Hamiltonicity of Random Graphs Produced by 2-Processes

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ABSTRACT: Suppose that a random graph begins with n isolated vertices and evolves by edges being added at random, conditional upon all vertex degrees being at most 2. The final graph is usually 2-regular, but is not uniformly distributed. Some properties of this final graph are already known, but the asymptotic probability of being a Hamilton cycle was not known. We answer this question along with some related questions about cycles arising in the process. © 2006 Wiley Periodicals, Inc. Random Struct. Alg., 30, 000–000, 2007

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1. INTRODUCTION

Suppose we begin with $n \geq 2$ isolated vertices and add edges randomly one by one such that no multiple edges are created and that the maximum degree of the graph thus induced

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is always at most d, where $d \geq 1$ is a given integer. These random processes were studied in [5], where it was shown that when no more edges can be added, the number of edges is almost surely equal to the maximum possible number, $\lfloor dn/2 \rfloor$, as $n \to \infty$. In the case $d = 2$, the final graph must be a collection of disjoint cycles, together perhaps with one isolated edge. This was studied in [6] in more detail and information about the distribution of cycles of bounded length in this graph was obtained. It was also shown that the expected number of cycles in total is at most $3 + \log n$. In the present paper we study the long cycles in the final graph and answer several of the questions raised about 2-processes by Erdős (see [6]). In particular we compute asymptotically the probability that the final graph is just an n-cycle. This tells us the probability that the final graph is connected. For d-processes with $d \geq 3$, it was shown [7] that the final graph is connected with probability tending to 1 as $n \to \infty$.

We assume that a graph g may have no multiple edges or loops and denote its edge set by $E(g)$. We say that g is 2-*maximal* if every vertex has degree at most 2, but no new edge can be added without violating this condition. The reason for this can be that every vertex, except perhaps one, already has degree 2, or that there are only two vertices of degree less than 2, which are already adjacent. Formally, we define a 2-*process* to be a sequence of graphs $(g_0, g_1, \ldots, g_n)$ on vertex set $[n] = \{1, 2, \ldots, n\}$ such that the following (i)–(iv) hold for some $w \leq n$:

(i) $|E(g_t)| = t, \qquad t = 0, 1, \ldots, w$;
(ii) $g_w = g_n$;
(iii) $\emptyset = E(g_0) \subseteq E(g_1) \subseteq \cdots \subseteq E(g_n)$; and
(iv) g_n is 2-maximal.

From (iv) it follows that the graph becomes 2-maximal at time $w = w(g_1, \ldots, g_n)$, which stands for the number of edges in the graph when no more can be added. The process remains static until time n, which is the maximum time a 2-process can possibly run. Clearly, $w = n - 1$ or n. Property (ii) is included merely for the convenience of having all sequences of equal length.

A *random* 2-*process* is a probabilistic space whose elements are 2-processes with probabilities assigned as follows. Define u_t to be the number of vertices of degree less than 2 (*unsaturated vertices*) in g_t and z_t the number of edges whose ends both have degree 1 (*isolated edges*). We assign the probability

$$\prod_{t=0}^{w-1} \frac{1}{\binom{u_t}{2} - z_t}$$

to the 2-process $(g_0, g_1, \ldots, g_n)$. It is convenient to think of g_t as being formed at time t by adding an edge e_t to g_{t-1}, where e_t is chosen uniformly at random from current available sites. Here an *available site* is a pair of unsaturated, non-adjacent vertices of g_{t-1}. The edges $e_1, \ldots, e_n$ of g_n are in the order in which they appear in the process, where e_n is left undefined if $w = n - 1$.

We use uppercase letters to denote random variables corresponding to the lowercase deterministic parameters. Thus, a random 2-process is denoted by $(G_0, G_1, \ldots, G_n)$, and U_t and Z_t are the numbers of unsaturated vertices and isolated edges at time t, respectively. From [5, Theorem 1.1] it follows that in a random 2-process $w = n$ occurs almost surely. In this case the final graph G_n is 2-regular and thus is a collection of disjoint cycles. If there is only one cycle, then G_n is hamiltonian. The main result in this paper, namely Theorem 1 below, gives asymptotically the probability of the hamiltonicity of G_n. Throughout the paper the asymptotics is made as $n \to \infty$. In particular, a random 2-process has a property Q *asymptotically almost surely* (a.a.s.) if $\lim_{n\to\infty} \mathbf{P}(Q) = 1$. Denote

$$\tau = \int_0^\infty \frac{\log(1+x)}{xe^x} dx$$

so that

$$\begin{aligned}\tau &= \frac{\gamma^2}{2} + \frac{\pi^2}{4} - \sum_{k=1}^{\infty} \frac{2k^{-1} + \psi(k)}{k \cdot k!} \\ &\approx 0.7452\end{aligned}$$

by [3, p. 530, eq. 6.], where $\gamma = \lim_{k\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log k\right)$ is Euler's constant and ψ is the digamma function.

Theorem 1.

$$\mathbf{P}(G_n \text{ is hamiltonian}) \sim \frac{c_1}{\sqrt{n}}$$

where $c_1 = \sqrt{\frac{\pi e^\tau}{2}} \approx 1.819$.

For random 2-regular graphs with uniform probability distribution, the probability of a hamilton cycle is easily computed from the results of Bender and Canfield [1] to be $\frac{1}{2}e^{3/4}\sqrt{\pi}n^{-1/2} \approx 1.876n^{-1/2}$, which is slightly different from the 2-process result above.

Theorem 1 is proved by obtaining a fairly precise estimate for the probability that the tth edge E_t completes a cycle. The graph G_n is hamiltonian if and only if the first time this happens is with E_n. We obtain this estimate by determining some parameters of the process fairly accurately, namely the numbers of isolated vertices (also called *isolates*) and of isolated edges. This is done by showing that a.a.s. these numbers are approximated by the solutions of some differential equations, as in [6] and [9]. The interesting new feature of the present analysis is that we can demonstrate this approximation until very near the end of the process (see Theorem 4, stated in Section 2 and proved in Section 3), due to the attractive nature of the solution of the differential equation. This is a necessary requirement in the present application, since the probability of forming a cycle at a given step increases throughout the process, to a constant value at the end, and therefore the end of the process plays an important role for hamiltonicity.

Our method has similarities with mean field theory used in statistical physics, where a complex interacting stochastic system is described by differential equations arising under the assumption that all the variables take their expected values. In this paper, we use conditional expectations and martingale arguments to make the heuristic mean field approach rigorous. In particular, this approximation works to near the very end of the process.

A simpler consequence of the analysis required for Theorem 1 leads to the following strengthening of [6, Theorem 3]. We use X_n to denote the number of cycles in G_n.

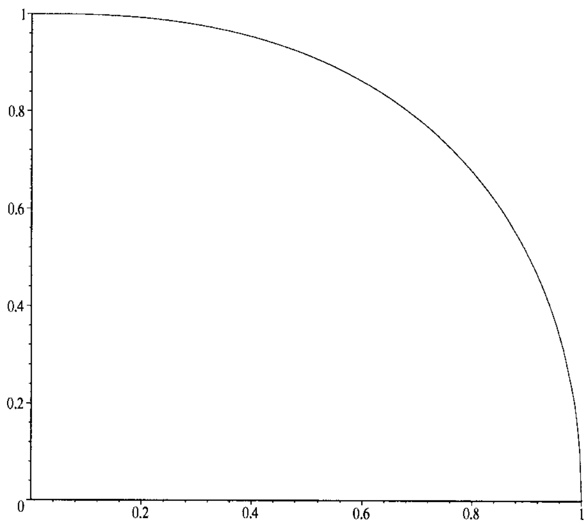

Fig. 1. Approximate probability of no cycles after xn steps.

Theorem 2.

$$\mathbf{E}(X_n) = \frac{1}{2}(\log n + \log 2 + \gamma - \tau) + O((\log n)^{-1/11})$$

where τ and γ are as above.

Another point of interest in the evolution of a 2-process is the time when the first cycle appears. We denote this time by T_n. Again, using little more than part of the proof of Theorem 1, we obtain the following result, where c_2 is a constant whose precise formula will be given in the proof.

Theorem 3.

$$\mathbf{E}(T_n) \sim c_2 n$$

where c_2 is constant, $c_2 \approx 0.829$.

The analysis required for Theorem 1 also gives rise to approximations of the probability that G_t contains no cycles, for given t. See Fig. 1 and the discussion at the end of Section 2.

The three theorems above answer Questions 6, 5, and 1 asked at the end of [6]. Probably the most interesting open problem remaining in this area is whether the final graph of this process is contiguous to a random 2-regular graph (see [8]). Unfortunately, the methods of the present paper are not strong enough to answer this.

2. PROOFS

It was the analysis of the number of isolated vertices that led to the main results in [5, 6]. Here we shall need more: sharp concentration not only of the number of isolated vertices but also of the number of isolated edges. Also we need knowledge of their behavior for longer than in [5, 6]: we need good upper and lower bounds almost until the very end of the process.

The proof of the concentration results will be deferred to Section 3. In the present section, we motivate these concentration results and use them to prove the theorems.

Let $\mathcal{C}_t$ denote the event that E_{t+1} creates a cycle when it is added to G_t and $\overline{\mathcal{C}_t}$ its complement. Then we have

$$\mathbf{P}(G_n \text{ is Hamiltonian}) = \mathbf{P}\left(\bigwedge_{t=0}^{n-2} \overline{\mathcal{C}_t}\right) \tag{2.1}$$

(assuming $n \geq 2$). We use Y_t to denote the (random) number of isolated vertices at time t. By counting vertex degrees, we know that the number of vertices of degree 1 in G_t is $2(n-t-Y_t)$ and hence

$$Y_t + U_t = 2(n-t). \tag{2.2}$$

Since the connected components of G_t are necessarily paths, cycles, and isolated vertices, the number of paths of length at least 2 is $n-t-Y_t-Z_t$. Hence, we have

$$Y_t + Z_t \leq n-t \leq U_t. \tag{2.3}$$

Note that $n-t-Y_t-Z_t$ is the number of edges that cause a cycle when added to G_t. Thus, given G_t, the probability of $\mathcal{C}_t$ is

$$\mathbf{P}(\mathcal{C}_t \mid G_t) = \frac{n-t-Y_t-Z_t}{\binom{U_t}{2} - Z_t} \tag{2.4}$$

$$= \frac{U_t - Y_t - 2Z_t}{U_t^2}\left(1 + O\left(\frac{1}{U_t}\right)\right), \tag{2.5}$$

where the $O(1/U_t)$ term is obtained by using $Z_t \leq U_t$ from (2.3).

The expected changes in Y_t and Z_t are given by

$$\mathbf{E}(Y_{t+1} - Y_t \mid G_t) = -\frac{2Y_t}{2n-2t-Y_t} + O\left(\frac{1}{n-t}\right)$$

$$\mathbf{E}(Z_{t+1} - Z_t \mid G_t) = \frac{Y_t^2 - 4(2n-2t-Y_t)Z_t}{(2n-2t-Y_t)^2} + O\left(\frac{1}{n-t}\right).$$

See Lemma 2 in the next section for details. Thus, we will approximate $\mathbf{E}(Y_{t+1} - Y_t \mid G_t)$ by $f(t/n, Y_t/n)$ and approximate $\mathbf{E}(Z_{t+1} - Z_t \mid G_t)$ by $g(t/n, Y_t/n, Z_t/n)$, where

$$f(x,y) = -\frac{2y}{2-2x-y} \tag{2.6}$$

$$g(x,y,z) = \frac{y^2 - 4(2-2x-y)z}{(2-2x-y)^2}. \tag{2.7}$$

The proof will be based on the following fact, which is *suggested* by putting $x = t/n$, $y = Y/n$, and $z = Z/n$ in the results above. For any time t, as long as the "remaining time," $n-t$, tends to ∞ as $n \to \infty$, the random variables Y_t and Z_t can be well approximated by $ny(t/n)$ and $nz(t/n)$ (see Theorem 4), where $y = y(x)$ and $z = z(x)$ $(0 \le x < 1)$ are the solutions of the following differential equations:

$$\frac{dy}{dx} = f(x, y), \qquad y(0) = 1, \tag{2.8}$$

$$\frac{dz}{dx} = g(x, y, z), \qquad z(0) = 0. \tag{2.9}$$

Thus, by (2.2), U_t will be approximated by $nu(t/n)$, where we define

$$u(x) = 2 - 2x - y(x). \tag{2.10}$$

Equation (2.8) and its analogue for a d-process in general were used heavily in [5] and [6], where some consequential properties of y were noted. We give these here, as well as similar properties of z.

Solving (2.8), by noting that dx/dy is linear in x for example, yields

$$y(2 - \log y) = 2(1 - x) \tag{2.11}$$

and thus

$$u = y(1 - \log y). \tag{2.12}$$

From this and (2.8) and (2.9) we have

$$\frac{dz}{dy} = -\frac{1}{2(1 - \log y)} + \frac{2z}{y}.$$

Using the initial condition $z(0) = 0$, we then obtain

$$z(x) = \frac{y(x)^2}{2} \int_{y(x)}^{1} \frac{d\eta}{\eta^2(1 - \log \eta)}. \tag{2.13}$$

For our later use we need the following approximations. Note that (2.11) (or (2.8)) implies

$$0 \le y(x) \le 1 - x. \tag{2.14}$$

Thus, $f(x, y) \le 0$ and so $y(x)$ is decreasing with x. From (2.14) and using the logarithmic inequality ($\log(1 - \alpha) \le -\alpha$ for $\alpha < 1$) twice we get

$$\begin{aligned} z(x) &\le \frac{y(x)^2}{2} \int_{y(x)}^{1} \frac{d\eta}{\eta^2(2 - \eta)} \\ &\le \frac{y(x)(1 - y(x))}{2} \qquad (2.15) \\ &\le \frac{1 - x}{2}. \qquad (2.16) \end{aligned}$$

Moreover, from (2.11) one can see that

$$y(x) \sim \frac{-2(1 - x)}{\log(1 - x)} \qquad \text{as} \quad x \to 1 \tag{2.17}$$

(see also [6, (2.10)]).

The object of this section is to show that the approximations of the random variables Y and Z by the corresponding functions y and z are sufficiently accurate. The proof of Theorem 1 is the most delicate, and for this the process is divided into two phases. For proving the other theorems, the treatment of the second phase could be made much simpler, although we obtain a smaller error in the result of Theorem 2 by careful consideration of Phase 2.

The two phases are $[0, t_1]$ and $(t_1, n-2]$, where we set

$$t_1 = n - \lceil (\log n)^N \rceil \tag{2.18}$$

for a large constant N, which we specify below. We will be able to estimate $\mathbf{P}(\mathcal{C}_t \mid G_t)$ to sufficient accuracy by the non-random value $\bar{r}(x)/n$, where

$$\bar{r}(x) = \begin{cases} r(x), & x \in [0, t_1/n] \\ \frac{1}{2-2x-n^{-1}}, & x \in (t_1/n, 1] \end{cases}$$

with

$$r(x) = \frac{u(x) - y(x) - 2z(x)}{u(x)^2}. \tag{2.19}$$

Note that for $t > t_1$, we arrive at this approximation by setting Y_t and Z_t equal to 0 in (2.4), using (2.2). Since $u(x) \geq 1 - x$ by (2.10) and (2.14), and since $y(x) \geq 0$ and $z(x) \geq 0$ (see (2.14) and (2.13), respectively), for any $x \in [0, 1)$ we have

$$r(x) \leq \frac{1}{u(x)} \leq \frac{1}{1-x}. \tag{2.20}$$

The following result will play a key role in our proof of Theorems 1, 2, and 3. It is of interest for its own sake since it might be useful in proving some other results concerning the random 2-process. It is in more generality than what is needed in this paper: we will use its special case where $\sigma(n) = t_1$ in the proof of Theorems 1–3. Note that it is an improvement of [6, Theorem 2]: the approximation is more accurate and lasts longer, reaching any point such that the remaining time tends to infinity as $n \to \infty$. Unfortunately, the accuracy of this theorem by itself is not good enough for our purposes in Phase 2, but it is sufficient for Phase 1.

Theorem 4. *For any $0 < \varepsilon < 1/4$ and any function $\sigma(n) > 0$ with $n - \sigma(n)$ tending to ∞ as $n \to \infty$, there exists $C_0 = C_0(\varepsilon)$, which is independent of $\sigma(n)$, such that with probability $1 - O\big(e^{-(n-\sigma(n))^{1/2-2\varepsilon}}\big)$ we have*

$$|Y_t - ny(t/n)| < C_0(n-t)^{1-\varepsilon} \tag{2.21}$$

$$|Z_t - nz(t/n)| < C_0(n-t)^{1-\varepsilon} \tag{2.22}$$

for all $0 \leq t \leq \sigma(n)$, where $y(x)$ and $z(x)$ are determined by (2.11) and (2.13), respectively.

By (2.2), (2.10), and (2.21), under the same conditions as in Theorem 4, we have

$$|U_t - nu(t/n)| < C_0(n-t)^{1-\varepsilon} \tag{2.23}$$

with probability $1 - O\big(e^{-(n-\sigma(n))^{1/2-2\varepsilon}}\big)$.

We now specify the values of a set of parameters beginning with any fixed ρ and K with $0 < \rho < 1$ and $K > 4$, as follows. Set $\varepsilon = 3\rho/16$ and $\zeta = 1/2 - 2\varepsilon = 1/2 - 3\rho/8$, so that $\zeta > 1/8$. For the definition (2.18) of t_1, select N such that $\max\{16/3, K/4\zeta\} < N < 2K$ so that in particular $(n-t_1)^\zeta > (\log n)^{K/4}$.

In order to introduce the next theorem we note that for large n (using (2.15), (2.11), and (2.14)), $C_0/(n-t_1)^\varepsilon \le C_0/(\log n)^{N\varepsilon} < 1/4(\log n)^{16\varepsilon/3} = 1/4(\log n)^\rho$, and $n(y(t_1/n) + z(t_1/n))/(n-t_1) < 2ny(t_1/n)/(n-t_1) = 4/(2 - \log y(t_1/n)) \le 4/(2 - \log(1 - t_1/n)) < 8/\log n < 1/2(\log n)^\rho$. Hence, with $\sigma = t_1$ in Theorem 4,

$$\mathbf{P}\left(Y_{t_1} + Z_{t_1} < \frac{n-t_1}{(\log n)^\rho}\right) = 1 - O\big(e^{-(\log n)^{K/4}}\big) = 1 - o(n^{-2}). \tag{2.24}$$

To cover the major part of phase 2 we first need to restrict to $t \le t_2$ where

$$t_2 = n - \lceil \log n \rceil.$$

For any graph G, denote by $Y(G)$ and $Z(G)$ the numbers of isolated vertices and isolated edges of G, respectively.

Theorem 5. *Let $0 < \rho < 1$ and $\mathcal{G}_1$ be any set of graphs such that for all $G \in \mathcal{G}_1$, $\mathbf{P}(G = G_{t_1}) \ne 0$ and $Y(G) + Z(G) < (n-t_1)/(\log n)^\rho$. Then for all $K > 0$*

$$\mathbf{P}\left(Y_t + Z_t \le \frac{2(n-t)}{(\log n)^\rho} \text{ for all } t \text{ with } t_1 < t \le t_2 \mid G_{t_1} \in \mathcal{G}_1\right) = 1 - O((\log n)^{-K}). \tag{2.25}$$

For the part of phase 2 from t_2 onward, we only need some quite simple observations. Lemma 3.1 of [5] was applied to the final stages of the process to show that the random 2-process a.a.s. has a final graph with $Y_n + Z_n = 0$. We need to sharpen this conclusion slightly here. For this purpose we choose δ, δ' such that

$$0 < \delta' < \delta < 1, \quad 2(1 - \rho + \delta') < \delta < \frac{1}{2} - \delta' < \rho - \frac{1}{2}. \tag{2.26}$$

(For example, choosing $\rho = 999/1000$, $\delta = 2/5$ and $\delta' = 99/1000$ satisfies all conditions above.) Let

$$t_3 = n - \lfloor (\log n)^\delta \rfloor, \qquad t_4 = n - \lfloor (\log n)^{\delta'} \rfloor.$$

Theorem 6. *Let $0 < \rho < 1$ and let $\mathcal{G}_2$ be any set of graphs such that for all $G \in \mathcal{G}_2$, $\mathbf{P}(G = G_{t_2}) \ne 0$ and $Y(G) + Z(G) = O((\log n)^{1-\rho})$. Then in the event that $G_{t_2} \in \mathcal{G}_2$,* *we have*

$$Y_t + Z_t = O((\log n)^{1-\rho}) \text{ for } t_2 \le t \le n. \tag{2.27}$$

Furthermore,

$$\mathbf{P}\left(Y_{t_3} = 0 \mid G_{t_2} \in \mathcal{G}_2\right) = 1 - O((\log n)^{\delta-\rho}) \tag{2.28}$$

and

$$\mathbf{P}\left(Z_{t_4} = 0 \mid Y_{t_3} = 0 \wedge G_{t_2} \in \mathcal{G}_2\right) = 1 - O((\log n)^{\delta'-\delta+1-\rho}). \tag{2.29}$$

The following result will be useful in computing both the probability of hamiltonicity and the expected number of cycles in the random 2-process.

Lemma 1. *For any $t_0 = t_0(n) > 0$ such that $n - t_0 = O(\sqrt{n})$ and $n - t_0 \to \infty$ as $n \to \infty$, we have*

$$\sum_{t=0}^{t_0} \frac{1}{n} r\left(\frac{t}{n}\right) = -\frac{1}{2}\log(2(n-t_0)/n) - \frac{\tau}{2} + O\left((n-t_0)^{-1} + (\log n)^{-1}\right).$$

Proof. Let $r_1(x) = (u(x) - y(x))/u(x)^2$ and $r_2(x) = -2z(x)/u(x)^2$, so that $r(x) = r_1(x) + r_2(x)$. Set $J_1(t) = \int_0^{t/n} r_1(x)dx$ and $J_2(t) = \int_0^{t/n} r_2(x)dx$. Using (2.12) we see that $dr_1/dy < 0$. Since $dy/dx = f(x,y) < 0$, it follows that $dr_1/dx > 0$. Hence,

$$\sum_{t=0}^{t_0} r_1(t/n)/n = J_1(t_0) + O(r_1(t_0/n)/n) = J_1(t_0) + O((n-t_0)^{-1}). \tag{2.30}$$

(Here we used $r_1(x) = -\log y(x)/y(x)(1 - \log y(x))^2 \sim 1/y(x)(1 - \log y(x)) = 1/(2(1 - x) - y(x)) \sim 1/2(1-x)$ as $x \to 1$, which follows from (2.12), (2.11), and the fact that $y(x) \to 0$ when $x \to 1$.) From (2.11), we have $\log(1-x) = \log y + \log(1 - (\log y)/2)$. Using this and (2.8), and noting $y(0) = 1$, we have

$$\begin{aligned}
J_1(t) &= \frac{1}{2}\int_1^{y(t/n)} \frac{\log y}{y(1-\log y)} dy \\
&= -\frac{1}{2}\log y(t/n) - \frac{1}{2}\log(1 - \log y(t/n)) \\
&= -\frac{1}{2}\log y(t/n) - \frac{1}{2}\log(1 - (\log y(t/n))/2) - \frac{1}{2}\log\left(\frac{1 - \log y(t/n)}{1 - (\log y(t/n))/2}\right) \\
&= -\frac{1}{2}\log(1 - t/n) - \frac{1}{2}\log\left(\frac{1 - \log y(t/n)}{1 - (\log y(t/n))/2}\right).
\end{aligned} \tag{2.31}$$

Since $t_0/n \to 1$, from (2.17) we have $y(t_0/n) \to 0$ as $n \to \infty$. Hence, using (2.11) and (2.17), we have

$$\begin{aligned}
J_1(t_0) &= -\frac{1}{2}\log(2(n-t_0)/n) - \frac{1}{2}\log\left(\frac{1-\log y(t_0/n)}{2-\log y(t_0/n)}\right)\\
&= -\frac{1}{2}\log(2(n-t_0)/n) + O\left(\frac{1}{2-\log y(t_0/n)}\right)\\
&= -\frac{1}{2}\log(2(n-t_0)/n) + O\left(\frac{y(t_0/n)}{2(1-t_0/n)}\right)\\
&= -\frac{1}{2}\log(2(n-t_0)/n) + O(-1/\log(1-t_0/n))\\
&= -\frac{1}{2}\log(2(n-t_0)/n) + O((\log n)^{-1}). \qquad (2.32)
\end{aligned}$$

Using (2.8) and (2.13), we get

$$\begin{aligned}
J_2(t) &= \int_1^{y(t/n)} \frac{z/y}{y(1-\log y)}\,dy\\
&= \frac{1}{2}\int_1^{y(t/n)} \frac{1}{1-\log y}\int_y^1 \frac{1}{\eta^2(1-\log\eta)}\,d\eta\,dy. \qquad (2.33)
\end{aligned}$$

This double integral can be evaluated as follows, first substituting $\xi = 1-\log\eta$, then $w = 1-\log y$, $a = w-\xi$, and then reversing the order of integration.

$$\begin{aligned}
J_2(t) &= \frac{1}{2}\int_{1-\log y(t/n)}^{1}\int_1^w \frac{e^{\xi-w}}{\xi w}\,d\xi\,dw\\
&= \frac{1}{2}\int_{1-\log y(t/n)}^{1}\int_0^{w-1} \frac{e^{-a}}{w(w-a)}\,da\,dw\\
&= \frac{1}{2}\int_{-\log y(t/n)}^{0} e^{-a}\int_{1+a}^{1-\log y(t/n)} \frac{1}{w(w-a)}\,dw\,da\\
&= \frac{1}{2}\int_{-\log y(t/n)}^{0} \frac{1}{ae^a}\log\left(\frac{(1-a-\log y(t/n))(1+a)}{1-\log y(t/n)}\right)da. \qquad (2.34)
\end{aligned}$$

Since $y(t_0/n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
J_2(t_0) &= -\frac{\tau}{2} - \frac{1}{2}\int_0^{-\log y(t_0/n)} \frac{1}{ae^a}\log\left(1-\frac{a}{1-\log y(t_0/n)}\right)da\\
&\quad + \frac{1}{2}\int_{-\log y(t_0/n)}^{\infty} \frac{\log(1+a)}{ae^a}da\\
&= -\frac{\tau}{2} + O(-1/\log(1-t_0/n)) + O\left(\int_{-\log y(t_0/n)}^{\infty} e^{-a}da\right)\\
&= -\frac{\tau}{2} + O(-1/\log(1-t_0/n)) + O\left(y(t_0/n)\right)\\
&= -\frac{\tau}{2} + O((\log n)^{-1}). \qquad (2.35)
\end{aligned}$$

Here the approximation of the first integral is obtained by using (2.11), (2.17), and the logarithmic inequality. The approximation of the second integral is obtained by using $\log(1+a) \le a$ and then (2.17).

Note that $r_2(0) = 0$ and $|r_2(t_0/n)| \le 2n/(n-t_0)$ since $z(0) = 0$, $z(t_0/n) \le (1-t_0/n)/2$ and $u(t_0/n) \ge 1-t_0/n$. Thus, using the Trapezoidal Rule and its error bound from numerical integration, we have

$$\begin{aligned}\sum_{t=0}^{t_0} r_2(t/n)/n &= J_2(t_0) + (r_2(0) + r_2(t_0/n))/2n + O(n^{-2})\\ &= J_2(t_0) + O((n-t_0)^{-1}).\end{aligned} \tag{2.36}$$

Therefore, combining (2.30) and (2.36), and using (2.32) and (2.35), we obtain

$$\begin{aligned}\sum_{t=0}^{t_0} r(t/n)/n &= \sum_{t=0}^{t_0} r_1(t/n)/n + \sum_{t=0}^{t_0} r_2(t/n)/n\\ &= -\frac{1}{2}\log(2(n-t_0)/n) - \frac{\tau}{2} + O((n-t_0)^{-1} + (\log n)^{-1})\end{aligned}$$

and the proof is complete. ■

Proof of Theorem 1. Define $\mathcal{H}_t$ to be the event that G_t contains no cycles, which is just $(\bigwedge_{i=0}^{t-1}\overline{\mathcal{C}_i})$. Set $t_5 = n-1$. For $t \le t_1$ define $\mathcal{E}_t$ to be the event that Y_t and Z_t satisfy the inequalities (2.21) and (2.22). For $t_1 < t \le t_2$ let $\mathcal{E}_t$ be the event that $Y_t + Z_t \le 2(n-t)/(\log n)^{\rho}$. Let $\mathcal{E}_{t_3}$ be the event that $Y_{t_3} = 0$ and $Z_{t_3} < (\log n)^{1-\rho}$ and $\mathcal{E}_{t_4}$ the event that $Y_{t_4} = Z_{t_4} = 0$. We define $\mathcal{F}_{t_i}$ to be an event that ensures that no cycles have been created and that Y_t and Z_t satisfy appropriate inequalities, for all $t \le t_i$. To be precise, we let $\mathcal{F}_{t_i} = \mathcal{H}_{t_i} \wedge \mathcal{E}_{t_1} \wedge \cdots \wedge \mathcal{E}_{t_i}$ for $1 \le i \le 4$ and $\mathcal{F}_{t_5} = \mathcal{F}_{t_4} \wedge \mathcal{H}_{t_5}$. Then $\mathcal{F}_{t_i} = \mathcal{F}_{t_{i-1}} \wedge \mathcal{H}_{t_i} \wedge \mathcal{E}_{t_i}$ for $i = 2, 3, 4$, and

$$\mathcal{H}_{t_5} = (\mathcal{H}_{t_5} \wedge \overline{\mathcal{F}}_{t_4}) \vee (\mathcal{H}_{t_5} \wedge \mathcal{F}_{t_4}) = (\mathcal{H}_{t_5} \wedge \overline{\mathcal{F}}_{t_4}) \vee \mathcal{F}_{t_5}.$$

Noting that $\mathcal{H}_{t_5} \wedge \overline{\mathcal{H}}_{t_4} = \emptyset$, we have

$$\mathcal{H}_{t_5} \wedge \overline{\mathcal{F}}_{t_4} = \bigvee_{i=1}^{4}\left(\mathcal{H}_{t_5} \wedge \overline{\mathcal{E}}_{t_i}\right) \subseteq \overline{\mathcal{E}}_{t_1} \vee \left(\bigvee_{i=1}^{3}\left(\mathcal{H}_{t_i} \wedge \overline{\mathcal{E}}_{t_{i+1}}\right)\right) \subseteq \overline{\mathcal{E}}_{t_1} \vee \left(\bigvee_{i=1}^{3}\left(\mathcal{F}_{t_i} \wedge \overline{\mathcal{E}}_{t_{i+1}}\right)\right)$$

and so

$$\mathcal{F}_{t_5} \subseteq \mathcal{H}_{t_5} \subseteq \overline{\mathcal{E}}_{t_1} \vee \left(\bigvee_{i=1}^{3}\left(\mathcal{F}_{t_i} \wedge \overline{\mathcal{E}}_{t_{i+1}}\right)\right) \vee \mathcal{F}_{t_5}.$$

Hence, by (2.1)

$$\mathbf{P}(G_n \text{ is hamiltonian}) = \mathbf{P}(\mathcal{F}_{t_5}) + O\left(\mathbf{P}(\overline{\mathcal{E}}_{t_1}) + \sum_{i=1}^{3}\mathbf{P}\left(\mathcal{F}_{t_i} \wedge \overline{\mathcal{E}}_{t_{i+1}}\right)\right). \tag{2.37}$$

Since the $\mathcal{F}_{t_i}$ are nested, the main term $\mathbf{P}(\mathcal{F}_{t_5})$ in this equation can be computed using

$$\mathbf{P}(\mathcal{F}_{t_5}) = \mathbf{P}\left(\mathcal{F}_{t_1}\right)\prod_{i=1}^{4}\mathbf{P}\left(\mathcal{F}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right). \tag{2.38}$$

In the following we will estimate the factors in this expression one by one.

Since $\mathcal{F}_{t_1} = \mathcal{H}_{t_1} \wedge \mathcal{E}_{t_1}$, the first factor in (2.38) is

$$\begin{aligned}\mathbf{P}(\mathcal{F}_{t_1}) &= \mathbf{P}(\mathcal{H}_{t_1}) - \mathbf{P}(\mathcal{H}_{t_1} \wedge \overline{\mathcal{E}_{t_1}}) \\ &= \mathbf{P}(\mathcal{H}_{t_1}) + O(\mathbf{P}(\overline{\mathcal{E}_{t_1}})).\end{aligned} \tag{2.39}$$

Similar to (2.1), since $\mathbf{P}(\mathcal{H}_0) = 1$,

$$\mathbf{P}(\mathcal{H}_t) = \mathbf{P}\left(\bigwedge_{i=0}^{t-1}\overline{\mathcal{C}_i}\right) = \prod_{i=0}^{t-1}\mathbf{P}(\overline{\mathcal{C}_i} \mid \mathcal{H}_i). \tag{2.40}$$

From (2.2) and (2.4), $\mathbf{P}(\mathcal{C}_i \mid G_i)$ is maximized for given i when $Y_i = Z_i = 0$, that is

$$\mathbf{P}(\mathcal{C}_i \mid G_i) \le \frac{1}{2n-2i-1}.$$

We therefore have the initial bound

$$\mathbf{P}(\mathcal{H}_t) \ge \prod_{i=0}^{t-1}\left(1 - \frac{1}{2n-2i-1}\right) = \prod_{j=n-t}^{n-1}\frac{2j}{2j+1}.$$

We note for here and later that

$$\prod_{j=1}^{s-2}\frac{2j}{2j+1} = \frac{2^{2s-3}(s-2)!(s-1)!}{(2s-2)!} \sim \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2}, \tag{2.41}$$

where the last estimate is by Stirling's formula. Thus, for all $t \le n-1$,

$$\mathbf{P}(\mathcal{H}_t) = \Omega\left(\left(\frac{n-t}{n}\right)^{1/2}\right). \tag{2.42}$$

For $t < t_1$, since $\mathcal{E}_t$ implies (2.21) and (2.22), we have

$$\begin{aligned}\mathbf{P}(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{E}_t) &= \frac{U_t - Y_t - 2Z_t}{U_t^2}\left(1 + O\left(\frac{1}{U_t}\right)\right) \\ &= \frac{u(t/n) - y(t/n) - 2z(t/n) + O((n-t)^{1-\varepsilon}/n)}{n(u(t/n) - O((n-t)^{1-\varepsilon}/n))^2}\left(1 + O\left(\frac{1}{U_t}\right)\right) \\ &= \left(\frac{\bar{r}(t/n)}{n} + O\left(\frac{(n-t)^{1-\varepsilon}}{n^2(u(t/n))^2}\right)\right)\left(1 + O\left(\frac{(n-t)^{1-\varepsilon}}{nu(t/n)}\right)\right)\left(1 + O\left(\frac{1}{U_t}\right)\right) \\ &= \left(\frac{\bar{r}(t/n)}{n} + O\left(\frac{(n-t)^{1-\varepsilon}}{n^2(u(t/n))^2}\right)\right)\left(1 + O\left(\frac{1}{U_t}\right)\right) \\ &= \left(\frac{\bar{r}(t/n)}{n} + O((n-t)^{-1-\varepsilon})\right)\left(1 + O\left(\frac{1}{U_t}\right)\right) \\ &= \frac{\bar{r}(t/n)}{n} + O((n-t)^{-1-\varepsilon}).\end{aligned} \tag{2.43}$$

Here in the third last step we used $\bar{r}(t/n) \leq 1/u(t/n)$, which follows from the left part of (2.20). In the second last step we used the right part of (2.20), and in the last step we used $\bar{r}(t/n)/n \leq 1/(n-t)$ from (2.20) and $U_t \geq n-t$ from (2.3). By (2.42) and Theorem 4 with $\sigma(n) = t_1$, and noting that $N > K/4\zeta$, similar to the argument leading to (2.24) we get $\mathbf{P}(\overline{\mathcal{E}_t})/\mathbf{P}(\mathcal{H}_t) = O(e^{-(\log n)^{K/4}}\sqrt{n}/\sqrt{n-t}) = O(n^{-2}\sqrt{n}/\sqrt{n-t})$. Combining this and (2.43), we have for $t < t_1$,

$$\begin{aligned}
\mathbf{P}(\mathcal{C}_t \mid \mathcal{H}_t) &\leq \mathbf{P}(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{E}_t) + \mathbf{P}(\mathcal{C}_t \wedge \overline{\mathcal{E}_t} \mid \mathcal{H}_t) \\
&\leq \mathbf{P}(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{E}_t) + \mathbf{P}(\overline{\mathcal{E}_t})/\mathbf{P}(\mathcal{H}_t) \\
&= \frac{\bar{r}(t/n)}{n} + O((n-t)^{-1-\varepsilon}) + O(n^{-3/2}) \qquad (2.44)\\
&= \frac{\bar{r}(t/n)}{n} + O((n-t)^{-1-\varepsilon}). \qquad (2.45)
\end{aligned}$$

Here the last step follows from the fact that $\varepsilon = 3\rho/16 < 1/4$. From (2.40) and (2.45), we have, for any $t \leq t_1$,

$$\mathbf{P}(\mathcal{H}_t) = \prod_{i=0}^{t-1}\left(1 - \frac{1}{n}r\left(\frac{i}{n}\right) + O\big((n-i)^{-1-\varepsilon}\big)\right) \sim \exp\left(-\sum_{i=0}^{t-1}\frac{1}{n}r\left(\frac{i}{n}\right)\right). \qquad (2.46)$$

Applying this to time $t = t_1$ and noting that $\mathbf{P}(\overline{\mathcal{E}_{t_1}}) = o(n^{-2})$ by Theorem 4, from (2.39) we have

$$\mathbf{P}(\mathcal{F}_{t_1}) \sim \exp\left(-\sum_{t=0}^{t_1-1}\frac{1}{n}r\left(\frac{t}{n}\right)\right) \sim \left(\frac{2(n-t_1)e^{\tau}}{n}\right)^{1/2}, \qquad (2.47)$$

where the last estimate is from Lemma 1.

We may treat the factor due to i in (2.38) using a similar argument, from (2.40) onward, conditioning on $\mathcal{F}_{t_i}$. This argument is entirely restricted to the time period from t_i to t_{i+1}. In fact, similar to (2.39) and (2.42), for $i = 1, 2, 3$ we have

$$\mathbf{P}(\mathcal{F}_{t_{i+1}} \mid \mathcal{F}_{t_i}) = \mathbf{P}(\mathcal{H}_{t_{i+1}} \mid \mathcal{F}_{t_i}) + O(\mathbf{P}(\overline{\mathcal{E}_{t_{i+1}}} \mid \mathcal{F}_{t_i})) \qquad (2.48)$$

and

$$\mathbf{P}\left(\mathcal{H}_t \mid \mathcal{F}_{t_i}\right) = \Omega\left(\left(\frac{n-t}{n-t_i}\right)^{1/2}\right) \qquad (2.49)$$

for $t_i < t \leq n-1$. For $i = 1$ and $t_1 < t \leq t_2$, we have $\bar{r}(t/n)/n = 1/(2(n-t)-1)$, $U_t \geq n - t \geq n - t_2 \geq \log n > (\log n)^{\rho}$ and $\mathcal{E}_t$ implies $Y_t + Z_t \leq 2(n-t)/(\log n)^{\rho}$. Thus, similar to (2.43), we have

$$\begin{aligned}\mathbf{P}\big(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{E}_t \wedge \mathcal{F}_{t_1}\big) &= \frac{\bar{r}\,(t/n)}{n}\left(1-\frac{Y_t+2Z_t}{2(n-t)-Y_t}\right)\frac{2(n-t)-1}{2(n-t)-Y_t}\left(1+O\Big(\frac{1}{U_t}\Big)\right)\\ &= \frac{\bar{r}\,(t/n)}{n}\left(1-O\Big(\frac{Y_t+Z_t}{n-t}\Big)\right)\left(1+O\Big(\frac{Y_t}{n-t}\Big)\right)\left(1+O\Big(\frac{1}{U_t}\Big)\right)\\ &= \frac{\bar{r}\,(t/n)}{n}\big(1+O((\log n)^{-\rho})\big)\left(1+O\Big(\frac{1}{U_t}\Big)\right)\\ &= \frac{\bar{r}(t/n)}{n}\big(1+O((\log n)^{-\rho})\big)\\ &= \frac{\bar{r}(t/n)}{n}+O\big((\log n)^{-1-\rho}\big). \end{aligned} \tag{2.50}$$

Since $\mathcal{F}_{t_1}$ implies $\mathcal{E}_{t_1}$, we have $Y_{t_1}+Z_{t_1} < (n-t_1)/(\log n)^{\rho}$. Thus, by an argument similar to that leading to (2.24), we get $\mathbf{P}(\overline{\mathcal{E}_t} \mid \mathcal{F}_{t_1}) = O((\log n)^{-K})$ for all t with $t_1 < t \le t_2$ by Theorem 5. From (2.49) we then have

$$\mathbf{P}(\overline{\mathcal{E}_t} \mid \mathcal{F}_{t_1})/\mathbf{P}(\mathcal{H}_t \mid \mathcal{F}_{t_1}) = O((\log n)^{-K}((n-t_1)/(n-t_2))^{1/2}) = O((\log n)^{-K+(N-1)/2}).$$

From this and (2.50), using an argument similar to that leading to (2.45), we obtain

$$\begin{aligned}\mathbf{P}\big(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{F}_{t_1}\big) &= \mathbf{P}\big(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{E}_t \wedge \mathcal{F}_{t_1}\big) + O\big(\mathbf{P}(\overline{\mathcal{E}_t} \mid \mathcal{F}_{t_1})/\mathbf{P}(\mathcal{H}_t \mid \mathcal{F}_{t_1})\big)\\ &= \frac{\bar{r}(t/n)}{n}+O\big((\log n)^{-1-\rho}\big)+O\big((\log n)^{-K+(N-1)/2}\big).\end{aligned}$$

It follows that the first term of (2.48) when $i=1$ is

$$\mathbf{P}(\mathcal{H}_{t_2} \mid \mathcal{F}_{t_1}) = \prod_{t=t_1}^{t_2-1}\mathbf{P}\big(\overline{\mathcal{C}_t} \mid \mathcal{H}_t \wedge \mathcal{F}_{t_1}\big) \sim \prod_{t=t_1}^{t_2-1}\left(1-\frac{\bar{r}(t/n)}{n}\right),$$

which is $\prod_{t=t_1}^{t_2-1}(2(n-t)-2)/(2(n-t)-1) = \Omega((\log n)^{-(N-1)/2})$ by (2.41). By Theorem 5, the second term $O(\mathbf{P}(\overline{\mathcal{E}_{t_2}} \mid \mathcal{F}_{t_1}))$ of (2.48) when $i=1$ is $O((\log n)^{-K})$, which is $o((\log n)^{-(N-1)/2})$ since $N < 2K+1$. From (2.48) we then obtain

$$\mathbf{P}(\mathcal{F}_{t_2} \mid \mathcal{F}_{t_1}) \sim \prod_{t=t_1}^{t_2-1}\left(1-\frac{\bar{r}(t/n)}{n}\right). \tag{2.51}$$

For the factor due to $i=2$ in (2.38), we condition on $\mathcal{F}_{t_2}$ and consider the time period from t_2 to t_3. Note that $\mathcal{F}_{t_2}$ implies $Y_{t_2}+Z_{t_2} \le 2(n-t_2)/(\log n)^{\rho} = 2(\log n)^{1-\rho}$ and hence $Y_t+Z_t = O((\log n)^{1-\rho})$ for $t_2 \le t < t_3$, since Y_t+Z_t is non-increasing with t (see (3.3)). Note also that $U_t \ge n-t \ge n-t_3 > (\log n)^{-1+\rho+\delta}$ for $t_2 \le t < t_3$ since $\rho < 1$ and $n-t_3 = \lfloor(\log n)^{\delta}\rfloor$. Thus, similar to (2.50), we have

$$\begin{aligned}\mathbf{P}\big(\mathcal{C}_t \mid \mathcal{H}_t \wedge \mathcal{F}_{t_2}\big) &= \frac{\bar{r}(t/n)}{n}\big(1+O((\log n)^{-\rho-\delta+1})\big)\\ &= \frac{\bar{r}(t/n)}{n}+O((\log n)^{-\rho-2\delta+1})\end{aligned} \tag{2.52}$$

for $t_2 \le t < t_3$. Consequently,

$$\mathbf{P}(\mathcal{H}_{t_3} \,|\, \mathcal{F}_{t_2}) = \prod_{t=t_2}^{t_3-1} \mathbf{P}\big(\overline{\mathcal{C}_t} \,|\, \mathcal{H}_t \wedge \mathcal{F}_{t_2}\big) \sim \prod_{t=t_2}^{t_3-1} \left(1 - \frac{\bar{r}(t/n)}{n}\right),$$

which is $\prod_{t=t_2}^{t_3-1}(2(n-t)-2)/(2(n-t)-1) = \Omega((\log n)^{-(1-\delta)/2})$ by (2.41). By Theorem 6 the second term of (2.48) when $i = 2$ is $O((\log n)^{\delta-\rho})$ and hence is $o((\log n)^{-(1-\delta)/2})$ since $\delta < 2\rho - 1$ by (2.26). Hence,

$$\mathbf{P}(\mathcal{F}_{t_3} \,|\, \mathcal{F}_{t_2}) \sim \prod_{t=t_2}^{t_3-1} \left(1 - \frac{\bar{r}(t/n)}{n}\right). \tag{2.53}$$

For the factor due to $i = 3$ in (2.38), similar to (2.52), we have

$$\mathbf{P}\big(\mathcal{C}_t \,|\, \mathcal{H}_t \wedge \mathcal{F}_{t_3}\big) = \frac{\bar{r}(t/n)}{n} + O((\log n)^{-\rho-2\delta'+1}) \tag{2.54}$$

for $t_3 \le t < t_4$. Thus,

$$\mathbf{P}(\mathcal{H}_{t_4} \,|\, \mathcal{F}_{t_3}) = \prod_{t=t_3}^{t_4-1} \mathbf{P}\left(\overline{\mathcal{C}_t} \,|\, \mathcal{H}_t \wedge \mathcal{F}_{t_3}\right) \sim \prod_{t=t_3}^{t_4-1} \left(1 - \frac{\bar{r}(t/n)}{n}\right),$$

which is $\prod_{t=t_3}^{t_4-1}(2(n-t)-2)/(2(n-t)-1) = \Omega((\log n)^{-(\delta-\delta')/2})$ by (2.41). By Theorem 6, when $i = 3$ the second term of (2.48) is $O((\log n)^{-\rho-(\delta-\delta')+1})$, and hence is $o((\log n)^{-(\delta-\delta')/2})$ as $(\delta - \delta')/2 + \rho > 1$. Therefore,

$$\mathbf{P}(\mathcal{F}_{t_4} \,|\, \mathcal{F}_{t_3}) \sim \prod_{t=t_3}^{t_4-1} \left(1 - \frac{\bar{r}(t/n)}{n}\right). \tag{2.55}$$

The factor due to $i = 4$ in (2.38) can be computed directly. Conditioning on $\mathcal{F}_{t_4}$, we have $Y_{t_4} = Z_{t_4} = 0$ and hence $Y_t = Z_t = 0$ for $t \ge t_4$ since $Y_t + Z_t$ is non-increasing with t (see (3.3)). Thus, as $\mathcal{F}_{t_5} = \mathcal{F}_{t_4} \wedge \mathcal{H}_{t_5}$, from (2.4) we have

$$\begin{aligned}\mathbf{P}(\mathcal{F}_{t_5} \,|\, \mathcal{F}_{t_4}) &= \mathbf{P}(\mathcal{H}_{t_5} \,|\, \mathcal{F}_{t_4}) \\ &= \prod_{t=t_4}^{t_5-1} \mathbf{P}(\overline{\mathcal{C}_t} \,|\, \mathcal{H}_t \wedge \mathcal{F}_{t_4}) \\ &= \prod_{t=t_4}^{n-2} \left(1 - \frac{1}{2(n-t)-1}\right) \\ &= \prod_{t=t_4}^{n-2} \left(1 - \frac{\bar{r}(t/n)}{n}\right). \end{aligned} \tag{2.56}$$

Plugging (2.47), (2.51), (2.53), (2.55), and (2.56) into (2.38) and using (2.41) we obtain

$$\begin{aligned}\mathbf{P}(\mathcal{F}_{t_5}) &\sim \left(\frac{2(n-t_1)e^\tau}{n}\right)^{1/2} \prod_{t=t_1}^{n-2}\left(1-\frac{\bar{r}(t/n)}{n}\right)\\ &= \left(\frac{2(n-t_1)e^\tau}{n}\right)^{1/2} \prod_{t=t_1}^{n-2}\frac{2(n-t)-2}{2(n-t)-1}\\ &\sim \left(\frac{2(n-t_1)e^\tau}{n}\right)^{1/2} \frac{1}{2}\left(\frac{\pi}{n-t_1}\right)^{1/2}\\ &= \sqrt{\frac{\pi e^\tau}{2}}\Big/\sqrt{n}.\end{aligned}$$

Thus, to complete the proof of Theorem 1, it remains to show that the second term in (2.37) is negligible, that is, it is $o(1/\sqrt{n})$. In fact, we have $\mathbf{P}(\overline{\mathcal{E}_{t_1}}) = O(e^{-(n-t_1)^\zeta}) = O\big(e^{-(\log n)^{K/4}}\big) = o(n^{-2})$ by Theorem 4. By Theorem 5 and (2.47), and since $N < 2K$ by our assumption, we have

$$\begin{aligned}\mathbf{P}(\mathcal{F}_{t_1} \wedge \overline{\mathcal{E}_{t_2}}) &= \mathbf{P}(\overline{\mathcal{E}_{t_2}}|\mathcal{F}_{t_1})\mathbf{P}(\mathcal{F}_{t_1})\\ &= O\left((\log n)^{-K}\left(\frac{n-t_1}{n}\right)^{1/2}\right)\\ &= O\left((\log n)^{-K+N/2}/\sqrt{n}\right)\\ &= o(1/\sqrt{n}).\end{aligned}$$

Similarly, by Theorem 6, (2.47), and (2.51), and noting that $\delta < \rho - 1/2$ by (2.26),

$$\begin{aligned}\mathbf{P}(\mathcal{F}_{t_2} \wedge \overline{\mathcal{E}_{t_3}}) &= \mathbf{P}(\overline{\mathcal{E}_{t_3}}|\mathcal{F}_{t_2})\mathbf{P}(\mathcal{F}_{t_2}|\mathcal{F}_{t_1})\mathbf{P}(\mathcal{F}_{t_1})\\ &= O\left((\log n)^{\delta-\rho+(1-N)/2+N/2}/\sqrt{n}\right)\\ &= o(1/\sqrt{n}).\end{aligned}$$

Again, by Theorem 6, (2.47), (2.51), and (2.53), and using $\delta/2 - \delta' + \rho > 1$ from (2.26), we have

$$\begin{aligned}\mathbf{P}(\mathcal{F}_{t_3} \wedge \overline{\mathcal{E}_{t_4}}) &= \mathbf{P}(\overline{\mathcal{E}_{t_4}}|\mathcal{F}_{t_3})\mathbf{P}(\mathcal{F}_{t_3}|\mathcal{F}_{t_2})\mathbf{P}(\mathcal{F}_{t_2}|\mathcal{F}_{t_1})\mathbf{P}(\mathcal{F}_{t_1})\\ &= O\big((\log n)^{(\delta'-\delta+1-\rho)+(\delta-1)/2+(1-N)/2+N/2}/\sqrt{n}\big)\\ &= o(1/\sqrt{n}).\end{aligned}$$

The proof of Theorem 1 is complete. ∎

The approximation results and the analysis in the proof of Theorem 1 contain essentially what we need in order to prove Theorem 2.

Proof of Theorem 2. Denote $Q_t = (n-t-Y_t-Z_t)/\left(\binom{U_t}{2} - Z_t\right)$. Then $Q_t = O((n-t)^{-1})$ for all t and by (2.4) the probability of creating a cycle at time t is equal to Q_t. Let $\mathcal{E}_t$ be as in the proof of Theorem 1. For t up to t_2 we will use the following to estimate $\mathbf{E}(Q_t)$:

$$\begin{aligned}\mathbf{E}(Q_t) &= \mathbf{E}(Q_t|\mathcal{E}_t)\mathbf{P}(\mathcal{E}_t) + \mathbf{E}(Q_t|\overline{\mathcal{E}_t})\mathbf{P}(\overline{\mathcal{E}_t}) \\ &= \mathbf{E}(Q_t|\mathcal{E}_t)\mathbf{P}(\mathcal{E}_t) + O((n-t)^{-1}\mathbf{P}(\overline{\mathcal{E}_t})). \end{aligned} \tag{2.57}$$

Taking $\sigma(n) = t_1$ in Theorem 4, we have $\mathbf{P}(\mathcal{E}_t) = 1 - o(n^{-2})$ for $t \le t_1$. By using arguments similar to that leading to (2.43), one can show that

$$\mathbf{E}(Q_t|\mathcal{E}_t) = \frac{r(t/n)}{n} + O((n-t)^{-1-\varepsilon})$$

for $t \le t_1$. Thus, by (2.57) and noting that $r(t/n)/n = O((n-t)^{-1})$, for $t \le t_1$ we have

$$\begin{aligned}\mathbf{E}(Q_t) &= \left(\frac{r(t/n)}{n} + O((n-t)^{-1-\varepsilon})\right)(1 - o(n^{-2})) + O((n-t)^{-1}n^{-2}) \\ &= \frac{r(t/n)}{n} + O((n-t)^{-1-\varepsilon}). \end{aligned} \tag{2.58}$$

Note that (2.21) and (2.22) imply (2.24), and in particular $Y_{t_1} + Z_{t_1} < (n - t_1)/(\log n)^\rho$ with probability $1-o(n^{-2})$. Hence, for $t_1 < t \le t_2$, $\mathbf{P}(\mathcal{E}_t) = 1-O((\log n)^{-K})$ by Theorem 5. Using an argument similar to that leading to (2.50), we get

$$\mathbf{E}(Q_t|\mathcal{E}_t) = \frac{\bar{r}(t/n)}{n}\big(1 + O((\log n)^{-\rho})\big)$$

and hence (2.57) gives

$$\begin{aligned}\mathbf{E}(Q_t) &= \frac{\bar{r}(t/n)}{n}\big(1 + O((\log n)^{-\rho})\big)\big(1 - O((\log n)^{-K})\big) + O((n-t)^{-1}(\log n)^{-K}) \\ &= \frac{\bar{r}(t/n)}{n}\big(1 + O((\log n)^{-\rho})\big) \end{aligned} \tag{2.59}$$

for $t_1 < t \le t_2$.

For $t_2 < t \le t_4$ we condition on the event $\mathcal{A}_t$ that $Y_t + Z_t \le C^*(\log n)^{1-\rho}$ for a sufficiently large C^*. Since $\mathbf{P}(\mathcal{E}_{t_2}) = \mathbf{P}(Y_{t_2} + Z_{t_2} \le 2(\log n)^{1-\rho}) = 1 - O((\log n)^{-K})$, from Theorem 6 it follows that $\mathbf{P}(\mathcal{A}_t) = 1 - O((\log n)^{-K})$. Similar to the proof of (2.52), for $t_2 < t \le t_3$ we have

$$\mathbf{E}(Q_t|\mathcal{A}_t) = \frac{\bar{r}(t/n)}{n}\left(1 + O((\log n)^{-\rho-\delta+1})\right).$$

Based on this and using an expression similar to (2.57) we have

$$\begin{aligned}\mathbf{E}(Q_t) &= \mathbf{E}(Q_t|\mathcal{A}_t)\mathbf{P}(\mathcal{A}_t) + O((n-t)^{-1}\mathbf{P}(\overline{\mathcal{A}_t})) \\ &= \frac{\bar{r}(t/n)}{n}\left(1 + O((\log n)^{-\rho-\delta+1})\right)\left(1 - O((\log n)^{-K})\right) + O((n-t)^{-1}(\log n)^{-K}) \\ &= \frac{\bar{r}(t/n)}{n}\left(1 + O((\log n)^{-\rho-\delta+1})\right). \end{aligned} \tag{2.60}$$

In a similar fashion, for $t_3 < t \le t_4$, we obtain

$$\mathbf{E}(Q_t) = \frac{\bar{r}(t/n)}{n}\big(1 + O((\log n)^{-\rho-\delta'+1})\big). \tag{2.61}$$

For $t_4 \le t < n$, we condition on the event $\mathcal{A}_t$ that $Y_t = Z_t = 0$, which holds with probability $1 - O((\log n)^{\delta'-\delta+1-\rho})$ by (2.28) and (2.29). Thus, since $\mathbf{E}(Q_t|\mathcal{A}_t) = \bar{r}(t/n)/n$, analogous to (2.60) we have

$$\mathbf{E}(Q_t) = \frac{\bar{r}(t/n)}{n}\big(1 + O((\log n)^{\delta'-\delta+1-\rho})\big). \tag{2.62}$$

Using (2.58)–(2.62) and noting (2.26), we have

$$\begin{aligned}
\mathbf{E}(X_n) &= \sum_{t=0}^{n-1} \mathbf{P}(\text{a cycle is formed at time } t)\\
&= \sum_{t=0}^{n-1} \mathbf{E}(Q_t)\\
&= \sum_{t=0}^{t_1} \frac{r(t/n)}{n} + \big(1 + O((\log n)^{-\rho-\delta'+1})\big) \sum_{t=t_1+1}^{n-1} \frac{\bar{r}(t/n)}{n} + O\left(\sum_{t=0}^{t_1} (n-t)^{-1-\varepsilon}\right)\\
&= \sum_{t=0}^{t_1} \frac{r(t/n)}{n} + \big(1 + O((\log n)^{-\rho-\delta'+1})\big) \sum_{t=t_1+1}^{n-1} \frac{\bar{r}(t/n)}{n} + O((\log n)^{-N\varepsilon}).
\end{aligned} \tag{2.63}$$

From Lemma 1,

$$\sum_{t=0}^{t_1} \frac{r(t/n)}{n} = -\frac{1}{2}\log(2(n-t_1)/n) - \frac{\tau}{2} + O((\log n)^{-1}).$$

Also, noting that the harmonic number $H(k) = \sum_{j=1}^{k} 1/k = \log k + \gamma + O(1/k)$, we have

$$\begin{aligned}
\sum_{t=t_1+1}^{n-1} \frac{\bar{r}(t/n)}{n} &= \sum_{t=1}^{n-t_1-1} \frac{1}{2t-1}\\
&= H(2(n-t_1-1)) - \frac{1}{2}H(n-t_1-1)\\
&= \log(2(n-t_1)) + \gamma - \frac{1}{2}(\log(n-t_1) + \gamma) + O\left(\frac{1}{n-t_1}\right)\\
&= \frac{1}{2}\log(2(n-t_1)) + \frac{1}{2}(\gamma + \log 2) + O\left(\frac{1}{(\log n)^N}\right).
\end{aligned}$$

Plugging the two sums above into (2.63) and noting that $N\varepsilon = N\rho/4 > \rho > \rho + \delta' - 1$, we obtain $\mathbf{E}(X_n) = \frac{1}{2}(\log n + \log 2 + \gamma - \tau) + O((\log n)^{-\rho-\delta'+1}\log\log n)$, where the $\log\log n$ factor is due to $\log(n - t_1)$ by noting (2.18). For the specific choice of ρ, δ and δ' after (2.26), we have $\rho + \delta' > 12/11$ and hence the error term is $O((\log n)^{-1/11})$, as required in Theorem 2. ∎

Proof of Theorem 3. We will use the notation in the proof of Theorem 1. Recall that $\mathcal{H}_t = \bigwedge_{i=0}^{t-1} \overline{\mathcal{C}_i}$ is the event that G_t contains no cycles. We have

$$\begin{aligned}\mathbf{E}(T_n) &= \sum_{t=1}^{n} \mathbf{P}(T_n \geq t) \\ &= \sum_{t=0}^{n-1} \mathbf{P}(\mathcal{H}_t) \\ &= \sum_{t=0}^{t_1} \mathbf{P}(\mathcal{H}_t) + \sum_{t=t_1+1}^{n-1} \mathbf{P}(\mathcal{H}_t). \end{aligned} \tag{2.64}$$

By the definition of t_1 and the fact that $\mathcal{H}_t \subseteq \mathcal{H}_{t-1}$ for any t, using (2.46) (for $t = t_1$) and then Lemma 1 (as in (2.47)) we have

$$\begin{aligned}\sum_{t=t_1+1}^{n-1} \mathbf{P}(\mathcal{H}_t) &\leq (\log n)^N \mathbf{P}(\mathcal{H}_{t_1}) \\ &\sim (\log n)^N \exp\left(-\sum_{t=0}^{t_1-1} \frac{1}{n} r\left(\frac{t}{n}\right)\right) \\ &\sim (\log n)^N \left(\frac{2(n-t_1)e^\tau}{n}\right)^{1/2} \\ &\sim \left(\frac{2(\log n)^{3N} e^\tau}{n}\right)^{1/2},\end{aligned}$$

which converges to 0 as $n \to \infty$. Thus, the second term of (2.64) is negligible.

Now let us deal with the first term of (2.64). Using (2.46) and the computation of $J_1(t)$ and $J_2(t)$ (the second line of (2.31) and (2.34)) in the proof of Lemma 1, we have

$$\begin{aligned}\sum_{t=0}^{t_1} \mathbf{P}(\mathcal{H}_t) &\sim \sum_{t=0}^{t_1} \exp\left(-\sum_{i=0}^{t-1} \frac{1}{n} r\left(\frac{i}{n}\right)\right) \\ &\sim \sum_{t=0}^{t_1} \exp(-J_1(t) - J_2(t)) \\ &\sim \sum_{t=0}^{t_1} \{y(t/n)(1-\log y(t/n))\}^{1/2} \exp(-J_2(t)) \\ &\sim \sum_{t=0}^{t_1} \{y(t/n)(1-\log y(t/n))\}^{1/2} \exp\left\{\frac{1}{2}\int_0^{-\log y(t/n)} \frac{\log\left(\frac{(1-a-\log y(t/n))(1+a)}{1-\log y(t/n)}\right)}{ae^a} da\right\} \\ &\sim \int_0^{t_1/n} \{y(x)(1-\log y(x))\}^{1/2} \exp\left\{\frac{1}{2}\int_0^{-\log y(x)} \frac{\log\left(\frac{(1-a-\log y(x))(1+a)}{1-\log y(x)}\right)}{ae^a} da\right\} dx.\end{aligned}$$

Now (2.6), (2.8), and (2.11) together imply

$$\frac{dy}{dx} = -\frac{2}{1-\log y(x)}.$$

Using this and taking $y = y(x)$ as a new variable in the integral above, we get

$$\sum_{t=0}^{t_1} \mathbf{P}(\mathcal{H}_t) \sim -\frac{1}{2}\int_1^{y(t_1/n)} y^{1/2}(1-\log y)^{3/2} \exp\left\{ \frac{1}{2}\int_0^{-\log y} \frac{\log\left(\frac{(1-a-\log y)(1+a)}{1-\log y}\right)}{ae^a} da \right\} dy$$
$$= \frac{1}{2}\int_0^{-\log y(t_1/n)} (1+\eta)^{3/2} \exp\left\{ -\frac{3\eta}{2} + \frac{1}{2}\int_0^{\eta} \frac{\log\left(\frac{(1-a+\eta)(1+a)}{1+\eta}\right)}{ae^a} da \right\} d\eta$$
$$\sim \frac{1}{2}\int_0^{\infty} (1+\eta)^{3/2} \exp\left\{ -\frac{3\eta}{2} + \frac{1}{2}\int_0^{\eta} \frac{\log\left(\frac{(1-a+\eta)(1+a)}{1+\eta}\right)}{ae^a} da \right\} d\eta$$
$$\approx 0.829.$$

Here in the third last step we set $\eta = -\log y$, and in the last step we used numerical integration. From this and (2.64) Theorem 3 follows immediately. ∎

In the proof above we actually computed the asymptotic probability $\mathbf{P}(\mathcal{H}_t)$ that G_t contains no cycle, for $t \le t_1$. In fact, for $0 \le x \le t_1/n$, we have

$$\mathbf{P}(\mathcal{H}_t) \sim \exp\left\{ I(x) - \frac{\eta}{2} \right\} \sqrt{1+\eta},$$

where $\eta = -\log y(x)$ and

$$I(x) = \frac{1}{2}\int_0^{\eta} \frac{\log\left(\frac{(1-a+\eta)(1+a)}{1+\eta}\right)}{ae^a} da.$$

The graph of this function (as function of x) is shown in Fig. 1 in Section 1. It is interesting that this graph is almost a circular shape.

3. APPROXIMATION OF *Y* AND *Z*

The variables Y and Z are obviously crucial, so we examine them closely, for t up until very nearly the end of the process.

Lemma 2. *For $0 \le t \le n-40$, we have*

$$\left| \mathbf{E}(Y_{t+1} - Y_t \mid G_t) - f\left(\frac{t}{n}, \frac{Y_t}{n}\right) \right| \le \frac{5}{n-t} \tag{3.1}$$

$$\left| \mathbf{E}(Z_{t+1} - Z_t \mid G_t) - g\left(\frac{t}{n}, \frac{Y_t}{n}, \frac{Z_t}{n}\right) \right| \le \frac{20}{n-t}, \tag{3.2}$$

where f and g are the functions defined in (2.6) and (2.7), respectively. Moreover, for any $t \ge 0$,

$$Y_t + Z_t - 2 \le Y_{t+1} + Z_{t+1} \le Y_t + Z_t. \tag{3.3}$$

Proof. We have

$$(Y_{t+1} - Y_t, Z_{t+1} - Z_t) \in \{(0,0),(0,-1),(0,-2),(-1,0),(-1,-1),(-2,1)\}$$

and the number of sites contributing to these changes are $4\binom{n-t-Y_t-Z_t}{2} + (n-t-Y_t-Z_t)$, $4Z_t(n-t-Y_t-Z_t)$, $4\binom{Z_t}{2}$, $2Y_t(n-t-Y_t-Z_t)$, $2Y_tZ_t$, and $\binom{Y_t}{2}$, respectively. In particular, $Y_t + Z_t$ is non-increasing with t and differs by at most 2 from $Y_{t+1} + Z_{t+1}$. Hence, (3.3) is true. Also, Y_t is non-increasing with t and differs by at most 2 from Y_{t+1}. From the above it can be checked that the expected change of the number of isolates at time t is $-(2Y_t/U_t) - \left(2Y_tZ_t/U_t\left(\binom{U_t}{2} - Z_t\right)\right)$. From (2.3) we have $\binom{U_t}{2} - Z_t = U_t^2/2 - (n-t-Y_t/2) - Z_t \geq U_t^2/2 - (n-t-Y_t/2) - (n-t-Y_t) \geq U_t(U_t-2)/2 \geq (n-t)(n-t-2)/2$. Using this and (2.3) one can show that the second term is bounded above in absolute value by $5/(n-t)$ (assuming $n-t \geq 10$), while the first term is $f(t/n, Y_t/n)$ by (2.2). So (3.1) is proved. In each step of the process, Z_t can remain unchanged, increase by 1, or decrease by 1 or 2, and the expected change of Z_t is $\left(\binom{Y_t}{2} - 2Z_t(U_t-2)\right) / \left(\binom{U_t}{2} - Z_t\right)$. From this and by a similar estimation we get (3.2) for $n - t \geq 40$. ∎

3.1. Phase 1: $t \leq t_1 = n - \lceil(\log n)^N\rceil$

The approach taken is to use large deviation inequalities as in [9] to show that the variables approximately satisfy differential equations. The solutions of these differential equations possess a stability that is used to show that the variables continue to concentrate near these solutions far past the point reached by the argument for this random process in [5,6].

We begin with some preliminary observations. Recall from (2.6) and (2.7) the functions $f(x,y) = -2y/(2-2x-y)$ and $g(x,y,z) = (y^2 - 4(2-2x-y)z)/(2-2x-y)^2$ in the differential equations (2.8) and (2.9). As before $y(x)$ and $z(x)$ denote the solutions of these equations. Then it is easy to see that

$$|f(x,y)| \leq 2, \ \ |g(x,y,z)| \leq 5, \tag{3.4}$$

provided $y, z \in [0, 1-x]$. Setting $p(x) = f(x, y(x))$ and $q(x) = g(x, y(x), z(x))$, then from (2.14) and (2.16) and estimating dy/dx and dz/dx using (3.4), we have

$$|p(x)| \leq 2, \ \ |q(x)| \leq 5. \tag{3.5}$$

Moreover, it is easily checked that $|dp/dx| \leq 4/(1-x)$ and $|dq/dx| \leq 56/(1-x)$ for $0 \leq x < 1$. Hence, by the Mean Value Theorem we have

$$|p(x_1) - p(x_2)| \leq \frac{4}{1-x_2}|x_1 - x_2| \tag{3.6}$$

$$|q(x_1) - q(x_2)| \leq \frac{56}{1-x_2}|x_1 - x_2|, \tag{3.7}$$

provided that $0 \leq x_1 \leq x_2 < 1$.

Proof of Theorem 4. The proof uses the main idea in the proof of [9, Theorem 5.1]. But we need more accurate estimations and judicious choices of the step lengths when bounding large deviations, and the error in the approximation needs to decrease during the process.

We prove (2.21) first. Let $0 = k_0 < k_1 < \cdots < k_\ell = k_\ell(n)$ be an integer time sequence such that $n - k_\ell \to \infty$ when $n \to \infty$. For $i = 0, 1, \ldots, \ell - 1$, denote $w_i = k_{i+1} - k_i$ and let $\alpha_i > 0$ be real numbers. At the moment we assume that $\{k_i\}$ and $\{\alpha_i\}$ are any sequences such that $\alpha_i \le w_i \le (n - k_i)/4$ and $w_i < n - k_{i+1}$ for each i. Later we will choose k_i's and α_i's in a particular way so that these conditions are satisfied. Denote

$$h(n,i) = \frac{(n-k_i)w_i}{(n-k_{i+1}-w_i)^2}.$$

Since $w_i < n - k_{i+1}$ by our assumption, we have $w_i < 3(n - k_{i+1})$ and using this one can check that

$$\frac{1}{n-k_{i+1}} \le \frac{w_i}{n-k_{i+1}} < h(n,i). \tag{3.8}$$

For each t with $0 \le t < w_i$, by the Mean Value Theorem there exist θ between k_i/n and $(k_i + t)/n$ and ψ between Y_{k_i+t}/n and Y_{k_i}/n such that

$$\begin{aligned}\left| f\left(\frac{k_i+t}{n}, \frac{Y_{k_i+t}}{n}\right) - f\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}\right)\right| &= \frac{4}{(2(1-\theta)-\psi)^2}\left\{\psi \cdot \frac{t}{n} + (1-\theta)\cdot \frac{|Y_{k_i+t} - Y_{k_i}|}{n}\right\} \\ &\le \frac{4}{\left(2\left(1-\frac{k_i+t}{n}\right) - \frac{Y_{k_i}}{n}\right)^2}\left\{\frac{Y_{k_i}}{n}\cdot\frac{t}{n} + \left(1 - \frac{k_i}{n}\right)\cdot \frac{2t}{n}\right\} \\ &\le \frac{4}{\left(2\left(1-\frac{k_{i+1}}{n}\right) - \frac{n-k_i}{n}\right)^2}\left\{\frac{n-k_i}{n}\cdot\frac{w_i}{n} + \frac{n-k_i}{n}\cdot \frac{2w_i}{n}\right\} \\ &= 12h(n,i). \end{aligned} \tag{3.9}$$

Here we used the facts that $|Y_{k_i+t} - Y_{k_i}| \le 2t$ and $Y_{k_i} \le n - k_i$. From (3.1), (3.9), and (3.8) we then have

$$\begin{aligned}\mathbf{E}(Y_{k_i+t+1} - Y_{k_i+t} \mid G_{k_i+t}) &\le f\left(\frac{k_i+t}{n}, \frac{Y_{k_i+t}}{n}\right) + \frac{5}{n-k_i-t} \\ &\le f\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}\right) + 12h(n,i) + \frac{5}{n-k_{i+1}} \\ &\le f\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}\right) + 17h(n,i). \end{aligned} \tag{3.10}$$

For $t = 0, 1, \ldots, w_i$, set

$$X_t = Y_{k_i+t} - Y_{k_i} - tf(k_i/n, Y_{k_i}/n) - 17t\,h(n,i).$$

Then (3.10) implies that $X_0, X_1, \ldots, X_{w_i}$ is a supermartingale (with respect to the sequence of σ-algebras generated by $\{(G_0, G_1, \ldots, G_t)\}_{t\ge 0}$) with $X_0 = 0$. As $Y_{k_i} \le n - k_i$, from (3.4) we have $|f(k_i/n, Y_{k_i}/n)| \le 2$ and thus

$$|X_{t+1} - X_t| \le 4 + 17h(n,i).$$

Therefore, from [9, Lemma 4.2] and since $\alpha_i \le w_i$, it follows that, conditioned upon G_{k_i},

$$\begin{aligned}\left| Y_{k_{i+1}} - Y_{k_i} - w_i f\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}\right)\right| &< 17 w_i h(n,i) + (4 + 17h(n,i))\sqrt{2w_i\alpha_i} \\ &< 51 w_i h(n,i) + 4\sqrt{2w_i\alpha_i} \qquad (3.11)\end{aligned}$$

holds with probability at least $1 - 2e^{-\alpha_i}$.

In the following we will show by induction that (2.21) holds at our chosen times k_i with the desired probability. Set

$$\begin{aligned} A_1 &= Y_{k_i} - ny\left(\frac{k_i}{n}\right) \\ A_2 &= Y_{k_{i+1}} - Y_{k_i} - w_i f\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}\right) \\ A_3 &= w_i p\left(\frac{k_i}{n}\right) + ny\left(\frac{k_i}{n}\right) - ny\left(\frac{k_{i+1}}{n}\right) \\ A_4 &= w_i f\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}\right) - w_i p\left(\frac{k_i}{n}\right), \end{aligned}$$

so that $Y_{k_{i+1}} - ny(k_{i+1}/n) = A_1 + A_2 + A_3 + A_4$. By the Mean Value Theorem and (3.6), (3.8), for some θ with $k_i < \theta < k_{i+1}$, we have

$$\begin{aligned} |A_3| &= w_i \left| p\left(\frac{k_i}{n}\right) - p\left(\frac{\theta}{n}\right)\right| \\ &\le \frac{4w_i}{1 - \frac{\theta}{n}} \left|\frac{\theta}{n} - \frac{k_i}{n}\right| \\ &\le \frac{4w_i^2}{n - k_{i+1}} \\ &\le 4w_i h(n,i). \qquad (3.12)\end{aligned}$$

Also by the Mean Value Theorem, there exists some ψ between Y_{k_i}/n and $y(k_i/n)$ (hence, $0 \le \psi < 1 - k_i/n$ by (2.3) and (2.14)) such that

$$\begin{aligned} A_4 &= w_i \cdot \frac{A_1}{n} \cdot \frac{\partial f}{\partial y}\left(\frac{k_i}{n}, \psi\right) \\ &= -\frac{4w_i\left(1 - \frac{k_i}{n}\right)}{n\left(2\left(1 - \frac{k_i}{n}\right) - \psi\right)^2} A_1. \end{aligned}$$

Since we assume $w_i \le (n - k_i)/4$, it is clear that the coefficient of A_1 here is between -1 and 0, and hence,

$$\begin{aligned} |A_1 + A_4| &= \left(1 - \frac{4w_i\left(1 - \frac{k_i}{n}\right)}{n\left(2\left(1 - \frac{k_i}{n}\right) - \psi\right)^2}\right)|A_1| \\ &\le \frac{n - k_{i+1}}{n - k_i}|A_1|. \qquad (3.13)\end{aligned}$$

Putting (3.11), (3.12), and (3.13) together, with probability at least $1 - 2e^{-\alpha_i}$, we have conditional upon any graph G_{k_i},

$$|Y_{k_{i+1}} - ny(k_{i+1}/n)| \leq \frac{n - k_{i+1}}{n - k_i}|A_1| + 55w_i h(n, i) + 4\sqrt{2w_i\alpha_i}. \tag{3.14}$$

Note that this is true for each $i = 0, 1, \ldots, \ell - 1$.

For each i up to ℓ, define

$$B_i = \sum_{j=1}^{i} \frac{n - k_i}{n - k_j} b_{j-1} \tag{3.15}$$

with

$$b_j = 55w_j h(n,j) + 4\sqrt{2w_j\alpha_j},$$

and note that $Y_0 - ny(0) = 0$. Consider conditioning for successive i on the event

$$|Y_{k_i} - ny(k_i/n)| \leq B_i, \tag{3.16}$$

with no conditioning in the case $i = 0$. Then, for each i, we obtain (3.14) with conditional probability $1-2e^{-\alpha_i}$. Thus, by induction and noting that $((n-k_i)/(n-k_{i-1}))B_{i-1}+b_{i-1} = B_i$, for $0 \leq i \leq \ell$ we have

$$\mathbf{P}(|Y_{k_i} - ny(k_i/n)| \leq B_i) \geq \prod_{j=0}^{i-1} (1 - 2e^{-\alpha_j}) \geq 1 - 2\sum_{j=0}^{i-1} e^{-\alpha_j}. \tag{3.17}$$

Now we choose k_i's in the following way. Let β be a real number with $0 < \beta \leq 1/2$ and denote $F(t) = t - \lceil (n-t)^\beta \rceil$ for $0 \leq t < n$. Set $k_0 = 0$. Inductively suppose $k_0, \ldots, k_i$ have been chosen for some $i \geq 0$. If $F(\sigma(n)) < k_i$ (that is, $\sigma(n) - k_i < \lceil (n - \sigma(n))^\beta \rceil$), then stop (this determines ℓ); otherwise, since F is strictly increasing with t, there exists a smallest integer k_{i+1} such that $F(k_{i+1}) \geq k_i$. In this way we define a sequence $0 = k_0, k_1, \ldots, k_\ell$ $(\leq \sigma(n))$, which is increasing since $F(k_i) < k_i$ implies $k_i < k_{i+1}$. Also, by definition we have $w_i = k_{i+1} - k_i \geq \lceil (n - k_{i+1})^\beta \rceil \geq (n - k_{i+1})^\beta$ and $F(k_{i+1} - 1) < k_i$, which implies $w_i < 1 + \lceil (n - k_{i+1} + 1)^\beta \rceil < 2(n - k_{i+1})^\beta$ as we assume $n - \sigma(n) \to \infty$. Hence, for $0 \leq i \leq \ell - 1$, we have

$$(n - k_{i+1})^\beta \leq w_i < 2(n - k_{i+1})^\beta, \tag{3.18}$$

which implies in particular that the requirements for the k_i's set at the beginning of the proof are satisfied. Moreover, for large n we have

$$w_i h(n, i) \leq \frac{6w_i^2}{n - k_{i+1}} < 24(n - k_{i+1})^{2\beta-1}. \tag{3.19}$$

Now for any ε with $0 < \varepsilon < 1/4$, choose β such that $0 < 2\varepsilon < \beta \leq 1/2$. With this choice of β, define k_i's as above, and define

$$\alpha_i = (n - k_{i+1})^{\beta-2\varepsilon} \tag{3.20}$$

for each $0 \le i \le \ell - 1$, so that $\alpha_i \le w_i$ is satisfied. Then (3.18) and (3.19) imply that $w_i h(n,i) < 24\sqrt{w_i\alpha_i}$, and hence, $b_i < C\sqrt{w_i\alpha_i}/2 \le C(n-k_{i+1})^{\beta-\varepsilon}$, $0 \le i \le \ell - 1$, where C is an absolute constant (e.g., $C = 1340$). Thus, by (3.18) we have for $0 \le i \le \ell$,

$$\begin{aligned}
B_i &\le C(n-k_i)\sum_{j=1}^{i}\frac{(n-k_j)^\beta}{(n-k_j)^{1+\varepsilon}}\\
&\le C(n-k_i)\sum_{j=1}^{i}\frac{w_{j-1}}{(n-k_j)^{1+\varepsilon}}\\
&= Cn^{1-\varepsilon}\left(1-\frac{k_i}{n}\right)\sum_{j=1}^{i}\frac{\frac{k_j}{n}-\frac{k_{j-1}}{n}}{(1-\frac{k_j}{n})^{1+\varepsilon}}.
\end{aligned}$$

The summation in this expression can be viewed as an upper Riemann sum of the increasing and convex function $1/(1-x)^{1+\varepsilon}$ on the interval $[0, k_i/n]$. So it can be approximated by the integral $\int_0^{k_i/n} dx/(1-x)^{1+\varepsilon}$. The error for this approximation is at most as large as this integral itself. To see this it suffices to show that

$$\frac{1}{\left(1-\frac{k_j}{n}\right)^{1+\varepsilon}} - \frac{1}{\left(1-\frac{k_{j-1}}{n}\right)^{1+\varepsilon}} \le \frac{1}{\left(1-\frac{k_{j-1}}{n}\right)^{1+\varepsilon}},$$

that is,

$$\left(\frac{n-k_{j-1}}{n-k_j}\right)^{1+\varepsilon} \le 2.$$

But this is guaranteed for large n since (3.18) implies

$$\frac{n-k_{j-1}}{n-k_j} < 1 + \frac{2}{(n-k_j)^{1-\beta}} \le 1 + \frac{2}{(n-\sigma(n))^{1-\beta}}$$

and since $n - \sigma(n) \to \infty$. Therefore, for $0 \le i \le \ell$, we have

$$B_i \le 2Cn^{1-\varepsilon}\left(1-\frac{k_i}{n}\right)\int_0^{k_i/n}\frac{dx}{(1-x)^{1+\varepsilon}} < \frac{2C}{\varepsilon}(n-k_i)^{1-\varepsilon}. \tag{3.21}$$

Now we estimate the bound in (3.17). By (3.18) we have $n-k_i \ge (n-k_{i+1})^\beta + n - k_{i+1} > n - k_{i+1} + 1$ for n sufficiently large, since $k_i \le \sigma(n)$ implies $n - k_i \to \infty$. Thus, $\alpha_{i-1} > \alpha_i$ for $0 \le i \le \ell - 1$. Therefore, noting that $\alpha_{\ell-1} = (n-k_\ell)^{\beta-2\varepsilon} \ge (n-\sigma(n))^{\beta-2\varepsilon}$, the right-hand side of (3.17) is $1 - O(e^{-\alpha_{\ell-1}}) = 1 - O(e^{-(n-\sigma(n))^\zeta})$ for $\zeta = \beta - 2\varepsilon > 0$ by (3.20).

We have now proved that $Y_{k_i} = ny(k_i/n) + 2C(n-k_i)^{1-\varepsilon}/\varepsilon$ holds with probability at least $1 - O(e^{-(n-\sigma(n))^\zeta})$, for each $i = 0, 1, \ldots, \ell$. This can easily be extended to Y_t for $0 \le t \le \sigma(n)$. In fact, for $0 \le t < k_\ell$ there exists a unique i with $0 \le i \le \ell - 1$ such that $k_i \le t < k_{i+1}$. We have $|Y_t - Y_{k_i}| \le 2(t-k_i)$ and $|y(t/n) - y(k_i/n)| \le 2(t-k_i)/n$ (by the first inequality in (3.5) and the Mean Value Theorem) deterministically. So with probability at least $1 - O(e^{-(n-\sigma(n))^\zeta})$ we have

$$
\begin{aligned}
|Y_t - ny(t/n)| &\le |Y_t - Y_{k_i}| + |Y_{k_i} - ny(k_i/n)| + n|y(t/n) - y(k_i/n)| \\
&\le B_i + 4(t - k_i) \\
&\le B_i + 4w_i \\
&\le B_i + 8(n - k_{i+1})^\beta \\
&\le 2C(n-k_i)^{1-\varepsilon}/\varepsilon + 8(n - k_{i+1})^{1-\varepsilon} \\
&< (6C/\varepsilon + 8)(n - k_{i+1})^{1-\varepsilon} \\
&< (6C/\varepsilon + 8)(n - t)^{1-\varepsilon}.
\end{aligned}
$$

Here we used (3.18) and its consequence $n - k_i \le 3(n - k_{i+1})$, as well as the fact $\beta \le 1 - \varepsilon$, which follows from $0 < 2\varepsilon < \beta \le 1/2$. For $k_\ell \le t \le \sigma(n)$, by a similar argument we have

$$
\begin{aligned}
|Y_t - ny(t/n)| &\le B_\ell + 4(t - k_\ell) \\
&\le B_\ell + 4(\sigma(n) - k_\ell) \\
&< 2C(n - k_\ell)^{1-\varepsilon}/\varepsilon + 8(n - \sigma(n))^\beta \\
&< (4C/\varepsilon + 8)(n - \sigma(n))^{1-\varepsilon} \\
&\le (4C/\varepsilon + 8)(n - t)^{1-\varepsilon}
\end{aligned}
$$

with probability at least $1 - O(e^{-(n-\sigma(n))^\zeta})$. Here we used the fact $\sigma(n) - k_\ell < \lceil (n - \sigma(n))^\beta \rceil$ (which follows from the definition of k_ℓ) and its consequence $n - k_\ell < 2(n - \sigma(n))$. This completes the proof of (2.21), in which we may choose C_0 to be any number no less than $6C/\varepsilon + 8$.

The proof of (2.22) can be done in a similar manner. For Z_t we will be conditioning on the analogue of (3.16) for Z_{k_i}, but also on the concentration given in (2.21) already established for Y_{k_i} with high probability. This is because the function g involves y (note that $g(x,y,z) = (f(x,y))^2/4 - 4z/(2 - 2x - y)$). Let k_i's and α_i's be as above. For $0 \le t < w_i$, using (3.9) and by an argument similar to the one used in its proof, one can check that

$$
\left| g\left(\frac{k_i + t}{n}, \frac{Y_{k_i+t}}{n}, \frac{Z_{k_i+t}}{n}\right) - g\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}, \frac{Z_{k_i}}{n}\right) \right| \le 44h(n,i).
$$

This together with (3.2) and (3.8) then gives

$$
\mathbf{E}(Z_{k_i+t+1} - Z_{k_i+t} \mid G_{k_i+t}) \le g\left(\frac{k_i}{n}, \frac{Y_{k_i}}{n}, \frac{Z_{k_i}}{n}\right) + 64h(n,i). \tag{3.22}
$$

Define A_1, A_2, A_3, and A_4 in a way similar to the previous (replacing $f(k_i/n, Y_{k_i}/n)$, $y(k_i/n)$, and $p(k_i/n)$ by $g(k_i/n, Y_{k_i}/n, Z_{k_i}/n)$, $z(k_i/n)$, and $q(k_i/n)$, respectively). By using a supermartingale and (3.22) we have $|A_2| < 192 w_i h(n,i) + 7\sqrt{2w_i\alpha_i}$ with probability at least $1 - 2e^{-\alpha_i}$, and by using (3.7) we get $|A_3| \le 56 w_i h(n,i)$. By the Mean Value Theorem (and using (2.16)) and (3.21) one can check that

$$
\begin{aligned}
|A_1 + A_4| &\le \frac{n - k_{i+1}}{n - k_i}|A_1| + \frac{6w_i}{n - k_i}|Y_{k_i} - ny(k_i/n)| \\
&\le \frac{n - k_{i+1}}{n - k_i}|A_1| + \frac{12C}{\varepsilon} w_i (n - k_i)^{-\varepsilon},
\end{aligned} \tag{3.23}
$$

where C is the absolute constant defined right after (3.20).

Here we are assuming that (2.21) holds for $t = k_i$, which we may do by imposing this extra restriction on the graph G_{k_i} which is being conditioned upon in the inductive argument for Z_t. With the particular choices of w_i and α_i as before, we have $w_i(n-k_i)^{-\varepsilon} < 2(n-k_{i+1})^{\beta}(n-k_i)^{-\varepsilon} < 2(n-k_{i+1})^{\beta-\varepsilon} \le 2\sqrt{w_i\alpha_i}$, and hence, the extra term $12Cw_i(n-k_i)^{-\varepsilon}/\varepsilon$ in (3.23) does not cause any extra cost in our proof. So, conditioning on (2.21) holding for $t = k_i$, we have

$$|Z_{k_{i+1}} - nz(k_{i+1}/n)| \le \frac{n-k_{i+1}}{n-k_i}|Z_{k_i} - nz(k_i/n)| + C_1\sqrt{w_i\alpha_i}/2$$

with probability at least $1 - 2e^{-\alpha_i}$, where $C_1 = 48C/\varepsilon + 6000$ for example.

Now define B_i as in (3.15) but with $b_j = C_1\sqrt{w_j\alpha_j}/2$ ($\le C_1(n-k_{j+1})^{\beta-\varepsilon}$). Condition for successive i on the event that $|Z_{k_i} - nz(k_i/n)| \le B_i$ and that (2.21) holds for $t = k_i$. The latter condition excludes a set of probability $O(e^{-(n-\sigma(n))^{\zeta}})$, using the result for Y. Then in place of (3.17) we have

$$\begin{aligned}
\mathbf{P}(|Z_{k_i} - nz(k_i/n)| \le B_i) &\ge \prod_{j=0}^{i-1} \left(1 - 2e^{-\alpha_j} - O(e^{-(n-\sigma(n))^{\zeta}})\right) \\
&= \prod_{j=0}^{i-1} (1 - 2e^{-\alpha_j})\left(1 - O(e^{-(n-\sigma(n))^{\zeta}})\right) \\
&\ge \left(1 - O(e^{-(n-\sigma(n))^{\zeta}})\right) \prod_{j=0}^{i-1} (1 - 2e^{-\alpha_j}) \\
&\ge \left(1 - O(e^{-(n-\sigma(n))^{\zeta}})\right)\left(1 - 2\sum_{j=0}^{i-1} e^{-\alpha_j}\right).
\end{aligned}$$

Based on this, the rest of the proof of (2.22) is essentially the same as for (2.21) and hence is omitted. The constant C_0 in (2.22) can be any number no less than $6C_1/\varepsilon + 14$ and hence is applicable to (2.21) as well. The desired probability $1 - O\left(e^{-(n-\sigma(n))^{1/2-2\varepsilon}}\right)$ in Theorem 4 is obtained by setting $\beta = 1/2$, specifically in $\zeta = \beta - 2\varepsilon$. ∎

3.2. Phase 2: $t_1 < t \le n-2$

Proof of Theorem 5. We can bound the value of Y for $t_1 < t \le t_2$ by considering the following events, where $t_1 \le s < i \le t_2$ and $n - i \ge \lfloor (n-s)/2 \rfloor$:

$$L_i = \left\{ \frac{Y_i + Z_i}{n-i} > \frac{2}{(\log n)^{\rho}} \right\}$$

$$J_{s,i} = \left\{ \frac{Y_i + Z_i}{n-i} \ge \frac{Y_s + Z_s}{n-s}(1 + \beta(n)) \text{ and } \frac{Y_j + Z_j}{n-j} \ge \frac{Y_s + Z_s}{n-s} \ge \frac{1}{(\log n)^{\rho}} \text{ for } s \le j < i \right\},$$

where $\beta(n) = 1/(\log\log n)^2$. Suppose that

$$Y_{t_1} + Z_{t_1} < \frac{n-t_1}{(\log n)^{\rho}}$$

but that L_u holds for some $u \le t_2$. Choose u to be minimum with this property. Then there is a minimum v_0 with $t_1 \le v_0 \le u$ such that $(Y_i + Z_i)/(n-i) \ge 1/(\log n)^{\rho}$ for $v_0 \le i \le u$.

It follows that $(Y_{v_0-1}+Z_{v_0-1})/(n-v_0+1) < 1/(\log n)^\rho$ and hence $(Y_{v_0}+Z_{v_0})/(n-v_0) \sim 1/(\log n)^\rho$ since $Y_{v_0}+Z_{v_0} \le Y_{v_0-1}+Z_{v_0-1}$ by the monotonicity of $Y+Z$. Defining inductively $n-v_k = \lfloor (n-v_{k-1})/2 \rfloor$, we have $n-v_k < n-u$ for $k \ge k'$, where

$$k' = \lceil \log_2((n-t_1)/(n-u)) \rceil = O(\log\log n).$$

Thus, the first inequality in $J_{s,i}$ must hold for some s and i with $v_0 \le s < i \le u$ and $n-i \ge \lfloor (n-s)/2 \rfloor$, since otherwise

$$\begin{aligned}
\frac{Y_u+Z_u}{n-u} &\le \frac{Y_{v_{k'-1}}+Z_{v_{k'-1}}}{n-v_{k'-1}}(1+\beta(n)) \\
&\vdots \\
&< \frac{Y_{v_0}+Z_{v_0}}{n-v_0}(1+\beta(n))^{k'} \\
&= (\log n)^{-\rho}(1+\beta(n))^{O(\log\log n)} \\
&= (\log n)^{-\rho}(1+o(1)),
\end{aligned}$$

contradicting L_u. Then we may assume that the inequalities $(Y_j+Z_j)/(n-j) \ge (Y_s+Z_s)/(n-s)$, $s \le j < i$, in the definition of $J_{s,i}$ are satisfied since otherwise s can be increased, holding j fixed, until this is the case. Note that $(Y_s+Z_s)/(n-s) \ge (\log n)^{-\rho}$ by the choice of v_0. So all the inequalities in $J_{s,i}$ hold for such s and i. In addition, by the minimality of u, L_s does not hold. Hence, for $u \le t_2$ as above,

$$L_u \wedge \left\{ Y_{t_1}+Z_{t_1} < \frac{n-t_1}{(\log n)^\rho} \right\} \subseteq \bigvee_{\substack{t_1 \le s < i \le t_2 \\ n-i \ge \lfloor (n-s)/2 \rfloor}} (J_{s,i} \wedge \overline{L_s}). \tag{3.24}$$

We next show that for all s and i in the range given, $\mathbf{P}(J_{s,i} \wedge \overline{L_s}) = O((\log n)^{-K})$ for all fixed $K > 0$. This requires a tail bound for a random sequence such as Y_t+Z_t, which, with probability close to 1, does not change in any given step. This is not easily accessible in the literature (although McDiarmid [2] has quite similar results for martingales) so we give the following lemma. Here we use a supermartingale, which with high probability, at least $1-O(\lambda)$, has a small difference, $O(\lambda)$ (where $\lambda \to 0$). The proof uses standard methods. In some sense it extends the argument in [4], which concerned the case that the sequence differences are independent and identically distributed.

Lemma 3. *Let $X_0, \ldots, X_r$ be a supermartingale with respect to the process $G_0, \ldots, G_r$, with $X_0 = 0$. Let C be a positive integer. Suppose that $W_t := X_t - X_{t-1}$ takes on at most C different values, u_j ($j = 0, \ldots$), and that $|u_j| < C$ for all j. Suppose furthermore that for some $0 < \lambda < 1$,*

(i) $\mathbf{E}(W_t^2 \mid G_0, \ldots, G_{t-1}) < \lambda$ *for all $t \le r$ and all $G_0, \ldots, G_{t-1}$,*
(ii) $|u_0| < \lambda$,
(iii) $\mathbf{P}(W_t \ne u_0 \mid G_0, \ldots, G_{t-1}) < \lambda$ *for all $t \le r$ and all $G_0, \ldots, G_{t-1}$.*

Then for any α with $0 < \alpha < \lambda r/C$,

$$\mathbf{P}(X_r \ge \alpha) < \exp(-\alpha^2/2\lambda r + C^3\alpha^3/\lambda^2 r^2).$$

Proof. At first, fix t and $G_0, \dots, G_{t-1}$. Letting $p_j = \mathbf{P}(W_t = u_j \mid G_0, \dots, G_{t-1})$, we have $|hu_j| < 1$ for $0 < h < 1/C$ (to be chosen shortly) and so

$$\mathbf{E}(e^{hW_t} \mid G_0, \dots, G_{t-1}) = \sum_j p_j e^{hu_j} = 1 + h \sum_j p_j u_j + \frac{1}{2}h^2 \sum_j p_j u_j^2 + \phi h^3 \sum_j p_j u_j^3,$$

where $|\phi| < 1/2$, by Taylor's formula. The first summation is $\mathbf{E}(W_t \mid G_0, \dots, G_{t-1}) \le 0$ since $\langle X_i \rangle$ is a supermartingale. The second summation is $\mathbf{E}(W_t^2 \mid G_0, \dots, G_{t-1}) < \lambda$ by assumption (i). The third summation is at most $2C^3\lambda$ by (ii) and (iii) and the assumptions about the u_j. So we have

$$\begin{aligned} \log \mathbf{E}(e^{hW_t} \mid G_0, \dots, G_{t-1}) &< \log \left(1 + \frac{1}{2}h^2\lambda + h^3C^3\lambda\right) \\ &< \frac{1}{2}h^2\lambda + h^3C^3\lambda. \end{aligned} \tag{3.25}$$

Now consider $0 < \alpha < \lambda r/C$. Using Markov's inequality for the second step,

$$\begin{aligned} \mathbf{P}(X_r \ge \alpha) &= \mathbf{P}\left(e^{hX_r} \ge e^{h\alpha}\right) \\ &\le e^{-h\alpha}\mathbf{E}\left(e^{hX_r}\right) \\ &= e^{-h\alpha}\mathbf{E}\left(e^{hX_{r-1}}e^{hW_r}\right) \\ &= e^{-h\alpha}\mathbf{E}\left(e^{hX_{r-1}}\mathbf{E}(e^{hW_r} \mid G_0, \dots, G_{r-1})\right) \\ &< \exp\left(-h\alpha + \frac{1}{2}h^2\lambda r + h^3C^3\lambda r\right) \end{aligned}$$

from (3.25) and by proceeding inductively. (Note: the second last step uses the fact that $\mathbf{E}(AB) = \mathbf{E}(A\mathbf{E}(B \mid C))$ for any random variables A, B, and C with A a function of C.) Selecting $h = \alpha/\lambda r$ to minimize the quadratic part, this becomes the bound in the lemma. ∎

We will apply Lemma 3 to the event $J_{s,i}$, conditioning on $\overline{L_s}$, where $\lfloor (n-s)/2 \rfloor \le n-i \le (\log n)^N$, for sufficiently large n. For all G_t, from the argument leading to (3.1) and (3.2), and noting (2.2) and (2.3), we have

$$\begin{aligned} \mathbf{E}(Y_{t+1} + Z_{t+1} - Y_t - Z_t \mid G_t) &= -\frac{U_t(Y_t + 2Z_t) - Y_t(Y_t+1)/2 - 4Z_t}{\binom{U_t}{2} - Z_t} \\ &\le -\frac{Y_t + Z_t}{n-t}. \end{aligned} \tag{3.26}$$

Similarly,

$$\begin{aligned} \mathbf{P}(Y_{t+1} + Z_{t+1} - Y_t - Z_t \ne 0 \mid G_t) &= \frac{U_t(Y_t + 2Z_t) - Y_t(Y_t+1)/2 - 2Z_t(Y_t + Z_t + 1)}{\binom{U_t}{2} - Z_t} \\ &\le \frac{U_t(Y_t + 2Z_t)}{\binom{U_t}{2} - Z_t} \\ &\le \frac{2(Y_t + 2Z_t)}{U_t - 2} \\ &\le \frac{4(Y_t + Z_t)}{n-t}. \end{aligned} \tag{3.27}$$

Here we used $\binom{U_t}{2} - Z_t \ge U_t(U_t - 2)/2$, which can be proved by using (2.3).

Let

$$\mu = \frac{Y_s + Z_s}{n - s}.$$

Let $D_0(\mu), D_1(\mu), \ldots$ be independent copies of the random variable $D(\mu)$ with $\mathbf{P}(D(\mu) = \mu) = 1 - \mu$ and $\mathbf{P}(D(\mu) = \mu - 1) = \mu$. At this point, take G_s fixed and consider the rest of the process, up to G_i. Note that for this process, we only need to condition on G_{s+t} rather than $G_s, \ldots, G_{s+t}$, since they both determine the same induced probability for $(G_{s+t}, G_{s+t+1}, \ldots)$. We are only concerned with $s + t \leq i$, so for the lemma, $r = i - s$. Define for $t \geq 0$

$$W_{t+1} = \begin{cases} (Y_{s+t+1} + Z_{s+t+1}) - (Y_{s+t} + Z_{s+t}) + \mu, & \text{if } Y_{s+t} + Z_{s+t} \geq (n - s - t)\mu \\ D_t(\mu), & \text{otherwise} \end{cases}$$

and

$$X_0 = 0, \qquad X_{t+1} = X_t + W_{t+1}.$$

If $Y_{s+t} + Z_{s+t} \geq (n - s - t)\mu$, then using (3.26) we have

$$\mathbf{E}(W_{t+1} \mid G_{s+t}) = \mathbf{E}(Y_{s+t+1} + Z_{s+t+1} - Y_{s+t} - Z_{s+t} \mid G_{s+t}) + \mu \leq 0;$$

while if $Y_{s+t} < (n-s-t)\mu$ then $\mathbf{E}(W_{t+1} \mid G_{s+t}) = \mathbf{E}D(\mu) = 0$. Thus, $\langle X_t \rangle$ is a supermartingale with respect to $\langle G_{s+t} \rangle$. In addition, if $J_{s,i}$ holds, then $Y_{s+t} + Z_{s+t} \geq (n - s - t)\mu$ for all t with $0 \leq t \leq i - s$, and hence we get

$$X_t = Y_{s+t} + Z_{s+t} - Y_s - Z_s + t\mu \tag{3.28}$$

by solving the difference equation $X_{t+1} - X_t = (Y_{s+t+1} + Z_{s+t+1}) - (Y_{s+t} + Z_{s+t}) + \mu, X_0 = 0$.

Note that $W_{t+1} = \mu$, $\mu - 1$ or $\mu - 2$ since $(Y_{s+t+1} + Z_{s+t+1}) - (Y_{s+t} + Z_{s+t}) = 0, -1$, or -2 as mentioned in the paragraph following (2.4). So with $u_0 = \mu$ and $\lambda = 34\mu$, we may let $C = 3$ in Lemma 3 (for n sufficiently large), and only the conditions (i) and (iii) still need to be checked (noting that $\lambda < 1$ by $\overline{L_s}$). By (3.27) we have

$$\mathbf{P}(W_{t+1} \neq u_0 \mid G_{s+t}) \leq \frac{4(Y_{s+t} + Z_{s+t})}{n - s - t} \leq \frac{8(Y_s + Z_s)}{n - s} = 8\mu < \lambda,$$

for n sufficiently large, using $n - s - t \geq n - i + 1 > (n - s)/2$ and $Y_{s+t} + Z_{s+t} \leq Y_s + Z_s$. This gives (iii) and also implies

$$\mathbf{E}(W_{t+1}^2 \mid G_{s+t}) < \mu^2 + \mathbf{P}(W_{t+1} \neq u_0 \mid G_{s+t}) \max W_{t+1}^2 < \mu^2 + 32\mu < \lambda$$

for n sufficiently large, since then $\mu < 1$ because $s \leq t_2$ and $\overline{L_s}$ holds. Thus, part (i) of the lemma also holds.

We may now apply the lemma with $r = i - s$, under the condition $J_{s,i} \wedge \overline{L_s}$. Note that the first condition in the definition of $J_{s,i}$ implies by monotonicity of $Y + Z$ that $n - s \geq (1 + \beta(n))(n - i)$ and hence

$$r = i - s \geq \beta(n)(n - i) = (n - i)/(\log\log n)^2.$$

Also note that $r < n - s - 1 \leq 2(n - i)$, and $\lambda = 34\mu \geq 34/(\log n)^\rho$, while $\overline{L_s}$ implies that $\lambda \leq 68/(\log n)^\rho$. Thus, with both these assumptions,

$$\frac{136(n-i)}{(\log n)^\rho} \geq \lambda r \geq \frac{34(n-i)}{(\log n)^\rho (\log\log n)^2}.$$

Since $n - i \geq n - t_2 \geq \log n$, applying Lemma 3 with $\alpha = (n - i)/(\log n)^{(1+2\rho)/3}$ gives

$$\mathbf{P}\left(X_r \geq \alpha | J_{s,i} \wedge \overline{L_s}\right) < \exp\left\{-\Theta\left(\frac{n-i}{(\log n)^{(2+\rho)/3}}\right)\right\} < \exp\left\{-\Theta((\log n)^{(1-\rho)/3})\right\}.$$

This implies

$$\mathbf{P}\left(\{X_r \geq \alpha\} \wedge J_{s,i} | \overline{L_s}\right) < \exp\left\{-\Theta((\log n)^{(1-\rho)/3})\right\} \tag{3.29}$$

as $\mathbf{P}(A \wedge B|C) \leq \mathbf{P}(A|B \wedge C)$ for any events A, B, C.

On the other hand, assuming $J_{s,i}$ and $\overline{L_s}$, by (3.28) we have

$$X_r = Y_i + Z_i - Y_s - Z_s + r\mu \geq (1 + \beta(n))(n - i)\mu - (n - i)\mu \geq \frac{\beta(n)(n-i)}{(\log n)^\rho} \geq \alpha$$

for n sufficiently large. Thus, $J_{s,i} \subseteq \{X_r \geq \alpha\} \wedge J_{s,i}$ conditioning on $\overline{L_s}$, and from (3.29) we have

$$\mathbf{P}\left(J_{s,i} \wedge \overline{L_s}\right) \leq \mathbf{P}\left(J_{s,i} | \overline{L_s}\right) < \exp\left\{-\Theta((\log n)^{(1-\rho)/3})\right\}.$$

Note that there are $O((\log n)^{2N})$ such events $J_{s,i} \wedge \overline{L_s}$. Note also that $G_{t_1} \in \mathcal{G}_1$ implies $Y_{t_1} + Z_{t_1} < (n - t_1)/(\log n)^\rho$. Thus, by (3.24) the probability that L_u holds for some u with $t_1 \leq u \leq t_2$, given $G_{t_1} \in \mathcal{G}_1$, is $O((\log n)^{-K})$ for all $K > 0$. In view of (2.25), this proves Theorem 5. ▪

To prove Theorem 6 we first note the following consequence of [5, Lemma 3.1].

Lemma 4. *Let $0 \leq t < n$, and suppose that G is a graph such that $\mathbf{P}(G_t = G) \neq 0$. Let v be a vertex of degree less than 2 in G. The probability that in a random 2-process, v is not incident with any of $E_{t+1}, \ldots, E_{t'}$, conditional upon $G_t = G$, is $O((n - t' + 1)/(n - t))$ as $n \to \infty$ uniformly over all $t' > t$.*

Proof of Theorem 6. Since $Y + Z$ is non-decreasing, (2.27) follows immediately from $G \in \mathcal{G}_2$. Applying Lemma 4 to $t = t_2$ and a graph $G \in \mathcal{G}_2$, the expected number of isolated vertices that remain isolated in G_{t_3} is $O((n - t_3)/(n - t_2)Y(G)) = O((\log n)^{\delta-\rho})$. Hence, (2.28) follows by the first moment principle. Since $Z_{t_3} < (\log n)^{1-\rho}$ by (2.27), the same argument gives (2.29). ▪

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