by practical applications, various gossiping models have been extensively studied; see [17] for a survey of the state-of-the-art on gossiping and broadcasting. In line with in [31], in the present paper we will analyze efficiency of second-kind Frobenius graphs in terms of gossiping under the store-and-forward, all-port and full-duplex model [4]: a vertex must receive a message wholly before retransmitting it to other vertices; a vertex can exchange messages (which may be different) with all of its neighbors at each time step; messages can traverse an edge in both directions simultaneously; no two messages can transmit over the same arc at the same time (an *arc* is an ordered pair of adjacent vertices); and it takes one time step to transmit any message over an arc. A procedure fulfilling the gossiping under this model is called a *gossiping scheme* for short. The *minimum gossip time* [4] of a graph Γ, denoted by $t(\Gamma)$, is the minimum number of time steps required by a gossiping scheme for Γ. Since a vertex of valency k can receive at most k messages at each time step, as noted in [4, Proposition 7] any graph Γ of minimum valency $\delta(\Gamma)$ satisfies

$$t(\Gamma) \ge \left\lceil \frac{|V(\Gamma)| - 1}{\delta(\Gamma)} \right\rceil. \tag{1}$$

An all-to-all routing (or an *routing* for short) of Γ is a set of oriented paths, one for each ordered pair of distinct vertices. The *load of an edge* is the number of times it is traversed by such paths in either direction; the *load of a routing* is the maximum load on an edge; and the *edge-forward index* $\pi(\Gamma)$ is [16] the minimum load over all possible routings of Γ. The *arc-forwarding index* $\overrightarrow{\pi}$ is defined similarly by taking the direction into account when counting the number of times an arc is traversed. A routing is a *shortest-path routing* if all paths used are shortest paths between their end-vertices. The *minimal edge-* and *arc-forwarding indices* [15], π_m, $\overrightarrow{\pi}_m$, are defined by restricting to shortest-path routings in the definitions of π and $\overrightarrow{\pi}$ respectively. Clearly,

$$\pi_m(\Gamma) \ge \pi(\Gamma) \ge \frac{\sum_{(u,v)\in V(\Gamma)\times V(\Gamma)} d(u, v)}{|E(\Gamma)|} \tag{2}$$

$$\overrightarrow{\pi}_m(\Gamma) \ge \overrightarrow{\pi}(\Gamma) \ge \frac{\sum_{(u,v)\in V(\Gamma)\times V(\Gamma)} d(u, v)}{2|E(\Gamma)|}. \tag{3}$$

The reader is referred to [12,24] for problems and results relating to various routing models.

In this paper we will first prove that, for any second-kind $K \rtimes H$-Frobenius graph Γ, $t(\Gamma)$ is at most twice as big as the right-hand side of (1), the latter being $(|K| - 1)/2|H|$ in this case. As a consequence the construction to be given to prove this bound implies a 2-factor approximation algorithm for computing $t(\Gamma)$. In the case when K is abelian, we will prove that $t(\Gamma)$ is at most $(|K| - 1)/2|H|$ plus the ratio of the number of involutions of K to $2|H|$. In particular, if K is abelian of odd order, then $t(\Gamma) = (|K| - 1)/2|H|$, which is exactly the lower bound (1) and hence is the best that one can hope. Moreover, in this case we will prove that there exist optimal gossiping schemes for Γ with the following properties: messages are always transmitted along shortest paths; each arc is used exactly once at each time step; at each step after the initial one, the arcs carrying the message originated from a given vertex form a perfect matching. We will give an algorithm for producing such optimal gossiping schemes.

In [10,27] it is proved that, for any Frobenius graph (of either kind), the equalities in (2) hold. In the present paper we will give an algorithm for producing routings which are optimal for π and π_m in a second-kind $K \rtimes H$-Frobenius graph with K abelian, and we will prove that such routings are also optimal for $\overrightarrow{\pi}$ and $\overrightarrow{\pi}_m$ when in addition K is of odd order. This algorithm and the one in the previous paragraph are based on the same subgraph structures, and both algorithms rely on knowledge of H-orbits on K. Given such H-orbits, both algorithms have complexity a polynomial of $|K|$. In some typical cases such as when K is a cyclic group, both algorithms can be easily implemented using, say, MAGMA [6].

Table 1
Properties of second-kind Frobenius graphs on Frobenius groups $K \rtimes H$.

Properties	Any $K \rtimes H$	K abelian	K abelian & $\|K\|$ odd
Order	$\|K\|$	$\|K\|$	$\|K\|$
Valency	$\|H\|$	$\|H\|$	$\|H\|$
Hamiltonian?	Maybe	Yes [23]	Yes [23]
Edge-forwarding index π	Best possible [10]	–	–
Minimal e-f index π_m	Best possible [10]	–	–
Optimal routing for π and π_m?	Unknown	Given in this paper	Given in this paper
Arc-forwarding index $\overrightarrow{\pi}$	Unknown	Unknown	Best possible
Minimal a-f index $\overrightarrow{\pi}_m$	Unknown	Unknown	Best possible
Optimal routing for $\overrightarrow{\pi}$ and $\overrightarrow{\pi}_m$?	Unknown	Unknown	Given in this paper
Gossiping time	$\leq 2 \cdot$ (trivial bound)	–	Best possible
Gossiping algorithm	2-factor approximation	–	Exact algorithm with nice properties given in this paper

The results above show that, although it is computationally difficult to determine $t, \pi_m, \pi, \overrightarrow{\pi}$ and $\overrightarrow{\pi}_m$ for general graphs, for any second-kind $K \rtimes H$-Frobenius graph with K abelian of odd order we know the exact values of these invariants and moreover we give algorithms for constructing optimal gossiping and routing schemes. Furthermore, such graphs achieve the smallest possible gossiping time (right-hand side of (1)) and forwarding indices (right-hand sides of (2)–(3)). They are thus very efficient for gossiping and routing in terms of the models considered. In addition, when H has a small order, the valency $2|H|$ of such a graph is small, meeting a key requirement in network design. Furthermore, such a second-kind Frobenius graph when K is abelian is Hamiltonian, meeting another desirable requirement in network design, because any Cayley graph on an abelian group of order at least three is Hamiltonian [23].

We remark that all results above for abelian K are valid for second-kind G-Frobenius graphs with G sharply 2-transitive, because in this case the kernel of G is known to be abelian (e.g. [9, Theorem 3.4B]).

Table 1 summarizes properties of second-kind Frobenius graphs. As we see in the table, in the case when K is abelian of odd order, second-kind $K \rtimes H$-Frobenius graphs have smallest possible forwarding indices and gossiping time; in this sense they are efficient for gossiping and routing from a theoretical point of view. Moreover, they have small valencies if in addition $|H|$ is small. A well-known conjecture (see e.g. [23]) asserts that any connected Cayley graph with at least three vertices is Hamiltonian. It would be interesting to investigate this conjecture for second-kind Frobenius graphs with nonabelian K. Another important remaining problem is to determine or estimate diameters of second-kind Frobenius graphs. This will be a challenging task since the class of second-kind Frobenius graphs is huge and different such graphs may behave significantly differently. It is believed that the diameter of such a graph depends not only on its order $|K|$ and valency $|H|$, but also on the structure of the group $K \rtimes H$ and the choice of the connection set $S = a^H \cup (a^{-1})^H$. It seems hopeless to find a uniform formula for diameters of all second-kind Frobenius graphs. Therefore, it may be more promising to focus on some concrete second-kind Frobenius graphs such as the ones to be discussed in Section 5.

From a practical point of view it would be desirable to explicitly construct second-kind $K \rtimes H$-Frobenius graphs of small valency with K abelian of odd order. To this end we will present in Section 5 a large family of second-kind $K \rtimes H$-Frobenius graphs with K abelian of odd order which contains all Paley graphs and connected generalized Paley graphs of odd order [22]. As a consequence we will see that, for any even integer $r \geq 4$, there exist second-kind Frobenius graphs with fixed valency r and order larger than any given number (Corollary 5.7). Thus when r is small we obtain large networks with small valency, and they are efficient in terms of gossiping and routing by our main results mentioned above.

We would like to emphasize that in this paper we only consider all-to-all routing and gossiping under the store-and-forward, all-port and full-duplex model. It would be interesting to investigate behavior of second-kind Frobenius graphs under other routing and gossiping models [14]. For example, one may consider gossiping under the store-and-forward, 1-port and full-duplex model [3,4,14]. (Here “1-port” means that a vertex can only communicate with one of its neighbors at any time.) Comparison of second-kind Frobenius graphs with other well-known interconnection networks is another interesting topic. In particular, it would be interesting to compare them with Knödel graphs since the latter are popular topologies for interconnection networks. The reader is referred to [19] for the original definition of Knödel graphs, [3] for their optimal gossiping (in the 1-port mode), [11] for a survey on Knödel graphs, and [13] for a logarithmic time 2-approximation algorithm for shortest paths in the Knödel graph on 2^d vertices with valency d.

The reader is referred to [9,25,26] for undefined notation and terminology on groups. We use $\{u, v\}$ to denote the edge between u and v, (u, v) the arc from u to v, and $A(\Gamma)$ the set of arcs of a graph Γ.

2. Shortest-path spanning trees

In this preliminary section we prove the existence and give constructions of a family of shortest-path spanning trees in a second-kind Frobenius graph which has properties needed in our later construction of gossiping and routing schemes.

Notation. Throughout this section $\Gamma = \mathrm{Cay}(K, S)$ is a second-kind G-Frobenius graph, where $G = K \rtimes H$ is a Frobenius group such that $|H|$ is odd and $S = a^H \cup (a^{-1})^H$ for some $a \in K$ satisfying $|a| \neq 2$ and $\langle a^H \rangle = K$. Whenever we require K to be abelian, we will state this explicitly. Denote by $d(x, y)$ the distance in Γ between x and y, and $d = d(\Gamma)$ the diameter of Γ. Denote $\Gamma_i(x) := \{y \in V(\Gamma) : d(x, y) = i\}$, $i = 0, 1, \ldots, d$. In particular, $\Gamma(x) := \Gamma_1(x)$ is the neighborhood of x in Γ (that is, the set of vertices adjacent to x). Since $H \le \mathrm{Aut}(\Gamma)_1$, each $\Gamma_i(1)$ is H-invariant and hence is a union of H-orbits on K. The number of H-orbits contained in $\Gamma_i(1)$ is denoted by n_i, and $(n_1, n_2, \ldots, n_d)$ is called [10] the *type* of Γ. For $X \subseteq K$, call $(\cup_{x \in X} \Gamma(x)) \setminus X$ the *neighborhood* of X in Γ.

Lemma 2.1. *For each $i = 1, \ldots, d$ and $x \in K \setminus \{1\}$, $x^H \subseteq \Gamma_i(1)$ if and only if $(x^{-1})^H \subseteq \Gamma_i(1)$. Moreover, $x^H = (x^{-1})^H$ if and only if $|x| = 2$.*

Proof. Suppose $d(x, 1) = i$, so that $x = s_i \cdots s_1$ for some $s_1, \ldots, s_i \in S$. Then $x^{-1} = s_1^{-1} \cdots s_i^{-1}$ and so $d(x^{-1}, 1) \le d(x, 1)$. Similarly, $d(x, 1) \le d(x^{-1}, 1)$. Hence $d(x^{-1}, 1) = d(x, 1)$; that is, $x^H \subseteq \Gamma_i(1)$ if and only if $(x^{-1})^H \subseteq \Gamma_i(1)$.

Clearly, if $|x| = 2$, then $x^H = (x^{-1})^H$. Conversely, if $x^H = (x^{-1})^H$ for some $x \in K \setminus \{1\}$, then $x^{-1} = h^{-1}xh = x^h$ for some $h \in H$, and hence $x^{h^2} = x$. Since H is semiregular on $K \setminus \{1\}$, it follows that $h^2 = 1$. However, H has no involution since $|H|$ is odd. Therefore, $h = 1$ and $|x| = 2$. □

Lemma 2.2. *There are exactly two G-orbits on the arcs of Γ, namely $A_+ = (1, a)^G$ and $A_- = (1, a^{-1})^G$. Moreover, if K is abelian, then $(x, y) \in A_+$ if and only if $(y, x) \in A_-$, which in turn is true if and only if $(x^{-1}, y^{-1}) \in A_-$. In particular, Γ is G-edge-transitive when K is abelian.*

Proof. Since Γ is G-vertex transitive, any G-orbit on the arcs of Γ is of the form $(1, y)^G$ for some arc $(1, y)$ of Γ, where $y \in S$. Since H fixes 1, all arcs $(1, y)$ for $y \in a^H$ are in the same G-orbit, and all arcs $(1, y)$ for $y \in (a^{-1})^H$ are in the same G-orbit. We claim that $(1, a)$ and $(1, a^{-1})$ are in different G-orbits. Suppose otherwise. Then there exists $hx \in G$ (where $x \in K$ and $h \in H$) such that $(1, a)^{hx} = (1, a^{-1})$. So $1 = 1^{hx} = x$ and $a^{-1} = a^{hx} = a^h x = a^h$. Similarly to the proof of Lemma 2.1, we obtain $h = 1$ and so $a^{-1} = a$. This contradicts the assumption that $|a| \neq 2$. Hence $(1, a)$ and $(1, a^{-1})$ are in different G-orbits. Therefore there are precisely two G-orbits on the arcs of Γ, which are $A_+ = (1, a)^G$ and $A_- = (1, a^{-1})^G$.

Now suppose K is abelian. If $(x, y) \in A_+$, then there exist $g \in K$ and $h \in H$ such that $(x, y) = (1, a)^{hg}$. Hence $(y, x) = (a, 1)^{hg} = (1, a^{-1})^{ahg} \in A_-$ and $(x^{-1}, y^{-1}) = (g^{-1}, g^{-1}(a^{-1})^h) = (g^{-1}, (a^{-1})^h g^{-1}) = (1, a^{-1})^{hg^{-1}} \in A_-$. Similarly, if $(y, x) \in A_-$, then $(y, x) = (1, a^{-1})^{h_1 g_1}$ for some $g_1 \in K$ and $h_1 \in H$ and so $(x, y) = (a^{-1}, 1)^{h_1 g_1} = (1, a)^{a^{-1} h_1 g_1} \in A_+$. Finally, if $(x^{-1}, y^{-1}) \in A_-$, then $(x^{-1}, y^{-1}) = (1, a^{-1})^{h_2 g_2} = (g_2, (a^{-1})^{h_2} g_2)$ for some $g_2 \in K$ and $h_2 \in H$, and hence $(x, y) = (g_2^{-1}, a^{h_2} g_2^{-1}) = (1, a)^{h_2 g_2^{-1}} \in A_+$ by using the assumption that K is abelian. □

The next lemma is obvious.

Lemma 2.3. *Suppose K is abelian. Then $u, v \in K$ are adjacent in Γ if and only if u^{-1}, v^{-1} are adjacent in Γ.*

Note that, since $H \leq \mathrm{Aut}(\Gamma)$, if u, v are adjacent in Γ, then u^h, v^h are adjacent in Γ for every $h \in H$. Note also that, since H is regular on K, $|x^H| = |H|$ for every $x \in K$.

Beginning with 1, one can conduct breadth first search on H-orbits on K to find the diameter d of Γ and $\Gamma_i(1)$ for $i = 1, 2, \ldots, d$. (More precisely, starting with 1 one performs breadth first search on the quotient graph of Γ with respect to the partition $\{\{1\}, x^H : x \in K \setminus \{1\}\}$ of K.) Based on this and Lemma 2.3, in the case where K is abelian we give the following algorithm for constructing a shortest-path subtree $T_{1,1}$ of Γ with root 1 which contains exactly one vertex from each H-orbit on K.

Algorithm 2.4. Suppose K is abelian.

1. Initially, set $V(T_{1,1}) := \{1, a, a^{-1}\}$, $E(T_{1,1}) := \{\{1, a\}, \{1, a^{-1}\}\}$ and $i := 1$.
2. If $i = d$, stop and output $T_{1,1}$. Otherwise, do:
   2.1. Set $Y := \emptyset$.
   2.2. If $Y = \Gamma_i(1)$, set $i := i + 1$ and go to Step 2. Otherwise, choose an arbitrary $u^H \subseteq \Gamma_i(1) \setminus Y$ (without loss of generality we may assume $u \in V(T_{1,1})$) and consider all H-orbits $v^H \subseteq \Gamma_{i+1}(1)$ which contain a neighbor of u (without loss of generality we may assume v is adjacent to u). If $|v| \neq 2$, set
   $$V(T_{1,1}) := V(T_{1,1}) \cup \{v, v^{-1}\},\ E(T_{1,1}) := E(T_{1,1}) \cup \{\{u, v\}, \{u^{-1}, v^{-1}\}\}; \tag{4}$$
   if $|v| = 2$, set
   $$V(T_{1,1}) := V(T_{1,1}) \cup \{v\},\ E(T_{1,1}) := E(T_{1,1}) \cup \{\{u, v\}\}. \tag{5}$$
   2.3. Set $Y := Y \cup u^H$ and go to Step 2.2.

In Step 2.2 above as long as $|v| \neq 2$ we add vertices v, v^{-1} and edges $\{u, v\}, \{u^{-1}, v^{-1}\}$ to the current $T_{1,1}$, regardless of the order of u.

Let $T_{1,1}$ be the subtree of Γ with root 1 when Algorithm 2.4 terminates. For $x \in K$ and $h \in H$, let $T_{x,h}$ be the graph with root x defined by

$$V(T_{x,h}) = \{u^h x : u \in V(T_{1,1})\}, \qquad E(T_{x,h}) = \{\{u^h x, v^h x\} : \{u, v\} \in E(T_{1,1})\}. \tag{6}$$

Define

$$T_x = \cup_{h \in H} T_{x,h} \tag{7}$$

to be the union of $T_{x,h}$ and take it as rooted at x. That is, T_x has vertex set $\cup_{h\in H} V(T_{x,h})$ and edge set $\cup_{h\in H} E(T_{x,h})$. (Note that for a fixed x any two subtrees $T_{x,h}$ have x as the unique common vertex.) Denote

$$\mathcal{T} = \{T_x : x \in K\}. \tag{8}$$

Note that different choices of $u^H \subseteq \Gamma_i(1) \setminus Y$ in each call of Step 2.2 in Algorithm 2.4 may result in different $T_{1,1}$ and hence different families $\mathcal{T}$.

A spanning tree T of Γ with root x is called a *shortest-path spanning tree of* Γ if the unique path in T from x to any vertex is a shortest path in Γ. The following lemma shows that $\mathcal{T}$ above is a family of shortest-path spanning trees of Γ as promised in the beginning of this section.

Lemma 2.5. *Suppose K is abelian. Then the following statements hold for every $x \in K$ and $h \in H$.*

(a) *T_x is a shortest-path spanning tree of Γ with root x, and T_x is the translation of T_1 by x, namely $E(T_x) = \{\{ux, vx\} : \{u, v\} \in E(T_1)\}$;*
(b) *x has valency 2 in each $T_{x,h}$ and the children of x in $T_{x,h}$ are $a^h x$ and $(a^{-1})^h x$, and $T_{x,h}$ contains exactly one vertex from each H^x-orbit on K;*
(c) *for $i = 1, 2, \ldots, d-1$, each vertex of $\Gamma_{i+1}(x)$ is adjacent to exactly one vertex of $\Gamma_i(x)$ in T_x, and whenever $|i - j| \geq 2$ there is no edge of T_x between $\Gamma_i(x)$ and $\Gamma_j(x)$;*
(d) *for every edge $\{ux, vx\}$ of T_x with $d(x, vx) > d(x, ux)$ (or equivalently $d(x, vx) = d(x, ux) + 1$), the neighborhood of $v^H x$ in T_x is precisely $u^H x$ and the edges of T_x between $u^H x$ and $v^H x$ are $\{u^h x, v^h x\}$, $h \in H$, which form a perfect matching between $u^H x$ and $v^H x$. Moreover, when $|x| \neq 2$ the edges of T_x between $(u^{-1})^H x$ and $(v^{-1})^H x$ are $\{(u^{-1})^h x, (v^{-1})^h x\}$, $h \in H$.*

Proof. We first prove (a)–(d) for $x = 1$.

Since each H-orbit on $K \setminus \{1\}$ contained in $\Gamma_{i+1}(1)$ is joined by edges of Γ to at least one H-orbit on $K \setminus \{1\}$ contained in $\Gamma_i(1)$, when $Y = \Gamma_i(1)$ in Step 2.2 of Algorithm 2.4, all H-orbits on $K \setminus \{1\}$ contained in $\Gamma_{i+1}(1)$ have been examined and exactly one vertex in each of them has been added to the growing $T_{1,1}$. Hence Algorithm 2.4 produces a subtree of Γ with root 1 which contains exactly one vertex from each H-orbit on K. From the algorithm one can see that the vertex 1 has valency two in $T_{1,1}$ and the unique path between 1 and any vertex of $T_{1,1}$ is a shortest path in Γ.

Since each $h \in H$ fixes $1 \in V(T_{1,1})$, by (6) we have $1 \in V(T_{1,h})$. Since 1 is adjacent to a and a^{-1} in $T_{1,1}$, and since H fixes 1 and is transitive on a^H and $(a^{-1})^H$, by (7) every vertex in S is adjacent to 1 in T_1. For any vertex $w \in K$ with $d(1, w) = i + 1 \geq 2$, by the construction of $T_{1,1}$, w^H contains a unique vertex $v \in V(T_{1,1})$ which is adjacent to a vertex $u \in \Gamma_i(1) \cap V(T_{1,1})$ in $T_{1,1}$. Assume $v = w^h$ for some $h \in H$. Then w is adjacent to $u^{h^{-1}} \in \Gamma_i(1) \cap V(T_{1,h^{-1}})$ in $T_{1,h^{-1}}$. Since this is true for every vertex w, it follows that there is a path in T_1 from 1 to any other vertex and therefore T_1 is a connected spanning subgraph of Γ with root 1. Moreover, if u is adjacent to v in $T_{1,1}$, then $\{u^h, v^h\}$, $h \in H$ are edges of T_1 between u^H and v^H. We claim that these are the only edges between these two H-orbits on $K \setminus \{1\}$. In fact, if u^h is adjacent to v^g in T_1, where $g \in H$, then since the edges of T_1 between u^H and v^H arc obtained from $\{u, v\}$ by the action of H, we may suppose $(u^h, v^g) = (u^f, v^f)$ for some $f \in H$. Then $hf^{-1} \in H_u$ and $gf^{-1} \in H_v$. However, we have $H_u = H_v = 1$ since G is a Frobenius group with Frobenius kernel H. Thus $hf^{-1} = gf^{-1} = 1$, which implies $f = g = h$. Hence v^h is the only vertex of v^H adjacent to u^h in T_1. So $\{u^h, v^h\}$, $h \in H$ are the only edges of T_1 between u^H and v^H, and they form a perfect matching between the two H-orbits. This together with the statements above implies that T_1 is a spanning tree of Γ. Moreover, any two of $T_{1,h}$, $h \in H$ have 1 as the unique common vertex, and each $T_{1,h}$ contains exactly one vertex from each H-orbit on K. Thus, by (7) and the fact that $T_{1,1}$ is a shortest-path tree, T_1 must be a shortest-path spanning tree of Γ with root 1. One can easily verify that (a)–(d) hold when $x = 1$ from the argument above and Algorithm 2.4.

From (7) it is evident that T_x is the translation of T_1 by x. Since $K \leq \mathrm{Aut}(\Gamma)$ and T_1 is a shortest-path spanning tree of Γ with root 1, T_x is a shortest-path spanning tree of Γ with root x and thus (a) holds for every $x \in K$. By the definition of T_x and noting that the H^x-orbits on K have the form $u^H x$, $u \in K$, the truth of (b), (c) and (d) can be extended from T_1 to T_x for every $x \in K$. □

Remark 2.6. In the general case where K is not necessarily abelian, the result in Lemma 2.3 may not hold. In this case we modify Algorithm 2.4 in such a way that we use rule (5) only in Step 2.2, regardless of the order of v. Using this modified algorithm we can construct $T_{1,1}$ and consequently T_x and $\mathcal{T}$ as in (7) and (8). Similar to the proof of Lemma 2.5, one can verify that $\mathcal{T}$ has all the properties as in Lemma 2.5 except the last statement in (d).

3. Gossiping in second-kind Frobenius graphs

A gossiping scheme is called a *shortest-path gossiping scheme* if the message originated from any vertex is transmitted to any other vertex along a shortest path. Denote by $I(K)$ the set of involutions of K. Recall that $t(\Gamma)$ denotes the minimum gossiping time of Γ.

Theorem 3.1. *Suppose that $G = K \rtimes H$ is a Frobenius group and $\Gamma = \mathrm{Cay}(K, S)$ is a second-kind G-Frobenius graph, where $S = a^H \cup (a^{-1})^H$ for some $a \in K$ such that $|a| \neq 2$, $|H|$ is odd and $\langle a^H \rangle = K$. Then*

$$\frac{|K| - 1}{2|H|} \leq t(\Gamma) \leq \frac{|K| - 1}{|H|}. \tag{9}$$

Moreover, if K is abelian, then

$$t(\Gamma) \leq \frac{|K| - 1 + |I(K)|}{2|H|}. \tag{10}$$

Furthermore, if K is abelian of odd order, then

$$t(\Gamma) = \frac{|K| - 1}{2|H|} \tag{11}$$

and there exists an optimal gossiping scheme for Γ which is a shortest-path gossiping scheme such that the following hold at any time $t = 1, 2, \ldots, (|K| - 1)/2|H|$:

(a) *each arc of Γ is used exactly once for data transmission;*
(b) *for every $x \in K$ exactly $|S|$ arcs are used to transmit messages with source x, and when $t \geq 2$ the set $A_t(x)$ of such arcs form a matching of Γ;*
(c) *K is transitive on the partition $\{A_t(x) : x \in K\}$ of $A(\Gamma)$.*

Note that the bounds in (9)–(11) are independent of the type of Γ.

Since H is a subgroup of $\mathrm{Aut}(K)$, if an H-orbit on K contains an involution then all of its elements are involutions. Let k_i be the number of H-orbits on K contained in $\Gamma_i(1)$ and consisted of involutions, $i = 1, 2, \ldots, d$, where as before d is the diameter of Γ. By Lemma 2.1, the H-orbits in $\Gamma_i(1)$ containing no involutions come up in pairs. Denote such H-orbits by

$$v_{i,1}^H, (v_{i,1}^{-1})^H, \ldots, v_{i,m_i}^H, (v_{i,m_i}^{-1})^H,$$

and denote the H-orbits in $\Gamma_i(1)$ consisting of involutions by

$$v_{i,m_i+1}^H, \ldots, v_{i,m_i+k_i}^H,$$

where $2m_i + k_i = n_i$. In particular, $m_1 = 1$, $k_1 = 0$ and $v_{1,1} = a$.

In the following proof of Theorem 3.1 we first deal with the case where K is abelian. In this case let $T_{1,1}$ be obtained from Algorithm 2.4. Without loss of generality we may assume $v_{i,j} \in \Gamma_i(1) \cap V(T_{1,1})$ for each pair (i, j). By Algorithm 2.4 and Lemma 2.5(c), for $j = 1, \ldots, m_i + k_i$, $v_{i,j}$ is adjacent to a vertex $u_{i,j} \in \Gamma_{i-1}(1) \cap V(T_{1,1})$ in $T_{1,1}$, and for $j = 1, \ldots, m_i$, we also have that $v_{i,j}^{-1}$ is adjacent to $u_{i,j}^{-1} \in \Gamma_{i-1}(1) \cap V(T_{1,1})$ in $T_{1,1}$. (Note that $u_{1,1} = 1$ by the construction of T_1.) Moreover, by Lemma 2.5, for $j = 1, \ldots, m_i + k_i$ the edges of T_1 between $u_{i,j}^H$ and $v_{i,j}^H$ are $\{u_{i,j}^h, v_{i,j}^h\}$, $h \in H$, and for $j = 1, \ldots, m_i$ the edges between $(u_{i,j}^{-1})^H$ and $(v_{i,j}^{-1})^H$ are $\{(u_{i,j}^{-1})^h, (v_{i,j}^{-1})^h\}$, $h \in H$. (It may happen that $u_{i,j}^H = u_{i,j'}^H$ for $j \neq j'$.) Note that for each $x \in K$ the H^x-orbits contained in $\Gamma_i(x)$ are

$$v_{i,1}^H x, (v_{i,1}^{-1})^H x, \ldots, v_{i,m_i}^H x, (v_{i,m_i}^{-1})^H x, v_{i,m_i+1}^H x, \ldots, v_{i,m_i+k_i}^H x.$$

Denote by M_x the message originated from x. Using $\mathcal{T}$ as defined in (6)–(8), we give the following d-phase algorithm such that the first phase consists of the first step only and in the ith phase ($i \geq 2$) M_x is transmitted along the arcs of T_x from $\Gamma_{i-1}(x)$ to $\Gamma_i(x)$ for all $x \in K$ simultaneously.

Algorithm 3.2. Suppose K is abelian.

1. In the first time step, for all $x \in K$, send M_x from x to $a^h x$ and $(a^{-1})^h x$ simultaneously for all $h \in H$.
2. Do the following:
   2.1. Set $i := 2$ initially.
   2.2. If $i = d$, stop. Otherwise, do the following successively: for $j = 1, 2, \ldots, m_i$, send M_x from $u_{i,j}^h x$ to $v_{i,j}^h x$ and $(u_{i,j}^{-1})^h x$ to $(v_{i,j}^{-1})^h x$ simultaneously for all $x \in K$ and $h \in H$; and for $j = m_i + 1, m_i + 2, \ldots, m_i + k_i$, send M_x from $u_{i,j}^h x$ to $v_{i,j}^h x$ simultaneously for all $x \in K$ and $h \in H$;
   2.3. Set $i := i + 1$ and return to Step 2.2.

Remark 3.3. In the general case when K is not necessarily abelian, we can use the modified Algorithm 2.4 as described in Remark 2.6 to construct $T_{1,1}$. For each $i = 1, 2, \ldots, d$, let $v_{i,j}^H$ be the H-orbits contained in $\Gamma_i(1)$, $j = 1, 2, \ldots, n_i$. Assume that $v_{i,j}$ is adjacent to $u_{i,j} \in \Gamma_{i-1}(1) \cap V(T_{1,1})$ in $T_{1,1}$ for each pair i, j. Since K is not necessarily abelian, it may not be possible to choose $u_{i,j}$ in such a way that $v_{i,j}^{-1}$ is adjacent to $u_{i,j}^{-1}$ in $T_{1,1}$ whenever $|v_{i,j}| \neq 2$. We can modify Algorithm 3.2 such that in Step 2.2 we do the following successively: for $j = 1, 2, \ldots, n_i$, send M_x from $u_{i,j}^h x$ to $v_{i,j}^h x$ simultaneously for all $x \in K$ and $h \in H$.

Proof of Theorem 3.1. Since Γ has order $|K|$ and valency $|S| = 2|H|$, by (1) we have $t(\Gamma) \geq (|K| - 1)/2|H|$.

Suppose K is abelian first. We use the notation set up before Algorithm 3.2. We claim that Algorithm 3.2 defines a gossiping scheme for Γ. In fact, by Lemma 2.5(d), it is clear that at the same time step a vertex sends at most one message to each of its neighbors. If an arc $(u_{i,j}^h x, v_{i,j}^h x)$ is used to transmit M_x as well as another message M_y at the same time, where $y \neq x$, then one of the following occurs: (i) $(u_{i,j}^h x, v_{i,j}^h x) = (u_{i,j}^g y, v_{i,j}^g y)$ for some $g \in K$; (ii) $(u_{i,j}^h x, v_{i,j}^h x) = ((u_{i,j}^{-1})^g y, (v_{i,j}^{-1})^g y)$ for some $g \in K$. In case (i), we have $w^{hg^{-1}} = w$, where $w = u_{i,j} v_{i,j}^{-1}$. Since H is semiregular on $K \setminus \{1\}$, it follows that $hg^{-1} = 1$ and so $h = g$. However, this and $u_{i,j}^h x = u_{i,j}^g y$ imply $y = x$, which is a contradiction. In case (ii), by noting that K is abelian, we obtain $w^{hg^{-1}} = w^{-1}$, where $w = u_{i,j} v_{i,j}^{-1}$. Hence $w^{(hg^{-1})^2} = (w^{-1})^{hg^{-1}} = (w^{hg^{-1}})^{-1} = w$. Since H is semiregular on $K \setminus \{1\}$, it follows that $(hg^{-1})^2 = 1$. Thus, since $|H|$ is odd, we have $hg^{-1} = 1$ and hence w is an involution. On the other hand, since $u_{i,j}, v_{i,j}$ are adjacent in Γ, we have $w \in S$. However, since $|a| \neq 2$ and $H \leq \mathrm{Aut}(K)$, S does not contain any involution. This contradiction shows that case (ii) cannot occur as well. So we have proved that an arc of the form $(u_{i,j}^h x, v_{i,j}^h x)$ cannot be used to transmit two messages at the same time. Similarly, one can show that an arc of the form $((u_{i,j}^{-1})^h x, (v_{i,j}^{-1})^h x)$ cannot be used to transmit two messages simultaneously. Therefore, Algorithm 3.2 is a gossiping scheme. Since $\sum_{i=1}^{d} k_i = |I(K)|/|H|$, this scheme requires $\sum_{i=1}^{d}(m_i + k_i) = \sum_{i=1}^{d}(n_i + k_i)/2$ time steps. Thus $t(\Gamma) \leq \sum_{i=1}^{d}(n_i + k_i)/2 = (|K| - 1 + |I(K)|)/2|H|$ and (10) is proved.

Now we assume that K is abelian of odd order. Then $I(K) = \emptyset$ and $k_i = 0$ for $i = 1, 2, \ldots, d$. Hence (11) follows from (10) and $t(\Gamma) \geq (|K| - 1)/2|H|$, and the gossiping scheme given by Algorithm 3.2 is optimal. Since by Lemma 2.5 each T_x is a shortest-path spanning tree of Γ, by our algorithm M_x is transmitted along shortest paths to vertices in $K \setminus \{x\}$. In other words, Algorithm 3.2 gives a shortest-path gossiping scheme. By Lemma 2.2, each arc of Γ is of the form $(1, a)^{hx} = (x, a^h x)$ or $(1, a^{-1})^{hx} = (x, (a^{-1})^h x)$ for some $x \in K$ and $h \in H$. Since these arcs are pairwise distinct, by Algorithm 3.2, at time $t = 1$ each arc of Γ is used exactly once for data transmission. At a later time $t \geq 2$, say, in the jth step of the ith phase, the arcs exploited are $(u_{i,j}^h x, v_{i,j}^h x) = (u_{i,j}, v_{i,j})^{hx}$ and $((u_{i,j}^{-1})^h x, (v_{i,j}^{-1})^h x) = (u_{i,j}^{-1}, v_{i,j}^{-1})^{hx}$, $x \in K$, $h \in H$. From Lemma 2.2 one can see that these are all arcs of Γ. Moreover, by a similar argument as in the previous paragraph, one can prove that these arcs are pairwise distinct. Therefore, each arc of Γ is used exactly once for data transmission at any time $t \geq 2$. Hence (a) holds. By Algorithm 3.2 the set of arcs used to transmit M_x at time $t \geq 2$ is $A_t(x) = \{(u_{i,j}^h x, v_{i,j}^h x), ((u_{i,j}^{-1})^h x, (v_{i,j}^{-1})^h x) : h \in H\}$, and by Lemma 2.5, $A_t(x)$ is a matching of Γ. From (a) it follows that $\{A_t(x) : x \in K\}$ is a partition of $A(\Gamma)$. It is clear that K is transitive on this partition.

Finally, in the general case where K is not necessarily abelian, one can verify that the modified Algorithm 3.2 as described in Remark 3.3 gives a gossiping scheme for Γ. Since this scheme takes $\sum_{i=1}^{d} n_i = (|K| - 1)/|H|$ time steps, the upper bound in (9) follows immediately. □

4. Routing in second-kind Frobenius graphs

A routing is *edge-uniform* (*arc-uniform*, respectively) if all edges (arcs, respectively) have the same load. If a subgroup M of $\mathrm{Aut}(\Gamma)$ leaves a routing $\mathcal{P}$ *invariant* (that is, $P^g \in \mathcal{P}$ for any $P \in \mathcal{P}$ and $g \in M$) and is transitive on $E(\Gamma)$ ($A(\Gamma)$, respectively), then $\mathcal{P}$ is said [21] to be an *M-edge transitive routing* (*M-arc-transitive routing*, respectively). The following lemma is extracted from [31, Lemma 6.2].

Lemma 4.1. *If $\mathcal{P}$ is a shortest-path routing of a graph Γ and there exists a subgroup M of $\mathrm{Aut}(\Gamma)$ such that $\mathcal{P}$ is M-edge-transitive, then $\pi(\Gamma) = \pi_m(\Gamma) = \sum_{(x,y)\in V(\Gamma)\times V(\Gamma)} d(x, y)/|E(\Gamma)|$ and $\mathcal{P}$ is edge-uniform and optimal with respect to π and π_m simultaneously.*

The following result was proved in [10].

Theorem 4.2. *Suppose that $G = K \rtimes H$ is a Frobenius group and Γ is a G-Frobenius graph with type $(n_1, n_2, \ldots, n_d)$. Then*

$$\pi(\Gamma) = \pi_m(\Gamma) = \begin{cases} 2\sum_{i=1}^{d} i\, n_i, & \text{if } \Gamma \text{ is of the first kind} \\ \sum_{i=1}^{d} i\, n_i, & \text{if } \Gamma \text{ is of the second kind.} \end{cases} \tag{12}$$

As a side remark, we notice that an immediate consequence of (12) is that the Wiener index of Γ is equal to $|G|(\sum_{i=1}^{d} i\, n_i)/2$ in both cases. (The *Wiener index* of a graph Σ is defined as $\sum_{x,y\in V(\Sigma)} d(x,y)$.)

In [31] optimal routings with attractive features were given for first-kind Frobenius graphs. For second-kind Frobenius graphs, we give the following result by exploiting the same family $\mathcal{T} = \{T_x : x \in K\}$ of shortest-path spanning trees as defined in (6)–(8).

Theorem 4.3. *Suppose that $G = K \rtimes H$ is a Frobenius group such that K is abelian. Suppose further that $\Gamma = \mathrm{Cay}(K,S)$ is a second-kind G-Frobenius graph, where $S = a^H \cup (a^{-1})^H$ for some $a \in K$ such that $|a| \neq 2$, $|H|$ is odd and $\langle a^H\rangle = K$. Then the routing $\mathcal{P}$ under which the path from x to y is the unique path from x to y in T_x is a shortest-path routing of Γ. Moreover, $\mathcal{P}$ is G-edge-transitive, edge-uniform and optimal for π and π_m simultaneously. Furthermore, if in addition $|K|$ is odd, then $\overrightarrow{\pi}(\Gamma) = \overrightarrow{\pi}_m(\Gamma) = \pi(\Gamma)/2$ and $\mathcal{P}$ is arc-uniform and optimal for $\overrightarrow{\pi}$ and $\overrightarrow{\pi}_m$ as well.*

Proof. By Lemma 2.5, $\mathcal{P}$ is a shortest-path routing of Γ. Since K is normal in G, for any $x \in K$ and $g \in H$, there exists $x' \in K$ such that $xg = gx'$. Since $T_x = \cup_{h\in H} T_{x,h} = \cup_{h\in H} T_{1,1}^{hx}$, for any $y \in K$, $T_x^{gy} = \cup_{h\in H}(T_{1,1}^{hx})^{gy} = \cup_{h\in H} T_{1,1}^{hg(x'y)} = \cup_{h\in H} T_{1,1}^{h(x'y)} = T_{x'y}$. Therefore, $\mathcal{T} = \{T_x : x \in K\}$ is G-invariant and hence $\mathcal{P}$ is G-invariant as well. Since $G \le \mathrm{Aut}(\Gamma)$ and Γ is G-edge-transitive by Lemma 2.2, from Lemma 4.1 we obtain (12) and that $\mathcal{P}$ is edge-uniform and optimal for π and π_m simultaneously.

Suppose $|K|$ is odd in the sequel. Then $k_i = 0$ and $n_i = 2m_i$ for $i = 1, 2, \ldots, d$. Since $\mathcal{P}$ is G-invariant and by Lemma 2.2, G is transitive on $A_+ = (1,a)^G$, all arcs in A_+ have the same load under $\mathcal{P}$. Similarly, all arcs in $A_- = (1,a^{-1})^G = (a,1)^G$ have the same load under $\mathcal{P}$. We now prove that $(1,a)$ and $(a,1)$ have the same load. Once this is achieved, then $\mathcal{P}$ is arc-uniform, $\overrightarrow{\pi}(\Gamma) = \overrightarrow{\pi}_m(\Gamma) = \pi(\Gamma)/2$, and $\mathcal{P}$ is optimal for both $\overrightarrow{\pi}$ and $\overrightarrow{\pi}_m$.

The arcs of T_x are $(u_{i,j}^h x, v_{i,j}^h x)$, $((u_{i,j}^{-1})^h x, (v_{i,j}^{-1})^h x)$, $i = 1, 2, \ldots, d$, $j = 1, 2, \ldots, m_i$, where $u_{1,1} = 1$ and $v_{1,1} = a$. If $(1,a) = (u_{i,j}^h x, v_{i,j}^h x)$, then $x = (u_{i,j}^{-1})^h$, $a = (v_{i,j}u_{i,j}^{-1})^h$ and so $(a,1) = ((u_{i,j}^{-1})^h y, (v_{i,j}^{-1})^h y)$, where $y = v_{i,j}^h$. Conversely, if $(a,1) = ((u_{i,j}^{-1})^h y, (v_{i,j}^{-1})^h y)$, then $(1,a) = (u_{i,j}^h x, v_{i,j}^h x)$, where $x = (u_{i,j}^{-1})^h$. Similarly, $(1,a) = ((u_{i,j}^{-1})^h x, (v_{i,j}^{-1})^h x)$ if and only if $(a,1) = (u_{i,j}^h y, v_{i,j}^h y)$, where it is necessary to have $x = u_{i,j}^h$ and $y = (v_{i,j}^{-1})^h$. Hence the number of times that $(1,a)$ appears on paths of $\mathcal{P}$ is equal to the number of times that $(a,1)$ appears on paths of $\mathcal{P}$. Therefore, $(1,a)$ and $(a,1)$ have the same load under $\mathcal{P}$ and the proof is complete. $\square$

5. Generalized Paley graphs

Theorems 3.1 and 4.3 suggest that second-kind $K \rtimes H$-Frobenius graphs with K abelian of odd order are efficient in terms of gossiping and routing under the models considered. It is thus desirable to construct such graphs with small valency. In this section we give a large family of second-kind Frobenius graphs with K abelian of odd order which contains all Paley graphs and connected generalized Paley graphs of odd order [22] as a proper subfamily. We will see that some graphs in our family have small valency as desired (Example 5.5 and Corollary 5.7).

Given a prime power $q \equiv 1 \pmod 4$, the *Paley graph* $P(q)$ is the Cayley graph on the additive group of the finite field $\mathbb{F}_q$ with respect to the set of non-zero squares in $\mathbb{F}_q$. In other words, $P(q)$ has vertex set $\mathbb{F}_q$ such that $x, y \in \mathbb{F}_q$ are adjacent if and only if $x - y$ is a non-zero square in $\mathbb{F}_q$. Paley graphs are self-complementary, distance-transitive and strongly regular [5], and they are well studied over many years. It is known [10,27] that Paley graphs are second-kind Frobenius graphs of type $(2, 2)$. Hence (11) in Theorem 3.1 and Algorithm 3.2 can be applied to obtain $t(P(q)) = 2$ and optimal gossiping schemes for $P(q)$. Thus Paley graphs are efficient for gossiping and routing in some sense. However, they are not attractive candidates for interconnection networks because of their large valency $(q-1)/2$. It would be helpful if we could construct graphs with similar structure which are efficient for gossiping and routing but have small valency. We will show that this is possible and such graphs exist in our family of 'generalized Paley graphs'.

A *near field* (see e.g. [9]) is a set F with at least two elements 0 and 1 which is equipped with two binary operations $+$ and $\cdot$ such that $(F,+)$ is an abelian group with identity 0; $(F^*,\cdot)$ is a group with identity 1 (where $F^* = F \setminus \{0\}$) and $\alpha \cdot 0 = 0 \cdot \alpha = 0$ for all $\alpha \in F$; and $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ for all $\alpha, \beta, \gamma \in F$. As usual we abbreviate $\alpha \cdot \beta$ to $\alpha\beta$ and denote the additive inverse of $\alpha \in F$ by $-\alpha$ and the multiplicative inverse of $\alpha \in F^*$ by α^{-1}. It is known (see e.g. [26, 10.6.3]) that for any near field F, $(F, +)$ is an elementary abelian group $\mathbb{Z}_p^n$. Obviously, any field is a near field.

The following theorem defines the family of graphs mentioned above.

Theorem 5.1. *Let $(F, +, \cdot)$ be a finite near field of odd order. Let $\beta \in F^*$ and let $\hat{H} \neq 1$ be a subgroup of $(F^*, \cdot)$ of odd order. If the left coset $\beta\hat{H}$ of $\hat{H}$ in $(F^*, \cdot)$ is a generating set of $(F, +)$, then the Cayley graph* $\mathrm{Cay}(F, \beta\hat{H} \cup (-\beta\hat{H}))$ *on the additive group of F is isomorphic to a second-kind G-Frobenius graph for some Frobenius group G whose kernel is abelian of odd order.*

The Frobenius group in this theorem is

$$G = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} : \alpha \in \hat{H}, \beta \in F \right\},$$

where the operation is the usual matrix multiplication. Let

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} : \beta \in F \right\}, \qquad H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : \alpha \in \hat{H} \right\}.$$

Then $K \cong (F, +)$ and $H \cong (\hat{H}, \cdot)$ are subgroups of G. The following lemma can be easily proved; see [9, Example 3.4.1] in the case when F is a field.

Lemma 5.2. *The group $G = K \rtimes H$ above is a Frobenius group with Frobenius kernel K and Frobenius complement H.*

The action of G on K is such that K acts on K by right multiplication and H acts on K by conjugation. More explicitly, for $u = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \in K$ and $x = \begin{pmatrix} \alpha & 0 \\ \gamma & 1 \end{pmatrix} \in G$, $u^x = \begin{pmatrix} 1 & 0 \\ \beta\alpha + \gamma & 1 \end{pmatrix}$. In the case when F is a field of order q, G is isomorphic to the subgroup of $AGL(1, d)$ formed by those affine transformations $\beta \mapsto \beta\alpha + \gamma$, $\beta \in F$ such that $\alpha \in \hat{H}$ and $\gamma \in F$.

Lemma 5.3. *Let $(F, +, \cdot)$ be a finite near field with $|F| = p^n$ for an odd prime p and an integer $n \geq 1$. Let $\hat{H} \neq 1$ be a subgroup of $(F^*, \cdot)$ and let $G = K \rtimes H$ be as above. Then there exists a second-kind G-Frobenius graph if and only if both p and $|H|$ are odd and there exists $\beta \in F^*$ such that the left coset $\beta\hat{H}$ in $(F^*, \cdot)$ is a generating set of $(F, +)$. Moreover, all second-kind G-Frobenius graphs are of the form* $\mathrm{Cay}(K, \hat{S})$, *where*

$$\hat{S} = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} : \gamma \in \beta\hat{H} \cup (-\beta\hat{H}) \right\}$$

for some $\beta \in F^*$ such that $\beta\hat{H}$ is a generating set of $(F, +)$.

Proof. Let

$$a = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \tag{13}$$

where $\beta \in F$. Suppose $\mathrm{Cay}(K, a^H \cup (a^{-1})^H)$ is a second-kind G-Frobenius graph. Then $|H|$ is odd, a is not an involution of G, and

$$a^H = \left\{ \begin{pmatrix} 1 & 0 \\ \beta\alpha & 1 \end{pmatrix} : \alpha \in \hat{H} \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} : \gamma \in \beta\hat{H} \right\}$$

is a generating set of K. Thus β is not an involution of $(F, +)$ (and so p is odd), and $\beta\hat{H}$ is a generating set of $(F, +)$ (and so $\beta \in F^*$).

Suppose conversely that both p and $|H|$ are odd, and $\beta \in F^*$ is such that $\beta\hat{H}$ is a generating set of $(F, +)$. Since $(F, +) \cong \mathbb{Z}_p^n$ and p is odd, $(F, +)$ has no involution and in particular β is not an involution of $(F, +)$. Thus a as defined in (13) is not an involution of G and a^H is a generating set of K. Hence $\mathrm{Cay}(K, \hat{S})$ is a second-kind G-Frobenius graph of valency $2|H|$, where

$$\hat{S} = a^H \cup (a^{-1})^H = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} : \gamma \in \beta\hat{H} \cup (-\beta\hat{H}) \right\}.$$

It is evident that all second-kind G-Frobenius graphs are of the form $\mathrm{Cay}(K, \hat{S})$. □

Proof of Theorem 5.1. Using the notation above, it is easily seen that $\mathrm{Cay}(K, \hat{S})$ is isomorphic to the Cayley graph $\mathrm{Cay}(F, \beta\hat{H} \cup (-\beta\hat{H}))$ on $(F, +) \cong \mathbb{Z}_p^n$ via the bijection $F \to K,\ \delta \mapsto \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$. Hence Theorem 5.1 follows from Lemma 5.3 immediately. □

The following example shows that the family of graphs defined in Theorem 5.1 contains all connected generalized Paley graphs of odd order [22] as a proper subfamily (which then contains all Paley graphs). There are graphs in our family but not in this subfamily as we will see in Examples 5.5 and 5.8.

Example 5.4 (*Paley Graphs and Generalized Paley Graphs [22]*)**.** Let $q = p^n$ be a prime power and $k \geq 2$ a divisor of $q - 1$ such that either q or $(q - 1)/k$ is even. Let A be the subgroup of $(\mathbb{F}_q^*, \cdot)$ of order $(q - 1)/k$. The *generalized Paley graph* $\mathrm{GPaley}(q, (q - 1)/k)$ is defined [22] as the Cayley graph $\mathrm{Cay}(\mathbb{F}_q, A)$ on $(\mathbb{F}_q, +)$. Note that if ω is a primitive element of $\mathbb{F}_q$, then $A = \langle \omega^k \rangle$. Since $\omega^{(q-2)(q-1)/2} = -1$ and either q or $(q - 1)/k$ is even, we have $-A = A$ and hence $\mathrm{GPaley}(q, (q - 1)/k)$ is an undirected graph. Such graphs and their automorphism groups are studied in [22] with motivations from homogeneous factorizations of complete graphs and links to symmetric cyclotomic association schemes. In particular, $\mathrm{GPaley}(q, (q - 1)/k)$ is connected if and only if k is not a multiple of $(q - 1)/(p^m - 1)$ for any proper divisor m of n [22, Theorem 2.2]. In this case, if q is odd, then $\mathrm{GPaley}(q, (q - 1)/k)$ is the second-kind Frobenius graph $\mathrm{Cay}(\mathbb{F}_q, 1A \cup (-1A))$ as in Theorem 5.1.

In particular, if $q \equiv 1 \pmod 4$, then $\mathrm{GPaley}(q, (q - 1)/2)$ is the Paley graph $P(q)$. Alternatively, $P(q)$ is given by $\mathrm{Cay}(\mathbb{F}_q, \beta\hat{H} \cup (-\beta\hat{H}))$, where $\hat{H}$ is the subgroup of $(\mathbb{F}_q^*, \cdot)$ with order $(q - 1)/4$ and β is a square in $\mathbb{F}_q^*$. □

Example 5.5. It is known that $f(x) = x^2 + x + 7$ is a primitive polynomial over $\mathbb{F}_{11}$. So we may take $\mathbb{F}_{121}$ as $\mathbb{F}_{11}[x]/(f)$, and f has a root ω in $\mathbb{F}_{121}$ which is a primitive element of $\mathbb{F}_{121}$. The unique subgroup of $\mathbb{F}_{121}^*$ with order 3 is $\hat{H} = \langle \omega^{40} \rangle = \{1, \omega^{40} = 4\omega + 7, \omega^{80} = 7\omega + 3\}$. Since $1 \in \hat{H}$ and $\omega = 2(4\omega + 7) - (7\omega + 3)$, $\hat{H}$ is a generating set of $(\mathbb{F}_{121}, +)$. Set $S_1 = 1\hat{H} \cup (-1\hat{H}) = \{1, 4\omega + 7, 7\omega + 3, 10, 7\omega + 4, 4\omega + 8\}$ and $S_1^* = \{(0, 1), (4, 7), (7, 3), (0, 10), (7, 4), (4, 8)\}$. By Theorem 5.1, $\mathrm{Cay}(\mathbb{F}_{121}, S_1) \cong \mathrm{Cay}(\mathbb{Z}_{11}^2, S_1^*)$ is a second-kind $\mathbb{Z}_{11}^2 \rtimes \mathbb{Z}_3$-Frobenius graph.

Consider another coset $(\omega + 1)\hat{H} = \{\omega + 1, 6\omega + 5, 3\omega + 9\}$. Since $1 = 6(\omega + 1) - (6\omega + 5)$ and $\omega = (\omega + 1) - 1$, $(\omega + 1)\hat{H}$ is a generating set of $(\mathbb{F}_{121}, +)$. Let $S_2 = (\omega + 1)\hat{H} \cup (-(\omega + 1)\hat{H}) = \{\omega + 1, 6\omega + 5, 3\omega + 9, 10\omega + 10, 5\omega + 6, 8\omega + 2\}$ and $S_2^* = \{(1, 1), (6, 5), (3, 9), (10, 10), (5, 6), (8, 2)\}$. By Theorem 5.1, $\mathrm{Cay}(\mathbb{F}_{121}, S_2) \cong \mathrm{Cay}(\mathbb{Z}_{11}^2, S_2^*)$ is a second-kind $\mathbb{Z}_{11}^2 \rtimes \mathbb{Z}_3$-Frobenius graph.

The two graphs above are not generalized Paley graphs in the sense of [22] (Example 5.4) since both $q = 121$ and $(q - 1)/k = 3$ are odd. They have valency 6 which is much smaller than the valency of the Paley graph $P(121)$ of the same order. □

The following is an immediate consequence of Theorems 3.1 and 5.1.

Corollary 5.6. *Let $\Gamma = \mathrm{Cay}(F, \beta\hat{H} \cup (-\beta\hat{H}))$ be the graph in Theorem 5.1. Then the minimum gossiping time of Γ is given by*

$$t(\Gamma) = (p^n - 1)/2|\hat{H}|$$

and there exist optimal gossiping schemes for Γ such that (a) *at any time t each arc of Γ is used exactly once for data transmission;* (b) *for each $x \in K$ exactly $2|\hat{H}|$ arcs are used to transmit messages with source*

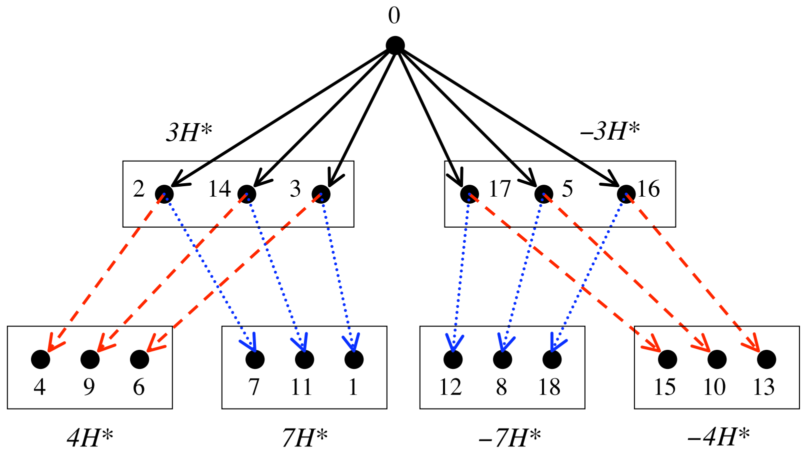

Fig. 1. A routing and gossiping tree for $\mathrm{Cay}(\mathbb{Z}_{19}, \{2, 14, 3, 17, 5, 16\})$ rooted at the identity element of $(\mathbb{F}_{19}, +)$.

x, and when $t \geq 2$ the set $A_t(x)$ of such arcs form a matching of Γ; (c) the group of translations induced by the additive group of F is transitive on the partition $\{A_t(x) : x \in K\}$ of $A(\Gamma)$.

In particular, for a connected generalized Paley graph $\mathrm{GPaley}(q, (q-1)/k)$ *as in Example 5.4, we have*

$$t(\mathrm{GPaley}(q, (q-1)/k)) = k$$

and there exists an optimal gossiping scheme for $\mathrm{GPaley}(q, (q-1)/k)$ *which has properties* (a)–(c) *above.*

The valency $2|\hat{H}|$ of the second-kind Frobenius graph $\mathrm{Cay}(F, \beta\hat{H} \cup (-\beta\hat{H}))$ in Theorem 5.1 is small when $\hat{H}$ has a small order. This is possible as we saw in Example 5.5. In fact, it is possible even when restricted to connected generalized Paley graphs as exemplified by the following corollary of Example 5.4, Theorem 5.1 and Dirichlet's Theorem on primes in an arithmetic progression.

Corollary 5.7. *For any even integer $r \geq 4$, there exist infinitely many odd primes p such that there is a second-kind Frobenius graph (connected generalized Paley graph) of order p^2 and valency r with the kernel of the underlying Frobenius group abelian.*

Proof. By the well-known Dirichlet prime number theorem (see e.g. [18]), there are infinitely many odd primes in the arithmetic progression $-1 + r, -1 + 2r, -1 + 3r, \ldots$. Let $p = -1 + \mathrm{tr}$ be such an odd prime, where $t \geq 1$ is an integer. Let $k = t(p-1)$ and $q = p^2$. Then $r = (q-1)/k$ and r is not a divisor of $p - 1$. Thus $p + 1 = (q-1)/(p-1)$ is not a divisor of k. By Example 5.4, $\mathrm{GPaley}(p^2, r)$ is a second-kind Frobenius graph of order p^2 and valency r whose underlying Frobenius group has an abelian kernel. □

Of course the even integer $r \geq 4$ in Corollary 5.7 is meant to be small for the purpose of network design. We remark that other connected generalized Paley graphs of small valency may be found by choosing appropriate p, n, k in Example 5.4.

Finally, by applying Algorithm 3.2 we can obtain optimal gossiping schemes for $\mathrm{Cay}(F, \beta\hat{H} \cup (-\beta\hat{H}))$ and $\mathrm{GPaley}(q, (q-1)/k)$. These graphs have forwarding indices given by (12) and they admit a routing optimal for the four forwarding indices simultaneously. We illustrate these by the following example.

Example 5.8. One can see that 3 is a primitive element of $\mathbb{F}_{19}$. The unique subgroup of $\mathbb{F}_{19}^*$ with order 3 is $\hat{H} = \langle 3^6 \rangle = \{3^6 = 7, 3^{12} = 11, 3^{18} = 1\}$. The coset of $\hat{H}$ in $\mathbb{F}_{19}^*$ containing 3, namely $3\hat{H} = \{7 \cdot 3 = 2, 11 \cdot 3 = 14, 3\}$, is a generating set of $(\mathbb{F}_{19}, +)$ since $2, 3 \in 3\hat{H}$ and $3 - 2 = 1$ generates $(\mathbb{F}_{19}, +)$. Since $3\hat{H} \cup (-3\hat{H}) = \{2, 14, 3, 17, 5, 16\}$, by Theorem 5.1, $\Gamma = \mathrm{Cay}(\mathbb{Z}_{19}, \{2, 14, 3, 17, 5, 16\})$ is a second-kind $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$-Frobenius graph (which is not a generalized Paley graph in the sense of [22]).

Fig. 1 depicts a spanning tree T_0 of Γ rooted at 0 constructed by using Algorithm 2.4. The spanning tree T_x of Γ rooted at $x \in \mathbb{Z}_{19}$ is obtained from T_0 by translation by x, namely, γ, δ are adjacent in T_0 if and only if $\gamma + x, \delta + x$ are adjacent in T_x. By Theorem 4.3, $\pi(\Gamma) = 2\overrightarrow{\pi}(\Gamma) = 2\overrightarrow{\pi}_m(\Gamma) = \pi_m(\Gamma) = 1 \cdot 2 + 2 \cdot 4 = 10$ and the routing whose (x, y)-path $(x, y \in \mathbb{Z}_n)$ is the unique path in T_x from x to y is optimal for these four indices simultaneously. By Theorem 3.1, $t(\Gamma) = (2 + 4)/2 = 3$.

Algorithm 3.2 gives the following optimal gossiping scheme for Γ: In the first step, send the message M_x at x from x to $x + 2, x + 14, x + 3, x + 17, x + 5, x + 16$ simultaneously for all $x \in \mathbb{Z}_{19}$. (M_0 is transmitted along the six heavy edges in Fig. 1.) In step 2, send M_x along the arcs $(x + 2, x + 4)$, $(x + 14, x + 9)$, $(x + 3, x + 6)$, $(x + 17, x + 15)$, $(x + 5, x + 10)$, $(x + 16, x + 13)$ (dashed arcs in Fig. 1 when $x = 0$) simultaneously for all x. In step 3, send M_x along the arcs $(x+2, x+7)$, $(x+14, x+11)$, $(x+3, x+1)$, $(x+17, x+12)$, $(x+5, x+8)$, $(x+16, x+18)$ (dotted arcs in Fig. 1 when $x = 0$) simultaneously for all x. $\square$

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