Gossiping and Routing in Undirected Triple-Loop Networks

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Given integers $n \geq 7$ and a, b, c with $1 \leq a, b, c \leq n-1$ such that $a, n-a, b, n-b, c, n-c$ are pairwise distinct, the (undirected) triple-loop network $TL_n(a, b, c)$ is the degree-six graph with vertices $0, 1, 2, \ldots, n-1$ such that each vertex x is adjacent to $x \pm a$, $x \pm b$, and $x \pm c$, where the operation is modulo n. It is known that the maximum order of a connected triple-loop network of the form $TL_n(a, b, n-(a+b))$ with given diameter $d \geq 2$ is $n_d = 3d^2 + 3d + 1$, which is achieved by $TL_{n_d} = TL_{n_d}(1, 3d+1, 3d^2-1)$. In this article, we study the routing and gossiping problems for such optimal triple-loop networks under the store-and-forward, all-port, and full-duplex model, and prove that they admit "perfect" gossiping and routing schemes which exhibit many interesting features. Using a group-theoretic approach we develop for TL_{n_d} a method for systematically producing such optimal gossiping and routing schemes. Moreover, we determine the minimum gossip time, the edge- and arc-forwarding indices, and the minimal edge- and arc-forwarding indices of TL_{n_d}, and prove that our routing schemes are optimal with respect to these four indices simultaneously. As a key step towards these results, we prove that TL_{n_d} is a Frobenius graph on a Frobenius group with Frobenius kernel $\mathbb{Z}_{n_d}$, and that TL_{n_d} is arc-transitive with respect to this Frobenius group. In addition, we show that TL_{n_d} admits complete rotations.
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1. INTRODUCTION

Let $n \geq 7$ be an integer. For integers a, b, c with $1 \leq a, b, c \leq n-1$ such that $a, n-a, b, n-b, c, n-c$ are pairwise distinct, the undirected triple-loop network $TL_n(a, b, c)$ is defined [13] to be the circulant graph with vertices $0, 1, 2, \ldots, n-1$ such that each vertex x is adjacent to $x \pm a$, $x \pm b$ and $x \pm c$, where the operation is modulo n. Thus, $TL_n(a, b, c)$ has degree six and each arc (x, y) of it receives a unique label $y - x \in \{\pm a, \pm b, \pm c\}$, where by an arc we mean an oriented edge or an ordered pair of adjacent vertices. Undirected triple-loop networks have been recognized as strong candidates for interconnection networks due to their low degree, simple structure, and symmetry with respect to vertices. Most research so far has been concerned with diameter, connectivity, and maximum order; see [13, Section 4] for an account of known results. In particular, restricted to the case $c = n-(a+b)$ (written $c = -(a+b)$ in the sequel), it has been known [18, Section 3] for more than two decades that, for any integer $d \geq 2$, the maximum number of vertices of an undirected triple-loop network $TL_n(a, b, -(a+b))$ of diameter d is

$$n_d := 3d^2 + 3d + 1$$

and the corresponding optimal network is

$$TL_{n_d} := TL_{n_d}(3d+1, 1, -(3d+2)).$$

In contrast, as far as we know, the efficiency of triple-loop networks with respect to communication and information dissemination is unknown even for such optimal networks. In this article, we will study these problems for TL_{n_d}, and prove that it admits "perfect" gossiping and routing schemes with interesting features. By a result of Morillo (see [1, Section 2] and Remark 2), up to isomorphism TL_{n_d} is the only undirected triple-loop network of order n_d and diameter d. Thus, without loss of generality we can focus on TL_{n_d} as far as optimal undirected triple-loop networks are concerned.

A routing of a network $\Gamma = (V(\Gamma), E(\Gamma))$ is a set $\mathcal{P}$ of oriented paths in Γ such that, for every ordered pair of vertices (x, y), there is exactly one path in $\mathcal{P}$ from x to y. The load of an edge under $\mathcal{P}$ is the number of paths in $\mathcal{P}$ going through the edge in either direction, and the load of an arc is defined similarly but taking into account the orientation. Let $\pi(\Gamma, \mathcal{P})$ ($\vec{\pi}(\Gamma, \mathcal{P})$, respectively) denote the maximum load on an edge (arc, respectively) of Γ under $\mathcal{P}$. The edge- and arc-forwarding indices of Γ are defined [9, 11] to be $\pi(\Gamma) = \min_{\mathcal{P}} \pi(\Gamma, \mathcal{P})$ and $\vec{\pi}(\Gamma) = \min_{\mathcal{P}} \vec{\pi}(\Gamma, \mathcal{P})$, respectively, with

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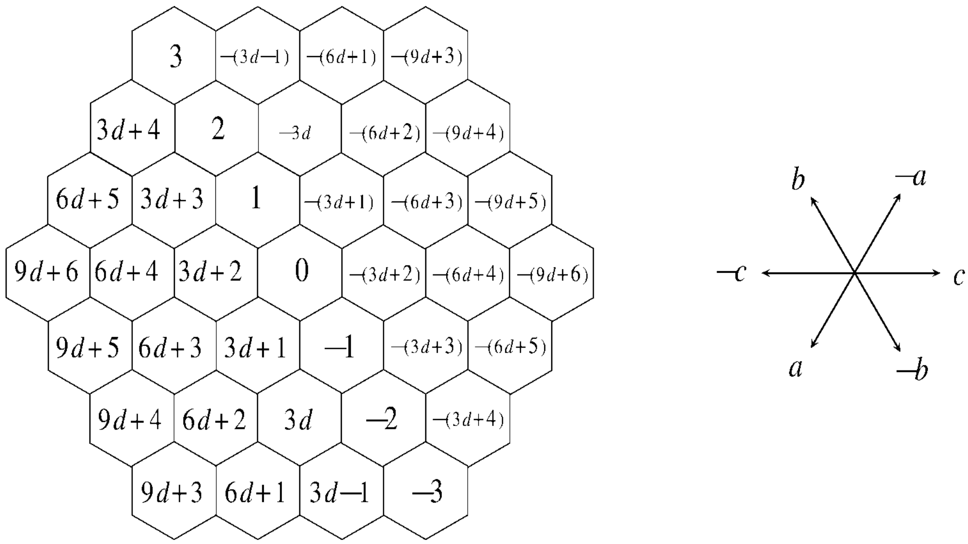

FIG. 1. Part of the plane tessellation associated with TL_{n_d}, where $a = 3d + 1, b = 1$ and $c = -(3d + 2)$.

the minimum taken over all possible routings of Γ. A routing $\mathcal{P}$ is called a shortest path routing if all paths in $\mathcal{P}$ are shortest paths, and the minimal edge- and arc-forwarding indices [9], $\pi_m(\Gamma)$, $\vec{\pi}_m(\Gamma)$, are defined by restricting attention to shortest path routings in the definitions of π and $\vec{\pi}$, respectively.

Gossiping is an information dissemination process in which every vertex has a distinct message to be sent to all other vertices. The gossiping problem has been studied extensively under various communication models [6, 8, 12]. In this article, we discuss the store-and-forward, all-port, and full-duplex model [2, 7]. That is, a vertex must receive a message wholly before retransmitting it to other vertices; a vertex can exchange messages (which may be different) with all of its neighbors during each time step; messages can traverse an edge in both directions simultaneously; and no two messages can transmit over the same arc at the same time. Further, we assume that it takes one time step to transmit any message over an arc; that is, the time required to transmit a message is independent of its length. A communication scheme which fulfils the gossiping task under these assumptions will be called a gossiping protocol. The minimum gossip time [2] of a network Γ, denoted by $t(\Gamma)$, is the minimum number of time steps required by a gossiping protocol for Γ. A gossiping protocol is called optimal if it uses $t(\Gamma)$ time steps.

This article solves the gossiping and routing problems above for TL_{n_d} completely for any integer $d \geq 2$, and gives a method for systematically producing optimal gossiping protocols as well as optimal routings for TL_{n_d}. More explicitly, we prove that the minimum gossip time of TL_{n_d} is given by $t(TL_{n_d}) = d(d + 1)/2$, and at the same time we prove by explicit construction that TL_{n_d} admits optimal gossiping protocols (not unique) with the following properties: (i) the message originating from any vertex is transmitted along shortest paths to other vertices; (ii) at any time each arc of TL_{n_d} is used exactly once for data transmission; (iii) for any vertex w, at any time exactly six arcs with different labels are used to transmit the message originating from w, and except for the first step the set of these six arcs is a matching of TL_{n_d}; (iv) the gossiping protocols are symmetric in the sense that, for any vertex w, at any time the six arcs carrying the message with source w are obtained from those carrying the message with source 0 by translation. Moreover, we determine the four forwarding indices of TL_{n_d}, that is, $\pi(TL_{n_d}) = 2\vec{\pi}(TL_{n_d}) = 2\vec{\pi}_m(TL_{n_d}) = \pi_m(TL_{n_d}) = d(d+1)(2d+1)/3$, and provide an algorithm for constructing optimal shortest path routings of TL_{n_d}. Such routings are not unique, and each of them loads both edges and arcs uniformly and is optimal for the four indices simultaneously. The complete version of these results with more technical details will be presented in Theorems 4 and 5, respectively.

Plane tessellations [18] are a useful tool for solving problems relating to undirected triple-loop networks. (See Figure 1 for an illustration.) In contrast, to achieve the results above, we will resort to an algebraic approach [19]. A key step is to prove that the automorphism group of TL_{n_d}, $\mathrm{Aut}(TL_{n_d})$, contains a subgroup which is a Frobenius group [4], and that TL_{n_d} is a Frobenius graph [5] with respect to this subgroup. Moreover, we will prove that TL_{n_d} admits complete rotations [2, 7, 10]. See Theorem 1 in the next section for details. These results are of interest for their own sake and we believe that they will be useful in solving other problems for TL_{n_d}. In the present article, we will use them together with a general framework introduced in [19] to obtain the results mentioned in the previous paragraph. In fact, once we have proved that TL_{n_d} is a Frobenius graph, we can then obtain good gossiping and routing schemes using this framework. To make the article self-contained, we will give detailed proofs of our results on gossiping and routing without relying on the abstract theories of [19]. However, we should point out that the ideas leading to such results are adapted from [19].

The reader is referred to [4] for notation and terminology on group actions. For a set V and a group G with identity 1, an action of G on V is a mapping $V \times G \to V$, $(v, g) \mapsto v^g$, such that $v^1 = v$ and $(v^g)^h = v^{gh}$ for any $v \in V$ and $g, h \in G$. The

G-orbit containing v is $v^G := \{v^g : g \in G\}$ and the stabilizer of v in G is $G_v := \{g \in G : v^g = v\}$. The group G is said to be semiregular on V if $G_v = 1$ for all $v \in V$, transitive on V if $v^G = V$ for some (and hence all) $v \in V$, and regular on V if it is both transitive and semiregular on V. For two groups K, H with H acting on K and respecting the structure of K, we use $K \rtimes H$ to denote the semidirected product of K by H.

For a group G and a subset S of $G \setminus \{1\}$ such that $S = S^{-1} := \{s^{-1} : s \in S\}$, the Cayley graph $\mathrm{Cay}(G, S)$ is defined [3] to have vertex set G and edge set $\{\{x, y\} : x, y \in G, xy^{-1} \in S\}$. A bijection $\omega : G \to G$ is called a complete rotation [2, 10] of $\mathrm{Cay}(G, S)$ if there exists an ordering of $S = S^{-1} = \{s_0, s_1, \ldots, s_{\delta-1}\}$ such that $\omega(1) = 1$, $\omega(xs_i) = \omega(x)s_{i+1}$ for all $x \in G$ and $i = 0, 1, \ldots, \delta - 1$, where subscripts are modulo δ. It is known [10, Proposition 2.2] that a bijection $\omega : G \to G$ is a complete rotation of $\mathrm{Cay}(G, S)$ if and only if ω is a group automorphism of G and ω fixes S setwise (that is, $s \in S$ implies $\omega(s) \in S$) and induces a cyclic permutation on S. A rotation [7] of $\mathrm{Cay}(G, S)$ is an inner automorphism of G which fixes S setwise and induces a cyclic permutation on S. In other words, a rotation of $\mathrm{Cay}(G, S)$ is a complete rotation induced by an inner automorphism of G.

The arc set and the automorphism group of a graph Γ will be denoted by $\mathtt{Arc}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$, respectively. For a subgroup G of $\mathrm{Aut}(\Gamma)$, Γ is said to be G-vertex transitive if G is transitive on the vertex set of Γ, and G-arc transitive if it is G-vertex transitive and G is transitive on $\mathtt{Arc}(\Gamma)$.

2. TL_{n_d} IS A FROBENIUS GRAPH

A Frobenius group is a transitive permutation group L on a set V which is not regular on V, but has the property that the only element of L which fixes two points of V is the identity of L. It is well known [4] that a finite Frobenius group L has a nilpotent normal subgroup G, called the Frobenius kernel, which is regular on V. Hence, L is the semidirect product $G \rtimes H$, where H is the stabilizer of a point of V; each such group H is called a Frobenius complement for G in L. Given a Frobenius group $L = G \rtimes H$, an L-Frobenius graph [5, 16] is a connected graph with vertex set V and edge set $\{\{x, y\} : (x, y) \in O\}$ for some L-orbit O on $\{(x, y) : x, y \in V, x \neq y\}$. It is known [5, Theorem 1.4] that any L-Frobenius graph is a Cayley graph $\mathrm{Cay}(G, S)$ on its Frobenius kernel G, where (i) $S = a^H$ if $|H|$ is even or $|a| = 2$, or (ii) $S = a^H \cup (a^{-1})^H$ if $|H|$ is odd and $|a| \neq 2$, for some $a \in G$ satisfying $\langle a^H \rangle = G$, where $|a|$ is the order of a and $x^H := \{h^{-1}xh : h \in H\}$ is the H-orbit containing $x \in G$ under the action of H on G (by conjugation). $\mathrm{Cay}(G, S)$ is called an L-Frobenius graph of the first or second type [19] according to whether S is given in (i) or (ii).

Let $\mathbb{Z}_{n_d} = \{[x] : x \in \mathbb{Z}\}$ be the additive group of integers modulo n_d, where $[x]$ is the residue class containing x. Then TL_{n_d} can be defined as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_{n_d}, S(d))$, where

$$S(d) = \{[3d + 1], -[3d + 1], [1], -[1], [3d + 2], -[3d + 2]\}. \tag{1}$$

(Since $n_d = 3d^2 + 3d + 1$, we have $[3d^2 + 3d] = -[1], [3d^2] = -[3d + 1]$ and $[3d^2 - 1] = -[3d + 2]$.) That is, TL_{n_d} has vertex set $\mathbb{Z}_{n_d}$ in which $[x], [y] \in \mathbb{Z}_{n_d}$ are adjacent if and only if $[x - y] \in S(d)$. With this notation the label of an arc $([x], [y]) \in \mathtt{Arc}(TL_{n_d})$ is $[y - x] \in S(d)$. It is well known that each $[m] \in \mathbb{Z}_{n_d}$ induces a translation $\hat{m} : [x] \mapsto [x + m]$, $[x] \in \mathbb{Z}_{n_d}$; in other words, $\mathbb{Z}_{n_d}$ induces a regular action on itself defined by $([x], [m]) \mapsto [x + m]$. Such translations form a subgroup of $\mathrm{Aut}(TL_{n_d})$ which is isomorphic to $\mathbb{Z}_{n_d}$. Henceforth, we will identify this subgroup with $\mathbb{Z}_{n_d}$ by identifying $\hat{m}$ with $[m]$ so that $\mathbb{Z}_{n_d} \leq \mathrm{Aut}(TL_{n_d})$. It is well known [15] that the automorphism group of $\mathbb{Z}_{n_d}$ is given by

$$\mathrm{Aut}(\mathbb{Z}_{n_d}) \cong \mathbb{Z}^*_{n_d} := \{[m] : 0 < m < n_d, \gcd(m, n_d) = 1\}, \tag{2}$$

where $\mathbb{Z}^*_{n_d}$ is the multiplicative group of units of ring $\mathbb{Z}_{n_d}$ and gcd means the greatest common divisor. As the automorphism group of $\mathbb{Z}_{n_d}$, $\mathbb{Z}^*_{n_d}$ acts on $\mathbb{Z}_{n_d}$ by the usual multiplication: $[x][m] = [xm]$, $[m] \in \mathbb{Z}^*_{n_d}$, $[x] \in \mathbb{Z}_{n_d}$. For any $[m] \in \mathbb{Z}_{n_d}$, we have $[m] \in \mathbb{Z}^*_{n_d} \Leftrightarrow \gcd(m, n_d) = 1 \Leftrightarrow$ there exist integers k, l such that $km + ln_d = 1$; in this case the inverse element of $[m]$ in $\mathbb{Z}^*_{n_d}$ is $[k]$. The semidirect product $\mathbb{Z}_{n_d} \rtimes \mathbb{Z}^*_{n_d}$ acts on $\mathbb{Z}_{n_d}$ in such a way that $\mathbb{Z}_{n_d}$ acts by addition and $\mathbb{Z}^*_{n_d}$ acts by multiplication. In other words, the action of $\mathbb{Z}_{n_d} \rtimes \mathbb{Z}^*_{n_d}$ on $\mathbb{Z}_{n_d}$ is defined by

$$[x]^{([z],[m])} := [(x + z)m] \tag{3}$$

for $[x] \in \mathbb{Z}_{n_d}$ and $([z], [m]) \in \mathbb{Z}_{n_d} \rtimes \mathbb{Z}^*_{n_d}$, where $[z] \in \mathbb{Z}_{n_d}$ and $[m] \in \mathbb{Z}^*_{n_d}$. In the following discussion, we will identify $\mathbb{Z}_{n_d} \rtimes \mathrm{Aut}(\mathbb{Z}_{n_d})$ with $\mathbb{Z}_{n_d} \rtimes \mathbb{Z}^*_{n_d}$.

The following results of an algebraic flavour are crucial to our design of optimal gossiping and routing schemes for TL_{n_d}, to be given in the next section.

Theorem 1. *Let $d \geq 2$ be an integer. Then the following hold:*

(a) *TL_{n_d} admits $[3d^2], [3d + 2] \in \mathrm{Aut}(\mathbb{Z}_{n_d})$ as complete rotations;*
(b) *the subgroup $H(d) := \langle [3d^2] \rangle = \{[1], [3d^2], [3d^2 - 1], [3d^2 + 3d], [3d + 1], [3d + 2]\}$ of $\mathrm{Aut}(\mathbb{Z}_{n_d})$ generated by $[3d^2]$ fixes $S(d)$ setwise;*
(c) *$\mathbb{Z}_{n_d} \rtimes H(d) \leq \mathrm{Aut}(TL_{n_d})$ is a Frobenius group with Frobenius kernel $\mathbb{Z}_{n_d}$, and TL_{n_d} is a $\mathbb{Z}_{n_d} \rtimes H(d)$-Frobenius graph of the first type;*
(d) *TL_{n_d} is $\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive, and all $\mathbb{Z}_{n_d} \rtimes H(d)$-Frobenius graphs are isomorphic to TL_{n_d}.*

Proof. Since $(3d + 2)(3d^2) - (3d - 1)n_d = 1$, we have $[3d^2] \in \mathrm{Aut}(\mathbb{Z}_{n_d})$ and the inverse element of $[3d^2]$ in $\mathrm{Aut}(\mathbb{Z}_{n_d})$ is $[3d + 2]$. In $\mathrm{Aut}(\mathbb{Z}_{n_d})$ we have $[3d^2]^2 = (-[3d + 1])^2 = [9d^2 + 6d + 1] = [3d^2 - 1] = -[3d + 2]$, $[3d^2]^3 = (-[3d + 2])(-[3d + 1]) = [9d^2 + 9d + 2] = [3d^2 + 3d] = -[1]$, $[3d^2]^4 = (-[1])(-[3d + 1]) = [3d + 1]$, $[3d^2]^5 = [3d + 1](-[3d + 1]) = -[9d^2 + 6d + 1] = [3d + 2]$, $[3d^2]^6 = [3d + 2](-[3d + 1]) = -[9d^2 + 9d + 2] = [1]$.

Thus, the subgroup $\langle[3d^2]\rangle$ of $\mathrm{Aut}(\mathbb{Z}_{n_d})$ generated by $[3d^2]$ is $H(d) = \{[1], [3d^2], [3d^2-1], [3d^2+3d], [3d+1], [3d+2]\}$. Because $\mathrm{Aut}(\mathbb{Z}_{n_d})$ acts on $\mathbb{Z}_{n_d}$ by multiplication, from the computation above it follows that $H(d)$ fixes $S(d)$ setwise and is transitive on $S(d)$. Since $|S(d)| = |H(d)|$, from the orbit-stabilizer lemma [4] it follows that $H(d)$ is regular on $S(d)$. Since $H(d)$ fixes $S(d)$ setwise, by [3, Proposition 16.2] each element of $H(d)$ preserves the adjacency of TL_{n_d}. Thus, $H(d)$ can be taken as a subgroup of $\mathrm{Aut}(TL_{n_d})$ and hence $\mathbb{Z}_{n_d} \rtimes H(d) \leq \mathrm{Aut}(TL_{n_d})$.

The computation above also tells us that $[3d^2]$ permutes the elements of $S(d)$ in a cyclic manner. Therefore, $[3d^2]$ is a complete rotation of TL_{n_d}. Similarly, $[3d+2]$ is a complete rotation of TL_{n_d}.

Since $(3d+1)(3d^2-1) + (-3d+2)n_d = 1$, we have $\gcd(3d^2-1, n_d) = 1$. Similarly, $d(3d^2-2) - (d-1)n_d = -(d+1)(3d) + n_d = -(3d+2)(3d+1) + 3n_d = 1$ and hence $\gcd(3d^2-2, n_d) = \gcd(3d, n_d) = \gcd(3d+1, n_d) = 1$. Also, $\gcd(3d^2+3d-1, n_d) = 1$ as $3d^2+3d-1$ and n_d are consecutive odd integers. Thus, we have proved that $[m-1] \in \mathrm{Aut}(\mathbb{Z}_{n_d})$ for any $[m] \in H(d) \setminus \{[1]\}$. Hence, for any $[x] \in \mathbb{Z}_{n_d} \setminus \{[0]\}$, we have $[m-1][x] \neq [0]$, that is, $[m][x] \neq [x]$. In other words, $H(d)$ is semiregular on $\mathbb{Z}_{n_d} \setminus \{[0]\}$.

As a subgroup of $\mathbb{Z}_{n_d} \rtimes \mathrm{Aut}(\mathbb{Z}_{n_d})$, $\mathbb{Z}_{n_d} \rtimes H(d)$ acts on $\mathbb{Z}_{n_d}$ as defined by (3). Obviously, this is a transitive action because $\mathbb{Z}_{n_d}$ is transitive on $\mathbb{Z}_{n_d}$ in its regular action (addition). Because the order of $\mathbb{Z}_{n_d} \rtimes H(d)$ is greater than n_d, $\mathbb{Z}_{n_d} \rtimes H(d)$ is not regular on $\mathbb{Z}_{n_d}$. Suppose that $([z],[m]) \in \mathbb{Z}_{n_d} \rtimes H(d)$ fixes $[a],[b] \in \mathbb{Z}_{n_d}$, where $[a] \neq [b]$. Then $[(a+z)m] = [a]$ and $[(b+z)m] = [b]$, and hence $[(a-b)m] = [a-b]$, that is, $[m]$ fixes $[a-b] \neq [0]$. Because $H(d)$ is semiregular on $\mathbb{Z}_{n_d} \setminus \{[0]\}$, it follows that $[m] = [1]$ and hence $[a+z] = [a]$. Thus $[z] = [0]$ and only the identity element $([0],[1])$ of $\mathbb{Z}_{n_d} \rtimes H(d)$ can fix two distinct elements of $\mathbb{Z}_{n_d}$. In other words, $\mathbb{Z}_{n_d} \rtimes H(d)$ is a Frobenius group with Frobenius kernel $\mathbb{Z}_{n_d}$.

The computation at the beginning of this proof shows that $S(d)$ is the $H(d)$-orbit containing $[3d^2]$. Also, $S(d)$ generates $\mathbb{Z}_{n_d}$ because $[1] \in S(d)$ and the operation of $\mathbb{Z}_{n_d}$ is addition. Because $|H(d)| = 6$ is even, from [5, Theorem 1.4] it follows that $TL_{n_d} = \mathrm{Cay}(\mathbb{Z}_{n_d}, S(d))$ is a $\mathbb{Z}_{n_d} \rtimes H(d)$-Frobenius graph, and evidently it is of the first type. Moreover, because the kernel of $\mathbb{Z}_{n_d} \rtimes H(d)$ is $\mathbb{Z}_{n_d}$, by [5, Corollary 3.6] all $\mathbb{Z}_{n_d} \rtimes H(d)$-Frobenius graphs are isomorphic to TL_{n_d}.

Let $([x],[y])$ and $([u],[v])$ be arcs of TL_{n_d}. Then $[x-y], [u-v] \in S(d)$. Because $H(d)$ is transitive on $S(d)$, there exists $[m] \in H(d)$ such that $[x-y][m] = [u-v]$. Let $[z] := [vk] - [y]$, where $[k]$ is the inverse element of $[m]$ in $\mathbb{Z}_{n_d}^*$, so that $[(y+z)m] = [vkm] = [v]$. Then $([x],[y])^{([z],[m])} = ([(x+z)m], [(y+z)m]) = ([(y+z)m + (u-v)], [(y+z)m]) = ([u],[v])$. Therefore, TL_{n_d} is $\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive. ∎

Remark 1.

(a) *$H(d)$ and $S(d)$ are identical as sets and $H(d)$ is generated by a complete rotation. This phenomenon occurs for another family* [17] *of circulant graphs with degree four.*

(b) *Because $[3d^2]$ and $[3d+2]$ are inverse elements of each other in $\mathrm{Aut}(\mathbb{Z}_{n_d})$, the fact that both of them are complete rotations of TL_{n_d} is not a coincidence. In general, for any Cayley graph $\mathrm{Cay}(G,S)$, if $\omega \in \mathrm{Aut}(G)$ is a complete rotation of $\mathrm{Cay}(G,S)$, then ω^{-1} is also a complete rotation of $\mathrm{Cay}(G,S)$ by* [10, Proposition 2.3(v)]. *Note that $[3d^2]$ and $[3d+2]$ are not rotations in the sense of* [7] *because they are not inner automorphisms of $\mathbb{Z}_{n_d}$.*

(c) *Because TL_{n_d} is a Cayley graph on $\mathbb{Z}_{n_d}$, it is vertex-transitive* [18]. *Here we proved the stronger result that TL_{n_d} is actually $\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive. In fact, this is a corollary of a general result* [19, Lemma 2.1].

(d) *In general, the holomorph $\mathbb{Z}_{n_d} \rtimes \mathrm{Aut}(\mathbb{Z}_{n_d})$ may not be a Frobenius group. For example, for $d = 7$ we have $n_7 = 13^2$. Hence, $[14] \in \mathrm{Aut}(\mathbb{Z}_{n_7})$ and $[14][13] = [13]$. Thus [14] fixes both [0] and [13], and therefore $\mathbb{Z}_{n_d} \rtimes \mathrm{Aut}(\mathbb{Z}_{n_d})$ is not a Frobenius group.*

Remark 2. *For any undirected triple-loop network $TL_{n_d}(a,b,c)$ with order n_d and diameter d, by* [18, Section III], *we have $a = d^2\alpha + (2d+1)\beta + d\gamma$, $b = d(d+1)\alpha - d\beta + (d+1)\gamma$, and $c = (d+1)^2\alpha - (d+1)\beta - (2d+1)\gamma$ for some integers α, β, and γ, where numbers are interpreted as residue classes. From these, it follows that $(a,b,c) = (a, -(3d+2)a, (3d+1)a) = ((3d+1)b, b, -(3d+2)b) = (-(3d+2)c, (3d+1)c, c)$. Thus, if at least one of a, b, and c is coprime to n_d, then $TL_{n_d}(a,b,c) \cong TL_{n_d}$ and the corresponding [a], [b], or [c] induces an isomorphism from TL_{n_d} to $TL_{n_d}(a,b,c)$. Note that a (b, c, respectively) is coprime to n_d if and only if it induces a "loop" (Hamiltonian cycle) of $TL_{n_d}(a,b,c)$. Therefore, up to isomorphism TL_{n_d} is the unique undirected triple-loop network with order n_d, diameter d and at least one "loop." This result was first proved in Morillo's Ph.D. thesis according to* [1, Section 2]. *Because $1, 3d+1$, and $3d+2$ are all coprime to n_d, TL_{n_d} consists* [18] *of three "loops."*

3. GOSSIPING AND ROUTING

In this section, we will give optimal gossiping protocols and routings for TL_{n_d} and determine $t(TL_{n_d})$ and the four forwarding indices of TL_{n_d} by using Theorem 1 and the approach developed in [19]. To use this approach we need detailed information about the orbits of $H(d)$ on $\mathbb{Z}_{n_d} \setminus \{[0]\}$, and this will be given in Lemma 2. Here as before $H(d) = \{[1], [3d^2], [3d^2-1], [3d^2+3d], [3d+1], [3d+2]\}$ denotes the subgroup of $\mathrm{Aut}(\mathbb{Z}_{n_d})$ generated by $[3d^2]$. The $H(d)$-orbit on $\mathbb{Z}_{n_d} \setminus \{[0]\}$ containing a given element $[x] \in \mathbb{Z}_{n_d} \setminus \{[0]\}$ is $H(d)[x] := \{[mx] : [m] \in H(d)\}$. Define

$$O_{i,j} := H(d)[(3d+2)j + (i-j)], \quad 1 \leq i \leq d, \quad 0 \leq j \leq i-1. \quad (4)$$

Given $[x] \in \mathbb{Z}_{n_d}$, for $0 \leq i \leq d$, let

$$TL_{n_d}([x], i) = \{[z] \in \mathbb{Z}_{n_d} : [z] \text{ is at distance } i \text{ from } [x] \text{ in } TL_{n_d}\}.$$

Thus, $TL_{n_d}([0], 1) = S(d) = O_{1,0}$.

Lemma 2. *Let $d \geq 2$ be an integer. Then the following hold:*

(a) *For $1 \leq i \leq d$, $TL_{n_d}([0], i)$ is the union of exactly i $H(d)$-orbits $O_{i,j}$ on $\mathbb{Z}_{n_d} \setminus \{[0]\}$, that is,*

$$TL_{n_d}([0], i) = \bigcup_{j=0}^{i-1} O_{i,j}; \tag{5}$$

(b) *for $1 \leq i \leq d-1$, $0 \leq j \leq i-1$ and $0 \leq j' \leq i$, in TL_{n_d} either there is no edge between $O_{i,j}$ and $O_{i+1,j'}$, or there are exactly six edges between them which form a matching; in the latter case the six arcs from $O_{i,j}$ to $O_{i+1,j'}$ have different labels.*

Proof. Because $H(d) \leq \mathrm{Aut}(TL_{n_d})$ and $H(d)$ fixes $[0]$, $TL_{n_d}([0], i)$ is invariant under the action of $H(d)$. Hence, $TL_{n_d}([0], i)$ is a union of $H(d)$-orbits on $\mathbb{Z}_{n_d} \setminus \{[0]\}$. Because $H(d)$ has order six and is semiregular on $\mathbb{Z}_{n_d} \setminus \{[0]\}$ by Theorem 1, each $H(d)$-orbit on $\mathbb{Z}_{n_d} \setminus \{[0]\}$ has length six by the well known orbit-stabilizer lemma [4].

(a) Consider the $H(d)$-orbits $O_{i,j}$ defined in (4) for $1 \leq i \leq d$ and $0 \leq j \leq i-1$. For $(i,j) \neq (i',j')$, we claim that $O_{i,j} \neq O_{i',j'}$, or equivalently $O_{i,j} \cap O_{i',j'} = \emptyset$. Suppose otherwise. Then there exists $[m] \in H(d)$ such that

$$[(3d+2)j' + (i' - j')] = [(3d+2)j + (i-j)][m]. \tag{6}$$

However, for each possibility of $[m]$ one can verify that this is impossible. For example, if $[m] = [3d+1]$, then since $[3d+1][3d+2] = -[1]$, (6) becomes $[(3d+1)(i-j-j') - (i'+j)] = [0]$. Because $(3d+1)(i-j-j') - (i'+j)$ is between $-3d^2$ and $3d^2+d-1$ for all possible (i,j) and (i',j'), the only possibility is that $(3d+1)(i-j-j') - (i'+j) = 0$. This implies that $3d+1$ is a divisor of $i'+j$, which is a contradiction since $1 \leq i'+j \leq 2d-1$. Similarly, one can show that the other five elements of $H(d)$ do not satisfy (6), and hence $O_{i,j} \neq O_{i',j'}$ for $(i,j) \neq (i',j')$. Therefore, the $H(d)$-orbits $O_{i,j}$ on $\mathbb{Z}_{n_d} \setminus \{[0]\}$ are pairwise distinct. Because there are $\sum_{i=1}^{d} i = d(d+1)/2$ such orbits each with length 6, they cover $6(d(d+1)/2) = 3d^2 + 3d$ elements of $\mathbb{Z}_{n_d} \setminus \{[0]\}$ collectively. Because $\mathbb{Z}_{n_d} \setminus \{[0]\}$ has exactly $3d^2 + 3d$ elements, $O_{i,j}$ $(1 \leq i \leq d, 0 \leq j \leq i-1)$ enumerate all $H(d)$-orbits on $\mathbb{Z}_{n_d} \setminus \{[0]\}$.

The expression $[(3d+2)j + (i-j)] = \overbrace{[3d+2] + \cdots + [3d+2]}^{j} + \overbrace{[1] + \cdots + [1]}^{i-j}$ attains the minimum length among all expressions of $[(3d+2)j + (i-j)]$ as a sum of elements of $S(d)$. Thus, $[(3d+2)j + (i-j)] \in TL_{n_d}([0], i)$. Because $O_{i,j}$ is an $H(d)$-orbit containing $[(3d+2)j + (i-j)]$ and $H(d) \leq \mathrm{Aut}(TL_{n_d})$ fixes $[0]$, it follows that $O_{i,j} \subseteq TL_{n_d}([0], i)$. This together with the result in the previous paragraph implies (5) for each i, and consequently $TL_{n_d}([0], i)$ is the union of exactly i $H(d)$-orbits on $\mathbb{Z}_{n_d} \setminus \{[0]\}$.

(b) Because $H(d) \leq \mathrm{Aut}(TL_{n_d})$ and $O_{i,j}$ and $O_{i+1,j'}$ are $H(d)$-orbits, if there is an edge between them, then in TL_{n_d} each vertex in $O_{i,j}$ is adjacent to at least one vertex in $O_{i+1,j'}$. Suppose $[x] \in O_{i,j}$ is adjacent to $[y], [z] \in O_{i+1,j'}$. Because $O_{i,j}$ and $O_{i+1,j'}$ are $H(d)$-orbits and TL_{n_d} is $\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive by Theorem 1(d), there exists $[m] \in H(d)$ such that $([x], [y])[m] = ([xm], [ym]) = ([x], [z])$. Because $H(d)$ is semiregular on $\mathbb{Z}_{n_d} \setminus \{[0]\}$, the only element of $H(d)$ which fixes $[x]$ is $[m] = [1]$. Thus, $[z] = [ym] = [y]$ and the edges between $O_{i,j}$ and $O_{i+1,j'}$ must form a matching of six edges. Moreover, the six arcs from $O_{i,j}$ to $O_{i+1,j'}$ have labels $[(y-x)m]$ where $[m] \in H(d)$, which are pairwise distinct by the semiregularity of $H(d)$ on $\mathbb{Z}_{n_d} \setminus \{[0]\}$. ■

For a spanning subgraph Γ_w of TL_{n_d} with root $[w]$, denote

$$\Gamma_w(i) = \{[z] \in \mathbb{Z}_{n_d} :\ [z] \text{ is at distance } i \text{ from } [w] \text{ in } \Gamma_w\}, i \geq 0.$$

A spanning tree Γ_w of TL_{n_d} with root $[w]$ is called a shortest path spanning tree if the unique path in Γ_w from $[w]$ to any vertex $[x]$ is a shortest path in TL_{n_d}, that is, $\Gamma_w(i) = TL_{n_d}([w], i)$, $0 \leq i \leq d$.

The following algorithm, which is based on Lemma 2, constructs essentially what we need for optimal gossiping protocols and optimal routings for TL_{n_d}.

Algorigthm 1 [Spanning Tree]. This algorithm constructs a shortest path spanning tree T_0 of TL_{n_d} with root $[0]$.

i. Starting with $T_0(0) = \{[0]\}$, add $S(d)$ and the six edges joining $[0]$ and the elements of $S(d)$ to T_0, so that $T_0(1) = S(d)$.
ii. Inductively, suppose that we have constructed T_0 up to level $T_0(i)$ for some i, $1 \leq i \leq d-1$. For each $H(d)$-orbit $O_{i+1,j} \subseteq TL_{n_d}([0], i+1)$, $0 \leq j \leq i$, choose exactly one $H(d)$-orbit $O_{i,j'} \subseteq TL_{n_d}([0], i)$ such that there are edges of TL_{n_d} between $O_{i,j'}$ and $O_{i+1,j}$. (The existence of $O_{i,j'}$ is ensured by the obvious fact that in TL_{n_d} each vertex of $TL_{n_d}([0], i+1)$ is adjacent to at least one vertex of $TL_{n_d}([0], i)$.) By Lemma 2(b), the edges between $O_{i,j'}$ and $O_{i+1,j}$ form a 6-matching, and we add $O_{i+1,j}$ together with these six edges to T_0.
iii. Stop if $i+1 = d$; otherwise set $i := i+1$ and repeat (ii).

■

Let

$$A_{1,0} = \{([0], [m]) : [m] \in S(d)\}. \tag{7}$$

For $1 \leq i \leq d-1, 0 \leq j \leq i$, let

$$A_{i+1,j} = \text{ the set of six arcs from } O_{i,j'} \text{ to } O_{i+1,j}, \tag{8}$$

where $O_{i,j'}$ to $O_{i+1,j}$ are as in (ii) of Algorithm 1.

Lemma 3. *The subgraph T_0 constructed in Algorithm 1 is a shortest path spanning tree of TL_{n_d} with root $[0]$, and for $0 \leq i \leq d-1$ the set of arcs of T_0 from $T_0(i)$ to $T_0(i+1)$ is $\cup_{0 \leq j \leq i} A_{i+1,j}$. Moreover, T_0 is the edge-disjoint union of six mutually isomorphic subtrees $T_{0,m}$, $[m] \in H(d)$, which have*

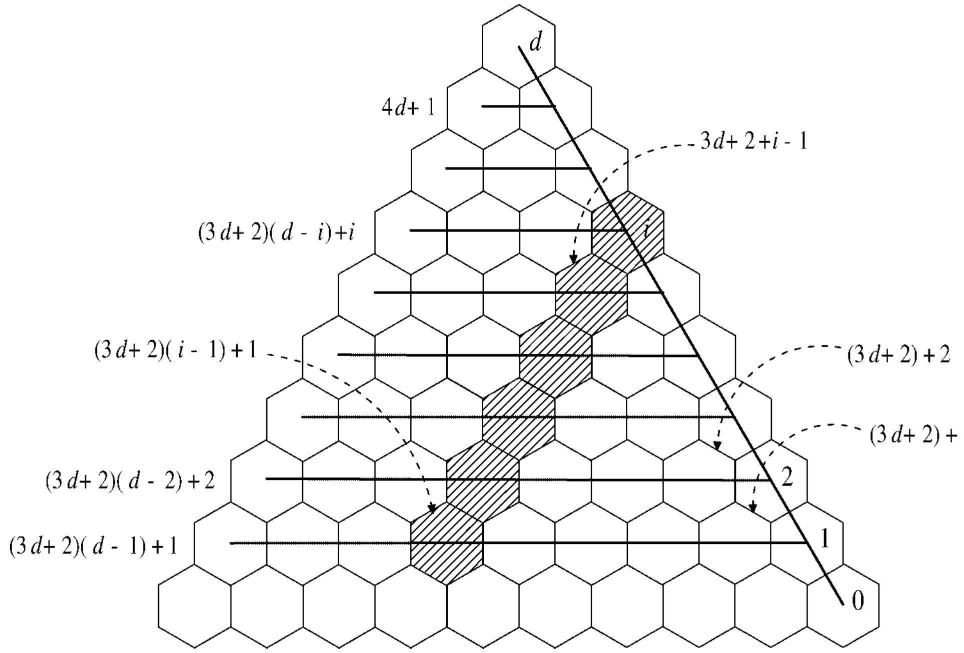

FIG. 2. Illustration of the $H(d)$-orbits $O_{i,j}$ and the branch $T_{0,1}$ of the "canonical" spanning tree obtained by using Algorithm 1. The cell with label i is in $O_{i,0}$, the next shaded cell is in $O_{i,1}$, etc., and the last shaded cell is in $O_{i,i-1}$. These shaded cells comprise one-sixth of the vertices of $TL_{n_d}([0], i)$, and the other ones are obtained from them by rotations. The other five branches $T_{0,3d+2}, T_{0,3d+1}, T_{0,3d^2+3d}, T_{0,3d^2-1}$, and $T_{0,3d^2}$ of this "canonical" spanning tree are obtained from this branch by rotation anticlockwise by 60°, 120°, 180°, 240° and 300°, respectively.

$[0]$ as the unique common vertex. Furthermore, $T_{0,m}$ can be obtained from $T_{0,1}$ by "multiplication," that is, the vertices of $T_{0,m}$ are $[xm]$ with $[x]$ running over all vertices of $T_{0,1}$, and the edges of $T_{0,m}$ are $\{[xm], [ym]\}$ with $\{[x], [y]\}$ running over all edges of $T_{0,1}$.

Proof. The statements in the first sentence follow from Algorithm 1 immediately. This algorithm can be modified to construct $T_{0,1}$: initially add the edge $\{[0], [1]\}$ to $T_{0,1}$; in each iteration (ii), by Lemma 2(b) there exists a unique vertex $[y] \in O_{i+1,j}$ such that $[y]$ is adjacent to a vertex $[x] \in O_{i,j'}$ which is in $T_{0,1}$ already, and we add $[y]$ together with the edge $\{[x], [y]\}$ to $T_{0,1}$. Because $H(d)$ is semiregular on $\mathbb{Z}_{n_d} \setminus \{[0]\}$, similar to the proof of Lemma 2(b) one can check that the edges of T_0 between $O_{i,j'}$ and $O_{i+1,j}$ are $\{[xm], [ym]\}$, where $[m] \in H(d)$. From this, the remaining statements in Lemma 3 follow without difficulty. ∎

Remark 3.

(a) *From Lemmas 2 and 3, we have*

$$T_0(i) = \bigcup_{j=0}^{i-1} O_{i,j}, \quad 1 \le i \le d.$$

(b) *In step (ii) of Algorithm 1, we do not require that in T_0 each O_{i,j^*} has neighbors in $T_0(i+1)$. In the case when this condition is satisfied, all but one O_{i,j^*} have neighbours in exactly one $O_{i+1,j}$, and this exceptional O_{i,j^*} must have neighbors in exactly two orbits $O_{i+1,j}$.*

(c) *By the definition of TL_{n_d}, for $1 \le i \le d-1$ and $1 \le j \le i$, $([i], [i+1])$ and $([(3d+2)(j-1)+(i-j+1)], (3d+2)j + (i+1-j)])$ are arcs of TL_{n_d}. Thus, as an example, in step (ii) of Algorithm 1 we may choose $0' = 0$ and $j' = j-1$ for $1 \le j \le i$. That is, $O_{i+1,0}$ is joined to $O_{i,0}$ and $O_{i+1,j}$ is joined to $O_{i,j-1}$, so that $A_{i+1,0} = H(d)([i], [i+1])$ and $A_{i+1,j} = H(d)([(3d+2)(j-1)+(i-j+1)], (3d+2)j + (i+1-j)])$ for $1 \le i \le d-1$ and $1 \le j \le i$, where we define $H(d)([x], [y]) := \{([xm], [ym]) : [m] \in H(d)\}$ for any arc $([x], [y])$ of TL_{n_d}. See Figure 2 for an illustration of the "canonical" spanning tree obtained this way.*

(d) *The spanning tree T_0 obtained by Algorithm 1 is not unique. For example, instead of $A_{2,0} = H(d)([1], [2])$ and $A_{2,1} = H(d)([1], [3d+3])$ as in (c) above, we may choose $A_{2,0} = H(d)([1], [2])$ and $A_{2,1} = H(d)([1], [3d^2+1])$ at distance two level.*

Given $[w] \in \mathbb{Z}_{n_d}$, $X \subseteq \mathbb{Z}_{n_d}$ and $A \subseteq \mathtt{Arc}(TL_{n_d})$, denote

$$X + [w] := \{[x+w] : [x] \in X\}$$
$$A + [w] := \{([x+w], [y+w]) : ([x], [y]) \in A\}.$$

Let T_0 be an arbitrary but fixed spanning tree of TL_{n_d} constructed by using Algorithm 1. Define

$$\mathcal{T} := \{T_w : [w] \in \mathbb{Z}_{n_d}\} \tag{9}$$

where T_w is the subgraph of TL_{n_d} with root $[w]$ which is obtained from T_0 by translation $[w]$, that is, $\mathtt{Arc}(T_w) = \mathtt{Arc}(T_0) + [w]$ and $T_w(i) = T_0(i) + [w]$, $0 \le i \le d$. In the remainder of this section, we will prove that $\mathcal{T}$ gives rise to an optimal gossiping protocol as well as an optimal routing for TL_{n_d}.

Algorithm 2 [Gossiping]. The algorithm proceeds in d phases. In the ith phase, $1 \leq i \leq d$, the message originating from $[w]$ is transmitted over T_w from $T_0(i-1)+[w]$ to $T_0(i)+[w]$ for all $[w] \in \mathbb{Z}_{n_d}$. The ith phase consists of i steps.

i. In the single step of the first phase, the message at $[w]$ is transmitted from $[w]$ to $T_0(1)+[w]$ $(= S(d)+[w])$ along the six arcs of $A_{1,0}+[w]$, and this is done for all $[w] \in \mathbb{Z}_{n_d}$ simultaneously.
ii. For $i = 1, 2, \ldots, d-1$ do the following: for $j = 0, 1, \ldots, i$, in the $(j+1)$th step of the $(i+1)$th phase, for all $[w] \in \mathbb{Z}_{n_d}$ send simultaneously the message with source $[w]$ from $O_{i,j'}+[w]$ to $O_{i+1,j}+[w]$ along the six arcs of $A_{i+1,j}+[w]$, where $O_{i,j'}$ and $O_{i+1,j}$ are as in (ii) of Algorithm 1.

■

In (ii) above, we treat $i = 1, 2, \ldots, d-1$ in this order so that the message originating from $[w]$ reaches level $T_0(i+1)+[w]$ before level $T_0(i'+1)+[w]$ whenever $i < i'$. Clearly, Algorithm 2 terminates in $\sum_{i=1}^{d} i = d(d+1)/2$ time steps. At time t, $1 \leq t \leq d(d+1)/2$, the set of arcs used to transmit the message originating from $[w]$ is given by

$$\Lambda_t([w]) = A_{i+1,j} + [w] \tag{10}$$

where i is the largest integer such that $t > i(i+1)/2$, and $j = t - i(i+1)/2 - 1$.

Theorem 4. *Let $d \geq 2$ be an integer. Then*

$$t(TL_{n_d}) = \frac{d(d+1)}{2}.$$

Moreover, for any shortest path spanning tree T_0 of TL_{n_d} constructed by Algorithm 1, Algorithm 2 gives an optimal gossiping protocol for TL_{n_d} such that:

(a) *the message originating from any vertex is transmitted along shortest paths to other vertices;*
(b) *for each vertex $[w]$ of TL_{n_d}, at any time $t \geq 1$ precisely six arcs with different labels are used to transmit the message originating from $[w]$, and for $t \geq 2$ the set $\Lambda_t([w])$ of these six arcs form a matching of TL_{n_d};*
(c) *at any time $t \geq 1$ each arc of TL_{n_d} is used exactly once for message transmission, that is, $\{\Lambda_t([w]) : [w] \in \mathbb{Z}_{n_d}\}$ is a partition of $\mathtt{Arc}(TL_{n_d})$.*

Proof. To show that Algorithm 2 gives a gossiping protocol it suffices to prove that $\{\Lambda_t([w]) : [w] \in \mathbb{Z}_{n_d}\}$ is a partition of $\mathtt{Arc}(TL_{n_d})$ for any $t \geq 1$, where $\Lambda_t([w]) = A_{i+1,j}+[w]$ as in (10). For $t = 1$, this is clearly true by (7) and step (i) of Algorithm 2. Let us then assume $t \geq 2$ and fix an arc $([x],[y])$ of $A_{i+1,j}$, where $[x] \in O_{i,j'}$ and $[y] \in O_{i+1,j}$ by the definition of $A_{i+1,j}$ in (8). Because TL_{n_d} is $\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive by Theorem 1(d), for any $([u],[v]) \in \mathtt{Arc}(TL_{n_d})$ there exists $([z],[m]) \in \mathbb{Z}_{n_d} \rtimes H(d)$, where $[z] \in \mathbb{Z}_{n_d}$ and $[m] \in H(d)$, such that $([u],[v]) = ([x],[y])^{([z],[m])} = ([xm+zm],[ym+zm])$. From the definitions of $O_{i,j'}$, $O_{i+1,j}$, and $A_{i+1,j}$, we know that $([xm],[ym]) \in A_{i+1,j}$ and hence $([u],[v]) \in \Lambda_t([zm])$. In other words, each arc of TL_{n_d} is contained in at least one $\Lambda_t([w])$, and hence $|\cup_{[w]\in\mathbb{Z}_{n_d}} \Lambda_t([w])| = |\mathtt{Arc}(TL_{n_d})|$. However, because TL_{n_d} has degree six and $|\Lambda_t([w])| = |A_{i+1,j}| = 6$ for all $[w] \in \mathbb{Z}_{n_d}$, we have $|\cup_{[w]\in\mathbb{Z}_{n_d}} \Lambda_t([w])| \leq \cup_{[w]\in\mathbb{Z}_{n_d}} |\Lambda_t([w])| = 6n_d = |\mathtt{Arc}(TL_{n_d})|$. This forces $\{\Lambda_t([w]) : [w] \in \mathbb{Z}_{n_d}\}$ to be a partition of $\mathtt{Arc}(TL_{n_d})$. Therefore, at any time no two messages compete for the same arc, and so Algorithm 2 is a gossiping protocol for TL_{n_d}. Moreover, at any time each arc is used exactly once for data transmission. This protocol requires $d(d+1)/2$ time steps. Hence, $t(TL_{n_d}) \leq d(d+1)/2$. However, $n_d - 1$ messages are to be sent to $[0]$, but at most six messages can reach $[0]$ at any time. Thus, $t(TL_{n_d}) \geq (n_d-1)/6 = d(d+1)/2$. Therefore, $t(TL_{n_d}) = d(d+1)/2$ and Algorithm 2 is an optimal gossiping protocol.

Because T_0 is a shortest path spanning tree of TL_{n_d} by Lemma 3, so is T_w and thus by Algorithm 2 the message originating from any vertex $[w]$ is transmitted along shortest paths to other vertices. Clearly, the labels of the six arcs of $A_{1,0}$ are pairwise distinct, and by Lemma 2(b), for $i \geq 1$ the labels of the arcs of $A_{i+1,j}$ are pairwise distinct as well. In view of (10), this implies that the six arcs in $\Lambda_t([w])$ carrying the message with source $[w]$ have different labels for any $t \geq 1$. From this and Lemma 2(b), the statements in (b) follow immediately. ■

Remark 4.

(a) *Because the spanning tree T_0 produced by Algorithm 1 is not unique, we may obtain many optimal gossiping protocols for TL_{n_d} by using Algorithms 1–2.*
(b) *In view of* (10)*, the optimal gossiping protocols given in Algorithm 2 are symmetric; that is, at any time the six arcs carrying the message with source* $[w]$ *are obtained from those carrying the message with source* $[0]$ *by translation. In other words, such an optimal gossiping protocol is obtained by using a broadcasting protocol at vertex* $[0]$.
(c) *The proof of Theorem 4 shows that the minimum gossip time of TL_{n_d} attains the trivial lower bound $t(TL_{n_d}) \geq (n_d-1)/6$, which itself is a specification of a general lower bound* [2, Proposition 7]*: $t(\Gamma) \geq \lceil (n-1)/\delta \rceil$ for any connected graph Γ with order n and minimum degree δ.*

Now let us move on to routings in TL_{n_d}. A routing $\mathcal{P}$ of a graph Γ is said to be edge-uniform (arc-uniform) if it loads all edges (arcs) uniformly. Call $\mathcal{P}$ a G-arc transitive routing [14], where $G \leq \mathrm{Aut}(\Gamma)$, if G is transitive on $\mathtt{Arc}(\Gamma)$ and leaves $\mathcal{P}$ invariant (that is, for any $g \in G$ the image of any path in $\mathcal{P}$ remains in $\mathcal{P}$). It is known [11, Theorem 3.2] that

$$\pi_m(\Gamma) \geq \pi(\Gamma) \geq \frac{\sum_{(u,v)\in V\times V} d(u,v)}{|E(\Gamma)|} \tag{11}$$

and the equalities hold if and only if Γ admits an edge-uniform shortest path routing. Similarly,

$$\vec{\pi}_m(\Gamma) \geq \vec{\pi}(\Gamma) \geq \frac{\sum_{(u,v)\in V\times V} d(u,v)}{2|E(\Gamma)|} \tag{12}$$

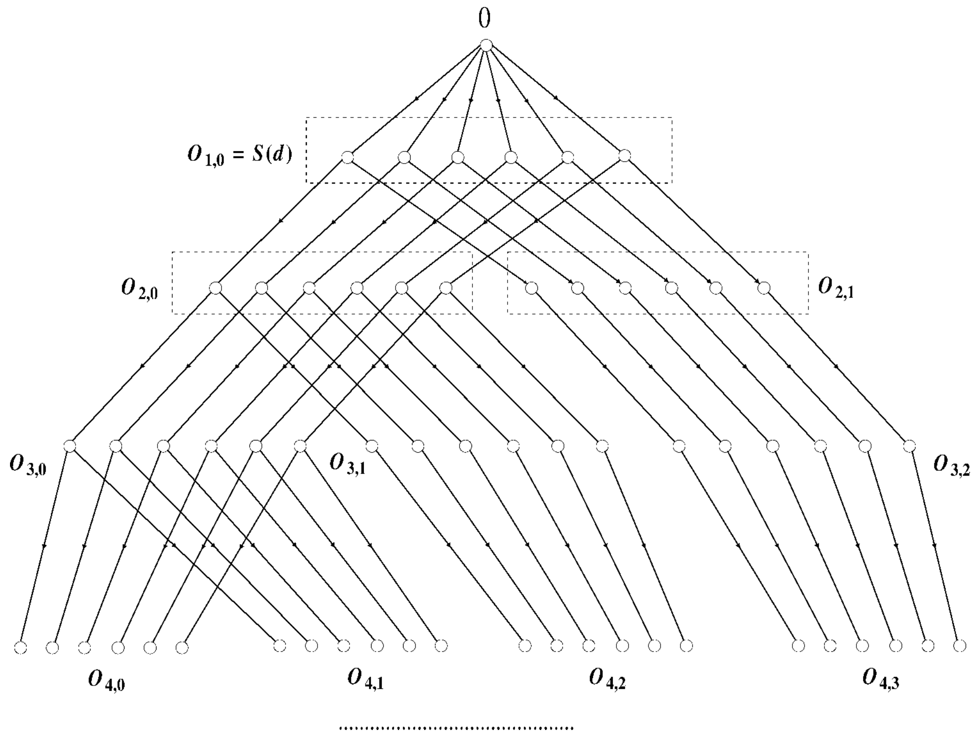

FIG. 3. Illustration of Algorithm 2 by using the "canonical" spanning tree shown in Figure 2. Arrows indicate the direction of data transmission. The figure shows the routing tree T_0 only. The vertices in levels 1–3 are $O_{1,0} = \{1, -1, 3d+1, -(3d+1), 3d+2, -(3d+2)\}$, $O_{2,0} = \{2, -2, 6d+2, -(6d+2), 6d+4, -(6d+4)\}$, $O_{2,1} = \{3d+3, -(3d+3), 3d, -3d, 6d+3, -(6d+3)\}$, $O_{3,0} = \{3, -3, 9d+3, -(9d+3), 9d+6, -(9d+6)\}$, $O_{3,1} = \{3d+4, -(3d+4), 6d+1, -(6d+1), 9d+5, -(9d+5)\}$, and $O_{3,2} = \{6d+5, -(6d+5), 3d-1, -(3d-1), 9d+4, -(9d+4)\}$, respectively.

with equalities precisely when Γ admits an arc-uniform shortest path routing. The following theorem shows that the forwarding indices of TL_{n_d} attain these lower bounds and, moreover, we can construct optimal routings explicitly.

Theorem 5. *Let $d \geq 2$ be an integer. Then*

$$\pi(TL_{n_d}) = 2\vec{\pi}(TL_{n_d}) = 2\vec{\pi}_m(TL_{n_d}) = \pi_m(TL_{n_d}) = \frac{d(d+1)(2d+1)}{3}. \quad (13)$$

Moreover, for any shortest path spanning tree T_0 of TL_{n_d} constructed by Algorithm 1, let $\mathcal{P}$ be the routing of TL_{n_d} such that for any $[w], [x] \in \mathbb{Z}_{n_d}$ the path from $[w]$ to $[x]$ is chosen to be the unique path in T_w from $[w]$ to $[x]$. Then $\mathcal{P}$ is

(a) *a shortest path routing;*
(b) *$\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive;*
(c) *both edge-uniform and arc-uniform; and*
(d) *optimal for π, $\vec{\pi}$, $\vec{\pi}_m$, and π_m simultaneously.*

Proof. Because by Lemma 3, T_w is a shortest path spanning tree of TL_{n_d} for all $[w] \in \mathbb{Z}_{n_d}$, $\mathcal{P}$ is a shortest path routing. From the construction of T_0 in Algorithm 1, one can check that T_0 is invariant under $H(d)$, that is, the image of T_0 under any $[m] \in H(d)$ is T_0 itself. From this and (9), it follows that $\mathbb{Z}_{n_d} \rtimes H(d)$ leaves $\mathcal{T}$ and hence $\mathcal{P}$ invariant. Because $\mathbb{Z}_{n_d} \rtimes H(d)$ is transitive on $\texttt{Arc}(TL_{n_d})$ (Theorem 1(d)), it follows that $\mathcal{P}$ is a $\mathbb{Z}_{n_d} \rtimes H(d)$-arc transitive routing. This in turn implies that $\mathcal{P}$ is arc-uniform and hence edge-uniform as well. Thus, all equalities in (11)–(12) hold for $\Gamma = TL_{n_d}$ and $\mathcal{P}$ is optimal with respect to all four indices. Because by Lemmas 2–3, there are exactly $6i$ vertices at distance i from [0] in T_0, $1 \leq i \leq d$, and because each T_w is a translation of T_0, under $\mathcal{P}$ the load of each edge of TL_{n_d} is equal to $(n_d \sum_{i=1}^{d} 6i^2)/3n_d = d(d+1)(2d+1)/3$, and hence (13) follows. ∎

Remark 5.

(a) *Again, because the spanning tree T_0 is not unique, we can obtain many optimal routings for TL_{n_d}. See Figure 3 for the routing based on the "canonical" T_0 given in Remark 3(c).*
(b) *Because each T_w is a translation of T_0, all such optimal routings are symmetric in the sense that, for any vertices $[w]$ and $[x]$, the route from $[w]$ to $[x]$ is obtained through translation of the route from [0] to $[x-w]$ by $[w]$.*

4. CONCLUDING REMARKS

As mentioned earlier, TL_{n_d} attains the maximum order among all undirected triple-loop networks $TL_n(a, b, -(a+b))$ of a given diameter $d \geq 2$. In this article, we proved that TL_{n_d} admits "perfect" gossiping and routing schemes which achieve the minimum possible edge-forwarding index and gossip time, respectively, and exhibit other very attractive features. Moreover, we found a systematic way of constructing such schemes explicitly by constructing a routing tree for TL_{n_d} which worked for both gossiping and routing problems. By choosing different routing trees, we can obtain different "perfect" gossiping and routing schemes. (However, we do not know whether all "perfect" gossiping and routing schemes can be obtained by using this method.) All these results were obtained by a group-theoretic method, which is an innovative approach to studying triple-loop networks. The results suggest that TL_{n_d} is a very strong candidate for interconnection networks, because not only is it optimal in the degree-diameter sense but also it is optimal for both gossiping and routing problems. This article considered only the gossiping problem under the store-and-forward, all-port, and full-duplex model. It would be interesting to study the gossiping problem for TL_{n_d} under other communication models. For this purpose, it is speculated that the algebraic properties of TL_{n_d} produced in Section 2 could be useful.

The Wiener index of a graph is the sum of the distances between all unordered pairs of vertices. With motivation from chemistry, this index has attracted considerable interest in chemical graph theory over sixty years. As a by-product, we obtain from Theorem 5 that the Wiener index of TL_{n_d} is equal to $d(d+1)(2d+1)(3d^2+3d+1)/2$ for any $d \geq 2$.

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