Two-arc transitive near-polygonal graphs

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In memory of Claude Berge

Abstract. For an integer $m \geq 3$, a near m-gonal graph is a pair $(\Sigma, \mathbf{E})$ consisting of a connected graph Σ and a set $\mathbf{E}$ of m-cycles of Σ such that each 2-arc of Σ is contained in exactly one member of $\mathbf{E}$, where a 2-arc of Σ is an ordered triple $(\sigma, \tau, \varepsilon)$ of distinct vertices such that τ is adjacent to both σ and ε. The graph Σ is call $(G, 2)$-arc transitive, where $G \leq \mathrm{Aut}(\Sigma)$, if G is transitive on the vertex set and on the set of 2-arcs of Σ. From a previous study it arises the question of when a $(G, 2)$-arc transitive graph is a near m-gonal graph with respect to a G-orbit on m-cycles. In this paper we answer this question by providing necessary and sufficient conditions in terms of the stabiliser of a 2-arc.

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1. Introduction

We consider finite, undirected and simple graphs only. For a graph $\Sigma = (V(\Sigma), E(\Sigma))$ and an integer $s \geq 1$, an *s-arc* of Σ is a sequence $(\sigma_0, \sigma_1, \ldots, \sigma_s)$ of $s+1$ vertices of Σ such that σ_{i-1} and σ_i are adjacent for $1 \leq i \leq s$ and $\sigma_{i-1} \neq \sigma_{i+1}$ for $1 \leq i \leq s-1$. For an integer $m \geq 3$, a *near m-gonal graph* [13] is a pair $(\Sigma, \mathbf{E})$, where Σ is a connected graph and $\mathbf{E}$ is a set of m-cycles of Σ, such that each 2-arc of Σ is contained in a unique member of $\mathbf{E}$. Here and in the following by an *m-cycle* we mean an undirected cycle of length m. In this case we also say that Σ is a near m-gonal graph with respect to $\mathbf{E}$, and we call cycles in $\mathbf{E}$ *basic cycles* of $(\Sigma, \mathbf{E})$. From the definition it follows that near m-gonal graphs are associated with the Buekenhout geometries [3, 13] of the following diagram:

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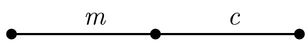

In such a geometry associated with $(\Sigma, \mathbf{E})$, the maximal flags are those triples (σ, e, C) such that $\sigma \in V(\Sigma)$, $e \in E(\Sigma)$ is incident with σ in Σ, and C is a member of $\mathbf{E}$ containing e. A near m-gonal graph with girth m is called an *m-gonal graph* [7]. (The *girth* of a graph Σ is the length of a shortest cycle of Σ if Σ contains cycles, and is defined to be ∞ otherwise.) In fact, the concept of a near-polygonal graph was introduced [13] as a generalisation of that of a polygonal graph. As a simple example, the (3-dimensional) cube together with its faces (taking as 4-cycles) is a 4-gonal graph. There are exactly four 6-cycles in the cube with the property that no three consecutive edges on the cycle belong to the same face; the cube together with these four 6-cycles is a near 6-gonal graph. Another example is the well-known embedding of the Petersen graph on the projective plane as the dual of K_6, which together with the six faces (taking as 5-cycles) is a near 5-gonal graph. The reader is referred to [7, 8, 9, 10, 11, 12, 16, 17] and [13, 14] respectively for results, constructions and more examples on polygonal graphs and near-polygonal graphs. For group-theoretic notation and terminology used in the paper, the reader may consult [1, 2].

This paper was motivated by a recent study [19] where the author found an intimate connection between near-polygonal graphs and a class of imprimitive symmetric graphs with 2-arc transitive quotients. Let Γ be a graph and G a group. If G acts on $V(\Gamma)$ as a group of automorphisms of Γ such that G is transitive on $V(\Gamma)$ and, in its induced action, transitive on the set of s-arcs of Γ, then Γ is said [1, 18] to be *(G, s)-arc transitive*. Usually, a 1-arc is called an *arc* and a $(G, 1)$-arc transitive graph is called a *G-symmetric graph*. A G-symmetric graph Γ is said to be *imprimitive* if G is imprimitive on $V(\Gamma)$, that is, $V(\Gamma)$ admits a partition $\mathbf{B}$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in \mathbf{B}$ for any *block* $B \in \mathbf{B}$ and element $g \in G$, where $B^g := \{\sigma^g : \sigma \in B\}$. In this case the *quotient graph* $\Gamma_{\mathbf{B}}$ of Γ with respect to this *G-invariant partition* $\mathbf{B}$ is defined to be the graph with vertex set $\mathbf{B}$ such that two blocks $B, C \in \mathbf{B}$ are adjacent if and only if there exists at least one edge of Γ between B and C. Denote by $\Gamma(B)$ the set of vertices of Γ adjacent to at least one vertex in B. In [19, Theorem 1.1] we proved that, if $(\Gamma, \mathbf{B})$ is an imprimitive G-symmetric graph with connected but non-complete $\Gamma_{\mathbf{B}}$ such that the subgraph (without including isolated vertices) induced by two adjacent blocks B, C of $\mathbf{B}$ is a matching of $|B| - 1 \geq 2$ edges and that $\Gamma(C) \cap B \neq \Gamma(D) \cap B$ for different blocks C, D of $\mathbf{B}$ adjacent to B, then $\Gamma_{\mathbf{B}}$ must be a $(G, 2)$-arc transitive near m-gonal graph with respect to a certain G-orbit on m-cycles of $\Gamma_{\mathbf{B}}$, where $m \geq 4$ is an even integer. Moreover, any $(G, 2)$-arc transitive near m-gonal graph (where $m \geq 4$ is even) with respect to a G-orbit on m-cycles can occur as such a quotient $\Gamma_{\mathbf{B}}$. Furthermore, the graph Γ can be reconstructed from $\Gamma_{\mathbf{B}}$ by using the 3-arc graph construction introduced in [6] by Li, Praeger and the author. For more about this construction, its extension and applications, see [6, 20], [21, 22] and [4, 5, 19, 21], respectively.

The result above motivated us to ask when a $(G, 2)$-arc transitive graph is a near m-gonal graph with respect to a G-orbit on m-cycles. In this paper we answer this question by giving necessary and sufficient conditions in terms of the stabiliser of a 2-arc.

2. Main Result

For a G-symmetric graph Σ and $\sigma, \tau, \varepsilon \in V(\Sigma)$, denote by $G_{\sigma\tau\varepsilon}$ the pointwise stabiliser of $\{\sigma, \tau, \varepsilon\}$ in G, that is, the subgroup of G consisting of those elements of G which fix each of σ, τ and ε. Denote by $\Sigma(\sigma)$ the subset of vertices of Σ which are adjacent to σ in Σ. For a subgroup H of G, let $N_G(H)$ denote the normalizer of H in G. For a near m-gonal graph $(\Sigma, \mathbf{E})$, define [13] $\mathrm{Aut}(\Sigma, \mathbf{E})$ to be the subgroup of $\mathrm{Aut}(\Sigma)$ consisting of those elements g of $\mathrm{Aut}(\Sigma)$ which leave $\mathbf{E}$ invariant, that is, $\mathbf{E}^g = \mathbf{E}$ under the induced action of $\mathrm{Aut}(\Sigma)$ on the set of m-cycles of Σ. Note that, for a near m-gonal graph $(\Sigma, \mathbf{E})$ such that Σ is $(G, 2)$-arc transitive, $G \leq \mathrm{Aut}(\Sigma, \mathbf{E})$ holds if and only if $\mathbf{E}$ is a G-orbit on m-cycles of Σ [19, Lemma 2.6].

Theorem 1. *Suppose that Σ is a connected $(G, 2)$-arc transitive graph, where $G \leq \mathrm{Aut}(\Sigma)$. Let $(\sigma, \tau, \varepsilon)$ be a 2-arc of Σ and set $H = G_{\sigma\tau\varepsilon}$. Then the following conditions* (a)-(c) *are equivalent:*

(a) *there exist an integer $m \geq 3$ and a G-orbit $\mathbf{E}$ on m-cycles of Σ such that $(\Sigma, \mathbf{E})$ is a near m-gonal graph;*

(b) *H fixes at least one vertex in $\Sigma(\varepsilon) \setminus \{\tau\}$;*

(c) *there exists $g \in N_G(H)$ such that $(\sigma, \tau)^g = (\tau, \varepsilon)$.*

Moreover, if one of these condition is satisfied, then $G \leq \mathrm{Aut}(\Sigma, \mathbf{E})$ and G is transitive on the maximal flags of the Buekenhout geometry associated with $(\Sigma, \mathbf{E})$.

Proof (a) $\Rightarrow$ (b) Suppose that $(\Sigma, \mathbf{E})$ is a near m-gonal graph for a G-orbit $\mathbf{E}$ on m-cycles of Σ, where $m \geq 3$. Let $C(\sigma, \tau, \varepsilon) = (\sigma, \tau, \varepsilon, \eta, \ldots, \sigma)$ be the basic cycle containing the 2-arc $(\sigma, \tau, \varepsilon)$. Then we have $\eta \in \Sigma(\varepsilon) \setminus \{\tau\}$. (Note that η coincides with σ when $m = 3$.) We claim that η is fixed by H. Suppose otherwise and let $\eta^g \neq \eta$ for some $g \in H$. Then, since $\mathbf{E}$ is a G-orbit on m-cycles of Σ, $(C(\sigma, \tau, \varepsilon))^g = (\sigma, \tau, \varepsilon, \eta^g, \ldots, \sigma)$ is a basic cycle containing $(\sigma, \tau, \varepsilon)$ which is different from $C(\sigma, \tau, \varepsilon)$. This contradicts with the uniqueness of the basic cycle containing a given 2-arc, and hence (b) holds.

(b) $\Rightarrow$ (c) Suppose H fixes $\eta \in \Sigma(\varepsilon) \setminus \{\tau\}$. Then we have $H \leq G_{\tau\varepsilon\eta}$. Since Σ is $(G, 2)$-arc transitive, there exists $g \in G$ such that $(\sigma, \tau, \varepsilon)^g = (\tau, \varepsilon, \eta)$ and hence $G_{\tau\varepsilon\eta} = H^g$. Therefore, $H^g = H$ and $g \in N_G(H)$.

(c) $\Rightarrow$ (a) Suppose that there exists $g \in N_G(H)$ such that $(\sigma, \tau)^g = (\tau, \varepsilon)$. Set $\eta := \varepsilon^g$. Then $\eta \in \Sigma(\varepsilon) \setminus \{\tau\}$, $(\sigma, \tau, \varepsilon)^g = (\tau, \varepsilon, \eta)$ and hence $G_{\tau\varepsilon\eta} = H^g = H$. Set $\sigma_0 = \sigma, \sigma_1 = \tau, \sigma_2 = \varepsilon$ and $\sigma_3 = \eta$, and set $\sigma_4 = \sigma_3^g$. Then $\sigma_4 \in \Sigma(\sigma_3) \setminus \{\sigma_2\}$ and $G_{\sigma_2\sigma_3\sigma_4} = (G_{\sigma_1\sigma_2\sigma_3})^g = H^g = H$. Now set $\sigma_5 = \sigma_4^g$, then similarly $\sigma_5 \in \Sigma(\sigma_4) \setminus \{\sigma_3\}$ and $G_{\sigma_3\sigma_4\sigma_5} = (G_{\sigma_2\sigma_3\sigma_4})^g = H^g = H$. Continuing this process, we obtain inductively a sequence $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \ldots$ of vertices of Σ with the following properties:

(1) $\sigma_i = \sigma_{i-1}^g$ for all $i \ge 1$, and hence $\sigma_{i+1} \in \Sigma(\sigma_i) \setminus \{\sigma_{i-1}\}$ for $i \ge 1$ and $\sigma_i = \sigma_0^{g^i}$ for $i \ge 0$; and
(2) $G_{\sigma_{i-1}\sigma_i\sigma_{i+1}} = H$ for all $i \ge 1$.

Since we have finitely many vertices in Σ, this sequence will eventually contain repeated terms. Suppose σ_m is the first vertex in this sequence which coincides with one of the preceding vertices. Without loss of generality we may suppose that σ_m coincides with σ_0 for if $\sigma_m = \sigma_i$ for some $i \ge 1$ then we can begin with σ_i and relabel the vertices in the sequence. Thus, we obtain an m-cycle

$$J := (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots, \sigma_{m-1}, \sigma_0)$$

of Σ. (It may happen that $m = 3$ if the girth of Σ is 3.) Let $\mathbf{E}$ denote the G-orbit on m-cycles of Σ containing J. In the following we will prove that each 2-arc of Σ is contained in exactly one of the "basic cycles" in $\mathbf{E}$ and hence $(\Sigma, \mathbf{E})$ is indeed a near m-gonal graph.

By the $(G, 2)$-arc transitivity of Σ, it is clear that each 2-arc $(\sigma', \tau', \varepsilon')$ of Σ is contained in at least one member J^x of $\mathbf{E}$, where $x \in G$ is such that $(\sigma', \tau', \varepsilon') = (\sigma, \tau, \varepsilon)^x$. So it suffices to show that if two members of $\mathbf{E}$ have a 2-arc in common then they are identical; or, equivalently, if J^x and J have a 2-arc in common then they are identical.

Suppose then that J^x and J have a 2-arc in common for some $x \in G$. Note that, for each $i \ge 0$, g^i maps each vertex σ_j to σ_{j+i} and so $\langle g \rangle$ leaves J invariant (subscripts modulo m here and in the rest of this proof). So, replacing J^x by J^{xg^i} for some i if necessary, we may suppose without loss of generality that $(\sigma_0, \sigma_1, \sigma_2)$ is a common 2-arc of J^x and J. Then $(\sigma_0, \sigma_1, \sigma_2) \in J^x$ implies that $(\sigma_0, \sigma_1, \sigma_2) = (\sigma_{i-1}, \sigma_i, \sigma_{i+1})^x$ for some $1 \le i \le m$. Thus, $(\sigma_0, \sigma_1, \sigma_2) = (\sigma_0, \sigma_1, \sigma_2)^{g^{i-1}x}$ and hence $g^{i-1}x \in H$. From the properties (1)-(2) above, we then have $\sigma_{j+i-1}^x = \sigma_j^{g^{i-1}x} = \sigma_j$ for each vertex σ_j on J. That is, $\sigma_j^x = \sigma_{j-i+1}$ for each j and hence $J^x = J$. Thus, we have proved that each 2-arc of Σ is contained in exactly one member of $\mathbf{E}$, and so $(\Sigma, \mathbf{E})$ is a near m-gonal graph.

So far we have proved the equivalence of (a), (b) and (c). Now assume that one of these conditions is satisfied, so that $(\Sigma, \mathbf{E})$ is a near m-gonal graph for a G-orbit $\mathbf{E}$ on m-cycles of Σ, where $m \ge 3$. Clearly, we have $G \le \mathrm{Aut}(\Sigma, \mathbf{E})$. Let (α, e, C), (α', e', C') be maximal flags of the Buekenhout geometry associated with $(\Sigma, \mathbf{E})$. Denote $e = \{\alpha, \beta\}$, $e' = \{\alpha', \beta'\}$, $C = (\alpha, \beta, \gamma, \ldots, \alpha)$ and $C' = (\alpha', \beta', \gamma', \ldots, \alpha')$. Since Σ is $(G, 2)$-arc transitive there exists $h \in G$ such that $(\alpha, \beta, \gamma)^h = (\alpha', \beta', \gamma')$. Hence $\alpha^h = \alpha'$, $e^h = e'$ and $C^h = C'$. That is, $(\alpha, e, C)^h = (\alpha', e', C')$, and thus G is transitive on the maximal flags of the Buekenhout geometry associated with $(\Sigma, \mathbf{E})$.

□

3. Remarks

The proof above gives a procedure for generating the near m-gonal graph $(\Sigma, \mathbf{E})$ guaranteed by Theorem 1. Unfortunately, it does not tell us any information about the relationship between m and the girth of Σ. Moreover, the basic cycles of $(\Sigma, \mathbf{E})$ are not necessarily induced cycles of Σ, that is, they may have chords. (See [19, Example 3.3, Proposition 3.4] for an example of such graphs. A *chord* of a cycle is an edge joining two non-consecutive vertices on the cycle.) Furthermore, from [19, Lemma 2.6(e)] such basic cycles contain chords only when either G_τ is sharply 2-transitive on $\Sigma(\tau)$ or $G_{\sigma\tau}$ is imprimitive on $\Sigma(\tau) \setminus \{\sigma\}$, where σ, τ are adjacent vertices of Σ, G_τ is the stabiliser of τ in G and $G_{\sigma\tau}$ is the pointwise stabiliser of $\{\sigma, \tau\}$ in G.

It is hoped that Theorem 1 would be useful in constructing 2-arc transitive near-polygonal graphs. In view of the 3-arc graph construction [6] and [19, Theorem 1.1], it would also be helpful in studying imprimitive G-symmetric graphs $(\Gamma, \mathbf{B})$ such that the subgraph (excluding isolated vertices) induced by two adjacent blocks B, C of $\mathbf{B}$ is a matching of $|B| - 1 \geq 2$ edges and that $\Gamma(C) \cap B \neq \Gamma(D) \cap B$ for different blocks C, D of $\mathbf{B}$ adjacent to B. A sufficient condition was given in [19] for a connected, non-complete, $(G, 2)$-arc transitive graph Σ of valency at least 3 to be a near m-gonal graph with respect to a G-orbit on m-cycles, where $m \geq 4$ is even. It was shown in [19, Corollary 4.1] that this is the case if G_σ is sharply 2-transitive on $\Sigma(\sigma)$ and one of the G-orbits on 3-arcs of Σ is self-paired. (A set A of 3-arcs is called *self-paired* if $(\sigma, \tau, \varepsilon, \delta) \in A$ implies $(\delta, \varepsilon, \tau, \sigma) \in A$.) Another sufficient condition was given in [13, Theorem 2.2] for a connected, non-complete, $(G, 2)$-arc transitive graph Σ to be a near m-gonal graph with respect to a G-orbit on m-cycles, where $m \geq 4$ is not necessarily even. Note that a near m-gonal graph $(\Sigma, \mathbf{E})$ is $(G, 2)$-arc transitive if and only if G is transitive on the maximal flags of the Buekenhout geometry associated with $(\Sigma, \mathbf{E})$. The "only if" part of this statement was proved in the last paragraph of the proof of Theorem 1, and the "if" part was part of [13, Theorem 1.8] and can be verified easily.

Finally, in the original definition [13] of a near m-gonal graph Σ, it was required that the girth of Σ be at least 4 and subsequently $m \geq 4$. In the definition given at the beginning of the introduction, we removed this requirement since the case of girth 3 is not entirely uninteresting when the graph is not 2-arc transitive. Of course for 2-arc transitive graphs this case is not so interesting, because a connected 2-arc transitive graph has girth 3 if and only if it is a complete graph (see e.g. [19, Lemma 2.5]). This is perhaps the main reason [15] for requiring girth ≥ 4 in a near-polygonal graph in [13], since the research in the area is focused on 2-arc transitive near-polygonal graphs.

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