Dynamic Domination in Fuzzy Causal Networks

Jian Ying Zhang, Zhi-Qiang Liu, and Sanming Zhou

***Abstract*—This paper presents a dynamic domination theory for fuzzy causal networks (FCN). There are three major contributions. First, we propose a new inference procedure based on dominating sets. Second, we introduce the concepts of dynamic and minimal dynamic dominating sets (DDS and MDDS) in an FCN. To reflect changes of dominance with time, we also introduce the concept of a dynamic dominating process (DDP) that has significant implications in many real-world problems. We pay a special attention to the minimal dynamic dominating process (MDDP) and develop rules for generating DDP and MDDP. Third, we investigate dynamic dominating sets with extended feedback, which we call effective dynamic dominating sets (EDDS), and related effective dynamic dominating process (EDDP). This study unveils a very important phenomenon in FCN: At any time *t*, either an EDDS exists or there is a dramatic change of the states of vertices. In the latter case we also identify the special structure of the sub-FCN induced by active vertices.**

***Index Terms*—Domination, dynamic system, fuzzy causal network (FCN), fuzzy cognitive map, intelligent system.**

I. Introduction

FUZZY causal networks (FCNs) can be used for decision support based on the principle of causal discovery in the presence of uncertainty and incomplete information [12], [14], [19], [20]. An FCN has two internal features that are very useful in modeling real-world problems. First, the dynamic behaviors of an FCN depend on the vertex states that vary with time, where each vertex stands for a concept with a fuzzy event or property described by words or phrases such as *young, tall, goals of a project, high revenue*, and so on. The associated vertex state value specifies the fuzzy event occurring to some degree at some discrete times [10]. Such fuzzy direct graphs can be regarded as a dynamic network. When some vertices receive a series of external stimuli [14], [15], [17], the vertex states of such a dynamic network are updated until a final equilibrium stable state is reached [7], [8]. Furthermore, the feedback mechanism in FCN enables the system to adjust (adapt) itself in response to the changing environment and to the information about the given goals and actual outcomes [17], [18].

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J. Y. Zhang is with the Department of Computer Science and Software Engineering, The University of Melbourne, Victoria 3010, Australia, and also with the Information and Communication Technologies, Swinburne University of Technology, Victoria 3122, Australia (e-mail: jyzhang@it.swin.edu.au).

Z.-Q. Liu is with the School of Creative Media, City University of Hong Kong, Kowloon, Hong Kong, China (e-mail: smzliu@cityu.edu.hk).

S. Zhou is with the Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia (e-mail: smzhou@ms.unimelb.edu.au).

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One of the major topics in the study of FCN is to understand FCN's dynamic properties and causal inference process. The study of dynamics of FCNs was evolved from that of fuzzy cognitive maps (FCM) [7]. For example, Miao and Liu [15], [16] proposed a theory of dynamic cognitive networks (DCN) based on FCM, where each vertex in a DCN can have its own value, depending on how precise it needs to be described in the networks. More recently, in their preliminary investigation, Liu and Zhang proposed a dynamic causal algebra for analysing FCNs in general [13].

In many systems, major changes are often caused by some "dominant" factors. For instance, the Internet and mobile communication systems have contributed significantly to our life styles in this digital era. The dominant factors that have caused such changes are the fast micro processors, high-speed communication lines, and the Internet infrastructure. Also, for example, scientists found significant ozone depletion. Although there may be other factors that may cause ozone depletion, scientists have identified a group of substances that causes the *most* damage to the ozone layer, which include CFCs, HydroCFCs (HCFCs), halons, methyl bromide, carbon tetrachloride, and methyl chloroform that were used widely as refrigerants, insulating foams, and solvents. This has enabled international environmental bodies to establish laws to ban the use of such substances which in turn helped slow down the rate of depletion and even possibly restore the ozone layer. For many real-world problems, identifying the dominant factors (players) will enable us to effectively design and analyze large inference of decision-making systems.

An FCN in most applications involves a large number of concepts that are interconnected as a directed graph [13]. In addition, being a causal inference system, FCN has to operate in a dynamic environment. As a result, dominant factors, which we call dynamic dominating sets, may change frequently. Therefore, we must answer the following basic questions.

a) For an arbitrary FCN, how do we measure accurately the strength of dominance of a dynamic dominating set at time t?
b) How do dynamic dominating sets evolve with time t? What can we say about the transition of dominance from t to $t+1$?
c) When will a dynamic dominating set have extended feedback? What will happen if such a set does not exist?

In this paper, we will develop a dynamic domination theory for FCN $\mho$ and answer in particular the aforementioned questions. The major contributions of the paper are as follows.

First, in Section III, we propose a new inference procedure, which is achieved by setting the vertices in a dominating set of the FCN active as the initial condition. As we will see, this inference procedure allows us to measure more effectively the impact of the initial condition on the operation of $\mho$.

Second, to understand better the dynamics of $\mho$ we introduce in Section IV the concepts of dynamic and minimal dynamic dominating sets (DDS and MDDS) of $\mho$ for $t \geq 1$. We will present an algorithm to generate such dominating sets and define their dominating strengths. In real-world applications a DDS or an MDDS can be interpreted as a group of key factors that collectively plays a dominating role in the inference of $\mho$ at t. This dominance changes with time t, just as the change of major players in the stock market. To reflect this dynamic nature, we introduce the concept of a dynamic dominating process (DDP): $(S_t : t = 0, 1, \ldots)$, which mimics many real-world scenarios. Its starting point (at $t = 0$) is the dominating set S_0 that was used to set initial condition. At $t \geq 1$, S_t is a DDS of $\mho$. When S_0 is a minimal dominating set of $\mho$ and S_t is an MDDS of $\mho$ for $t \geq 1$, the DDP above is called a minimal dynamic dominating process (MDDP). We will pay a particular attention to MDDP and present a few rules for generating DDP and MDDP.

The third major contribution of this paper is the recognition and study of an important and interesting phenomenon which we call *revolution*. This is unveiled in our study of the effective dynamic dominating set (EDDS) and the effective dynamic dominating process (EDDP), where by an EDDS we mean a DDS with extended feedback. The phenomenon tells us that, at $t \geq 1$, either an EDDS exists in the FCN $\mho$ or the states of vertices undergo a dramatic change that suddenly makes all the active dominating vertices inactive at the next time $t+1$. In other words, the currently active key factors may all be driven out of the club of major players in the next round. When this *revolutionary* change happens, we identify the special structure of the sub-FCN induced by the active vertices at t.

In Section VI we will conduct simulations to demonstrate the proposed inference procedure and its advantages over the traditional inference procedure.

II. Concepts and Notations

A. FCNs

An FCN is a dynamic system whose topological structure is a directed graph $\mho = (V, A)$, where V is the set of vertices and A the set of directed arcs of $\mho$. In the following we will use $n = |V|$ to denote the number of vertices of $\mho$. (In general, for any finite set S, $|S|$ denotes the size of S, that is, the number of elements of S.) Each vertex of $\mho$ represents a concept whose state varies with (discrete) time, and each arc indicates a causal relationship from the tail to the head of the arc. Thus, associated with $\mho$ is the *vertex state space* whose elements are the n-dimensional vectors of $[0,1]^n$

$$\boldsymbol{x}_{\mho}(t) = (x_v(t))_{v \in V}$$

where $x_v(t) \in [0,1]$ is the value of the *state* of vertex v at time t, which can be continuous or discrete.

For example, in the binary case the state $x_v(t) \in \{0,1\}$ for each $v \in V$, and $x_{\mho}(t)$ is in the *binary state space* $\{0,1\}^n$. For both continuous and discrete cases, if $x_v(t) > 0$, then the vertex v is said to be *active* at t; and if $x_v(t) = 0$ then v is said to be *inactive* at t. Thus, according to the state values the set V of vertices of $\mho$ is partitioned into two subsets: active and inactive. We will use V_t to denote the subset of active vertices

$$V_t = \{v \in V \mid v \text{ is active at } t\} = \{v \in V \mid x_v(t) > 0\}. \quad (1)$$

As usual we assume throughout the paper that $\mho$ contains no loops and multiple arcs, where a loop is an arc from a vertex to itself and multiple arcs are distinct arcs with the same initial and terminal vertices. Also associated with $\mho$ is its *weight function*

$$(u, v) \mapsto w_{uv}, \qquad u, v \in V$$

which specifies the *weight* w_{uv} of each ordered pair (u, v). As usual we will assume throughout that all the weights $w_{uv} \in [-1, 1]$. The weight function defines uniquely the $n \times n$ weight matrix

$$W_{\mho} = (w_{uv})_{u,v \in V}$$

of $\mho$ (also called adjacency matrix in the literature). Note that we define $w_{uv} = 0$ if (u, v) is not an arc of $\mho$, which means that the vertex u has no influence on v at any t. In particular, since we assume the loop (v, v) is not an arc of $\mho$ for any vertex $v \in V$, we have $w_{vv} = 0$. We will adopt the usual convention[1] that, if (u, v) is an arc of $\mho$, then the weight $w_{uv} \neq 0$ and u may influence v. The strength of such an influence at time t is

$$\rho_{uv}(t) = x_u(t) w_{uv}. \quad (2)$$

Thus, we have the $n \times n$ *strength matrix*

$$G_{\mho}(t) = (\rho_{uv}(t))_{u,v \in V}$$

of $\mho$ at time t. Note that $\rho_{uv}(t) = 0$ if $w_{uv} = 0$ or the state value $x_u(t) = 0$ (which indicates that u is inactive at time t). In particular, since $w_{vv} = 0$ for each v, the diagonal entries $p_{vv}(t)$ of the strength matrix $G_{\mho}(t)$ are all equal to 0. In view of (2), we have the following linear relationship between the vertex states $x_v(t)$, the weight matrix $W_{\mho}$ and the strength matrix $G_{\mho}(t)$ at time t:

$$G_{\mho}(t) = X_{\mho}(t) \cdot W_{\mho}$$

where $X_{\mho}(t) = \mathrm{diag}(x_v(t))_{v \in V}$ is the (diagonal) matrix with $x_v(t)$ as the diagonal entries and all other entries being 0, and the dot means matrix product.

The dynamics of $\mho$ is as follows. First, an initial condition $\boldsymbol{x}_{\mho}(0) = x_0$ is set at $t = 0$, where $x_0 = (x_v)_{v \in V}$ is an n-dimensional vector in $[0,1]^n$. which specifies the initial state of $\mho$ and the initial set

$$V_0 = \{v \in V \mid x_v > 0\}$$

of active vertices. At t each vertex v receives a number of inputs (stimuli) from other vertices. The *total input* received by v is given by

$$y_v(t) = \sum_{u \in V - \{v\}} \rho_{uv}(t). \quad (3)$$

[1]This has been used in the literature but not stated explicitly. As a matter of fact, if $w_{uv} = 0$ holds for some arcs (u, v) of $\mho$, we may simply delete all such arcs from $\mho$ to get a new FCN. The study of $\mho$ is equivalent to the study of this new FCN, as they have the same dynamics and inference. So assuming $w_{uv} \neq 0$ for all arcs (u, v) will not sacrifice generality.

Note that, by (2) and the definition of weights w_{uv}, those vertices u which are either inactive at t or have no arc to v contribute nothing to the sum (3). In the following, we will denote:

$$\mathbf{y}_{\mho}(t) = (y_v(t))_{v\in V}$$

and call it the *input vector* of $\mho$ at t. In view of (2) and (3), we have

$$\mathbf{y}_{\mho}(t) = \mathbf{x}_{\mho}(t)W_{\mho}.$$

The state transition is governed by n functions, one for each $v \in V$. The function f_v for v transforms the input $y_v(t)$ received by v at t into the next state $x_v(t+1)$ of v:

$$\begin{aligned} f_v &: R \to [0,1] \\ y_v(t) &\mapsto x_v(t+1) \end{aligned} \tag{4}$$

or, equivalently

$$x_v(t+1) = f_v(y_u(t))$$

where R is the set of real numbers. Define $f = (f_v)_{v\in V}$ to be the vector function acting *coordinate-wise*, that is, $f(\mathbf{y}) = (f_v(y_v))_{v\in V}$ for any $\mathbf{y} = (y_v)_{v\in V}$. Then, we have

$$\mathbf{x}_{\mho}(t+1) = f(\mathbf{y}_{\mho}(t)) = f(\mathbf{x}_{\mho}(t)\cdot W_{\mho}). \tag{5}$$

By this formula, once the initial condition, $\mathbf{x}_{\mho}(0) = (x_v)_{v\in V}$, has been set, the state of v at any time t can be determined recursively.

Different state transition functions f_v have been proposed in the literature. For example, in [10] Kosko suggested the use of bounded signal functions f_{T_v} relating to a given threshold T_v, which applies to the case of continuous states. In the case of binary states, the function f_v is usually chosen [17], [20] to be a threshold function.

In our discussions, the vertices of $\mho$ are sometimes indexed as $V = \{v_1,\ldots,v_n\}$. In this case, we simply write $x_i(t)$, w_{ij}, $\rho_{ij}(t)$, T_i, $y_i(t)$, etc., in place of $x_{vi}(t)$, $w_{v_iv_j}$, $p_{v_iv_j}(t)$, T_{v_i}, $y_{v_i}(t)$, etc., respectively. Thus

$$\begin{aligned} &\mathbf{x}_{\mho}(t) = (x_1(t),\ldots,x_n(t)) \quad W_{\mho} = (w_{ij}) \\ &\rho_{ij}(t) = x_i(t)w_{ij} \quad \mathbf{y}_{\mho}(t) = (y_1(t),\ldots,y_n(t)) \end{aligned}$$

and so on.

B. Neighborhood and Domination

In this section, we discuss the topological aspects of an FCN $\mho = (V,A)$. For this purpose, we may simply consider $\mho$ as a directed graph, since no dynamics of $\mho$ will be involved at this stage. A *directed path* of $\mho$ with *length* r is a sequence $(v_0,v_1,\ldots,v_r)$ of $r+1$ distinct vertices such that (v_i, v_i+i) is an arc of $\mho$ for $i = 0,1,\ldots,r-1$. If $v_0 = v_r$ in this sequence and $v_0,v_1,\ldots,v_{r-1}$ are pairwise distinct, then the sequence defines a *directed cycle* $(v_0,v_1,\ldots,v_{r-1},v_0)$ of $\mho$ with *length* r. Note that a directed cycle of length 2, which is allowed in the FCN, consists of two arcs incident with the same pair of vertices but with opposite directions. We use $P(u,v)$ to denote a directed path of $\mho$ from u to v.

For any vertex $v \in V$, we define

$$\begin{aligned} N^-(v) &= \{u \mid u \in V, (u,v) \in A\} \\ N^+(v) &= \{u \mid u \in V, (v,u) \in A\} \end{aligned}$$

and call them the *in-neighborhood* and *out-neighborhood* of v in $\mho$. The sizes of them, namely

$$d^-(v) = |N^-(v)| \quad d^+(v) = |N^+(v)|$$

are called the *in-degree* and *out-degree* of v in $\mho$, respectively. The vertices in $N^-(v)$ are called the *in-neighbors* of v in $\mho$ and those in $N^+(v)$ are *out-neighbors* of v in $\mho$. Denote by

$$\begin{aligned} \partial^-(v) &= \{(u,v) \in A \mid u \in N^-(v)\} \\ \partial^+(v) &= \{(u,v) \in A \mid u \in N^+(v)\} \end{aligned}$$

the sets of arcs of $\mho$ *coming to* v and *leaving from* v, respectively. In general, for any subset S of V, we define

$$\begin{aligned} N^-(S) = \{u \in V - S \mid \text{ there exists } v \in S \\ \text{such that } (u,v) \in A\} \\ N^+(S) = \{u \in V - S \mid \text{ there exists } v \in S \\ \text{such that } (v,u) \in A\} \end{aligned}$$

and call them the *in-neighborhood* and *out-neighborhood* of S, respectively. Let

$$\begin{aligned} \partial^-(S) &= \{(u,v) \in A \mid v \in S, u \in V - S\} \\ \partial^+(S) &= \{(u,v) \in A \mid v \in S, u \in V - S\} \end{aligned}$$

be the set of arcs of $\mho$ from $V-S$ to S, and the set of arcs of $\mho$ from S to $V-S$, respectively. Note that in the case where $S = \{v\}$ is a singleton we have $N^-(S) = N^-(v)$, $N^+(S) = N^+(v)$, $\partial^-(S) = \partial^-(v)$ and $\partial^+(S) = \partial^+(v)$. For any $S \subseteq V$, from the above we have

$$\begin{aligned} N^-(S) &= \left(\bigcup_{v\in S} N^-(v)\right) \cap (V-S) \\ N^+(S) &= \left(\bigcup_{v\in S} N^+(v)\right) \cap (V-S) \\ \partial^-(S) &= \partial^+(V-S) \quad \partial^+(S) = \partial^-(V-S) \\ \partial^-(S) &= \left(\bigcup_{v\in S} \partial^-(v)\right) \cap \left(\bigcup_{u\in V-S} \partial^+(u)\right) \\ \partial^+(S) &= \left(\bigcup_{v\in S} \partial^+(v)\right) \cap \left(\bigcup_{u\in V-S} \partial^-(u)\right). \end{aligned}$$

With the notation above, we give the following definition, which is crucial to our subsequent discussions.

Definition 2.1: Let $\mho = (V,A)$ be an FCN. A subset S of V is called a dominating set of $\mho$ if $N^+(S) = V - S$, in other words, for any $u \in V - S$ there exists at least one $v \in S$ such that (v,u) is an arc of $\mho$. A dominating set S of $\mho$ is called a minimal dominating set of $\mho$ if, for any $v \in S$, $S - \{v\}$ is not a dominating set of $\mho$.

The counterpart concept of domination for undirected graphs has been studied extensively in the past two decades by researchers in computer science and mathematics [2], [5], [21]. One of the major concerns so far for both directed and undirected cases is to find dominating sets of minimum size. Such dominating sets are called *minimum dominating sets*, and the size of them is called the *domination number* of the (directed or undirected) graph, denoted by γ. A minimum dominating set is always a minimal dominating set, but the converse is generally not true. It has been shown that the problem for determining the domination number is NP-complete (see [3] for definition) in both undirected and directed cases [1], [5]. Any FCN $\mho$ has at least one dominating set—the vertex set V itself is a dominating set—and by deleting "redundant" vertices in a dominating set we can obtain a minimal dominating set. Usually, an FCN may contain more than one dominating sets and minimal dominating sets. For relatively few results on domination in directed graphs, the interested reader is referred to the survey paper [4].

III. New Inference Procedure

When using FCN to model real-world applications, we usually have to estimate the impact of the initial condition on the whole FCN in decision support or causal discovery. The impact comes into being through an inference process, which we have summarized in Section II-A. For most applications, setting an appropriate initial condition for the FCN plays a crucial role in obtaining a reliable inference pattern. In the literature, most researchers randomly set a vertex active and leave the remaining vertices inactive, and then calculate inference patterns [7], [9], [12], [13], [17]. In other words, for an FCN $\mho = (V, A)$, usually a vertex v with state x_v, $0 < x_v \leq 1$, is chosen and the initial condition is set

$$\mathbf{x}_{\mho}(0) = (0, \ldots, x_v, \ldots, 0)$$

where the state of v at $t = 0$ is $x_v(0) = x_v$.

A. Inference Procedure

In this section, we propose a new procedure for setting initial conditions. This is achieved by considering the position of a subset of vertices. The basic approach is to choose, based on expert knowledge, a subset $S \subseteq V$ that has a significant influence on the whole FCN $\mho = (V, A)$, and then set vertices in S active as the initial condition. We first measure the influence of a subset by its dominance in the FCN. At the same time, in many cases, to reduce working loads and costs we may require that the number of vertices in S be reasonable. We choose a minimal dominating set of $\mho$, which may be a set of *major* investors of a company, the key substances that cause ozone layer depletion, and so on. We can use the following simple algorithm to find a minimal dominating set of $\mho$.

Algorithm 3.1 Input: An FCN $\mho = (V, A)$;
Output: A minimal dominating set S of $\mho$.
1. Set initially $S = V$;
2. for each $v \in S$, check whether v satisfies $N^+(v) \cap (V - S) \subseteq N^+(S - \{v\})$ and $N^-(v) \cap S \neq \emptyset$ simultaneously;
3. if no such a v exists, then stop and output S; otherwise, choose a vertex $v \in S$ such that $|N + (v) \cap (V - S)| \leq |N^+(u) \cap (V - S)|$ for all $u \in S - \{v\}$, set $S := S - \{v\}$ and go to Step 2.

Note that $|N^+(v) \cap (V - S)|$ is required to have the minimum size in order to output a minimal dominating set with maximum number of out-going arcs. The correctness of this algorithm is ensured by the following theorem.

Theorem 3.2: For any $\mho = (V, A)$, Algorithm 3.1 produces a minimal dominating set of $\mho$.

Proof: As mentioned at the end of Section II-B, the initial set $S \in V$ is a dominating set of $\mho$. Assume inductively that at some round of iteration the set S is a dominating set of $\mho$. Suppose that no v satisfies the conditions in Step 2 simultaneously, that is, for all $v \in S$ either $N^+(v) \cap (V - S) \nsubseteq N^+(S - \{v\})$ or $N^-(v) \cap S = \emptyset$. In the former case, there exists $u \in V - S$ such that $u \in N^+(v)$ but $u \notin N^+(w)$ for any $w \in S - \{v\}$. In other words, v is the only member of S such that $u \in N^+(v)$. Hence $S - \{v\}$ is not a dominating set of $\mho$. Also, in the latter case $S - \{v\}$ is not a dominating set of $\mho$ since there is no arc from $S - \{v\}$ to v. Thus, for all $v \in S$, $S - \{v\}$ is not a dominating set of $\mho$ if no v satisfies the conditions in Step 2. This is equivalent to saying that in this case the algorithm outputs a minimal dominating set S of $\mho$. On the other hand, if there exists $v \in S$ such that $N^+(v) \cap (V - S) \subseteq N^+(S - \{v\})$ and $N^-(v) \cap S \neq \emptyset$, then $(V - S) \cup \{v\} = N^+(S) \cup \{v\} = N^+(S - \{v\}) \cup (N^+(v) \cap (V - S)) \cup \{v\} = N^+(S - \{v\})$. Herethe first equality is due to the fact that S is a dominating set of $\mho$, and the last one is implied by the two conditions. Therefore, $S - \{v\}$ is a dominating set of $\mho$ and the algorithm proceeds to the next iteration. □

After finding a minimal dominating set S of $\mho$, we set all vertices in S active as the initial condition. With the inference process being initialized, we can use the following procedure to estimate the impact of the initial condition on the whole FCN.

a) Construct an FCN $\mho$ to model the system with which we are dealing.
b) Find out a minimal dominating set S of $\mho$ by using Algorithm 3.1.
c) Set all vertices in S active as the initial condition.
d) Calculate inference pattern by using (4) or (5).

The second way we propose to measure the influence of a subset is to calculate the maximum possible strength of the impact it can have on the FCN. This measure is

$$\sigma(S) = \sum_{(v,u) \in \partial^+(S)} m_v w_{vu}$$

where $m_v \in (0, 1]$ is the maximum value allowed for the state of v. For binary states, $m_v = 1 \, \forall \, u \in V$, which leads to $\sigma(S) = \sum_{(v,u) \in \partial^+(S)} w_{vu}$.

In general, $\sigma(S)$ gives the total strength of arcs of $\mho$ from S to $V - S$ when each vertex of S exerts its maximum influence. The larger $\sigma(S)$ is, the stronger the impact. To further improve the above strategy, we propose to find a minimal dominating set S such that $\sigma(S)$ achieves the maximum value. Once we have found such an S, we apply a procedure similar to a)–d) to calculate the inference pattern.

B. Discussion

Compared with the conventional inference procedures [7], [9], [12], [13], [17] mentioned at the beginning of this section, the new inference procedure we suggested has the following advantages. First, it counts the total impact of the initial active set S by requiring that it dominates the FCN. Moreover, we require further that S has the maximum influence $\sigma(S)$ on the FCN. In this way, we can obtain more accurate measurement of the impact of the initial condition. In Section VI, we will illustrate our inference procedure and its advantages by using a simple example.

An important problem in applying the inference procedure is: How small could a minimal dominating set of $\mho$ be? This is related to the estimation of the domination number $\gamma(\mho)$. Let

$$\delta^-(\mho) = \min_{v\in V} d^-(v) \ \text{ and } \Delta^+(\mho) = \max_{v\in V} d^+(v)$$

be the minimum in-degree and maximum out-degree of vertices of $\mho$, respectively. By using the results from the domination theory on directed graphs, we have the following lower and upper bounds for $\gamma(\mho)$, which are due to [11] (see also [4, Th. 15.49] and [4, Th. 15.57], respectively).

Theorem 3.3: For any FCN $\mho = (V, A)$ with n vertices, we have

$$1 \le \gamma(\mho) \le \frac{\delta^-(\mho)+1}{2\delta^-(\mho)+1} n$$
$$\frac{n}{\Delta^+(\mho)+1} \le \gamma(\mho) \le n - \Delta^+(\mho).$$

In [12], an FCN $\mho$ is called *simple* if it contains no directed cycles (acyclicity). Following the standard terminology in graph theory, we call $\mho$ *strongly connected* if, for any two vertices u and v, there exists a *directed path* of $\mho$ from u to v. Intuitively, in a simple FCN there is no feedback, but in a strongly connected FCN there are very strong causal relationships among the concepts of $\mho$. The following theorem shows that, in these two extreme cases, we can find dominating sets with certain extra properties. A subset S of vertices of $\mho = (V, A)$ is called an *independent set* of $\mho$ if $(u, v) \notin A$, $\forall\, u, v \in S$; that is, S is independent if there is no causal relationship between any two vertices of S.

Theorem 3.4: Let $\mho = (V, A)$ be an FCN with n vertices.

a) If $\mho = (V, A)$ is simple, then there exists a subset of V which is both dominating and independent [4].

b) If $\mho$ is strongly connected, then $\gamma(\mho) \le n/2$ [11].

Part a) is a classic result of Von Neumann and Morgenstern proved initially in terms of game theory [4, Sec. 15.6]. The same result as in a) is true if $\mho$ has a large number of arcs, namely, at least $n^2 - 2n + 1$ ($n \ge 3$), proved by Harary and Behzad (see [4, Th. 15.45]). Part b) is due to Lee [11] and can be found in [4, Th. 15.59] as well.

IV. Dynamic Domination in FCN

In the previous section, we proposed a new inference procedure by setting vertices in a minimal dominating set active as its initial condition, after which the state $x_\mho(t)$ of $\mho$ is updated recursively, following the rule (4) or (5). In the case of binary states, since there are only 2^n possible states, the system will converge to a static state after less than 2^n steps. (See [22] for a recent study on the speed of convergence.) In this section, we will investigate the dynamic behavior of the FCN after the initial setting. We focus on dynamic dominating sets and the related dynamic dominating processes which have important implications in applications, e.g., computer networks, social science, personnel assignments, stock market analysis, and so on.

Recall that in (1) we defined V_t to be the set of vertices of $\mho$ activated at t. In the following, we use $\mho_t$ to denote the sub-FCN induced by V_t:

$$\mho_t = (V_t, A_t)$$

where

$$A_t = \{(u, v) \in A \mid u, v \in\in V_t\}.$$

$\mho_t$ inherits from $\mho$ the causal relationships and weights. Thus, the weight matrix of $\mho_t$ is $W_{\mho_t} = (w_{uv})_{u,v\in V_t}$, which can be obtained from the weight matrix $W_\mho$ of $\mho$ by deleting those rows and columns indexed by vertices of $V - V_t$. This sub-FCN $\mho_t$ alone determines the inference of $\mho$ since all vertices outside $\mho_t$ are inactive at t.

A. Active Subsets and Active Vertices

In this section, we will discuss the role of an active subset S of $\mho$ by considering the relationship between S and the vertices outside S. This is preliminary to our study of dynamic dominating sets and processes. We start with the following definition of an active subset.

Definition 4.1: Let $\mho \in (V, A)$ be an FCN. A subset S of V is called an active subset of $\mho$ at time t if every vertex in S is active at t. In other words S is active at t if $S \subseteq V_t$.

For such an active subset S, we define

$$N_t^-(S) = N^-(S) \cap V_t \quad N_t^+(S) = N_t^+(S) \cap V_t$$

which are the sets of in-neighbors and out-neighbors, outside S and active at t, of the vertices of S, respectively. Equivalently, they are the in-neighborhood and out-neighborhood of S in the sub-FCN $\mho_t$. Let

$$\partial_t^-(S) = \partial^-(S) \cap A_t \quad \partial_t^+(S) = \partial^+(S) \cap A_t.$$

Then they are, respectively, the sets of arcs of $\mho$ from $V_t - S$ to S, and S to $V_t - S$, whose end-vertices are both active. Thus

$$N_t^-(S) = N_{\mho_t}^-(S) \quad N_t^+(S) = N_{\mho_t}^+(S)$$
$$\partial_t^-(S) = \partial_{\mho_t}^-(S) \quad \partial_t^+(S) = \partial_{\mho_t}^+(S)$$

where the subscript $\mho_t$ refers to the operations N^-, N^+, ∂^- and ∂^+ in the sub-FCN $\mho_t$.

Definition 4.2: For an active subset S of $\mho = (V, A)$ at t, we define

$$\Gamma_t^-(S) = \sum_{(u,v)\in\partial^-(S)} \rho_{uv}(t)$$
$$\Gamma_t^+(S) = \sum_{(u,v)\in\partial^+(S)} \rho_{uv}(t) \tag{6}$$

and call them the in-strength and out-strength of S at t, respectively.

For $u \in V - S$, $v \in S$ such that (u, v) is not an arc of $\mho$, we have $w_{uv} = 0$ and hence $\rho_{uv}(t) = 0$ in view of (2). Thus, $\Gamma_t^-(S) = \sum_{u \in V-S, v \in S} \rho_{uv}(t)$. Also, for $u \in (V - S) \cap (V - V_t)$, we have $x_u(t) = 0$ and, hence, $\rho_{uv}(t) = 0$ by (2). Therefore, we have

$$\begin{aligned}\Gamma_t^-(S) &= \sum_{u \in V-S, v \in S} \rho_{uv}(t) \\ &= \sum_{(u,v) \in \partial_t^- S} \rho_{uv}(t) + \sum_{u \in V-S, v \in S \cap (V-V_t)} \rho_{uv}(t).\end{aligned} \tag{7}$$

Recall that $\partial^-(S) = \partial^+(V - S)$ and $\partial^+(S) = \partial^-(V - S)$. From this and Definition 4.2, it follows that

$$\Gamma_t^-(S) = \Gamma_t^+(V - S) \quad \Gamma_t^+(S) = \Gamma_t^-(V - S). \tag{8}$$

Since $\partial_t^-(V - S) = \partial^-(V - S) \cap A_t = \partial^+(S) \cap A_t = \partial_t^+(S)$, (7) and (8) imply

$$\begin{aligned}\Gamma_t^+(S) &= \sum_{v \in S, u \in V-S} \rho_{vu}(t) \\ &= \sum_{(v,u) \in \partial_t^+(S)} \rho_{vu}(t) + \sum_{v \in S, u \in (V-S) \cap (V-V_t)} \rho_{vu}(t).\end{aligned} \tag{9}$$

Therefore, $\Gamma_t^-(S)$ and $\Gamma_t^+(S)$, respectively, represent the total impact strength received by S from $V - S$ and the total impact strength that S has on $V - S$. An important part of the out-strength of S is the second term on the right-hand side of (9), which is the strengths of S on those vertices $u \in V - S$ inactive at t. It is this part that may activate some of such inactive vertices u in the next time, namely $t + 1$.

In order to recognize the role of an active subset in $\mho$, we may classify active subsets S at t into the following three categories by comparing the values of the in-strength $\Gamma_t^-(S)$ and out-strength $\Gamma_t^+(S)$: (a) $\Gamma_t^+(S) > \Gamma_t^-(S)$; (b) $\Gamma_t^+(S) < \Gamma_t^-(S)$; (c) $\Gamma_t^+(S) = \Gamma_t^-(S)$. In the first case, the out-strength of S is larger than the in-strength of S, which indicates that at time t the influence of S on $V - S$ is stronger than the influence of $V - S$ on S. The other two cases can be interpreted in a similar way.

It is obvious that the smallest active subsets are singletons, $S = \{v\}$, for which we write $N_t^-(v)$, $N_t^+(v)$, $\partial_t^-(v)$, $\partial_t^+(v)$, $\Gamma_t^-(v)$ and $\Gamma_t^+(v)$. Thus, from the previous notation we have

$$\begin{aligned}&N_t^-(v) = N^-(v) \cap V_t \quad N_t^+(v) = N^+(v) \cap V_t \\ &\partial_t^-(v) = \partial^-(v) \cap A_t \quad \partial_t^+(v) = \partial^+(v) \cap A_t \\ &\Gamma_t^-(v) = \sum_{u \in N^-(v)} \rho_{uv}(t) \quad \Gamma_t^+(v) = \sum_{u \in N^+(v)} \rho_{uv}(t).\end{aligned}$$

We call $\Gamma_t^-(v)$ and $\Gamma_t^+(v)$ the *in-strength* and *out-strength* of v at t, respectively. The former is the total input received by v from all in-neighbors of v (but only active $u \in N^-(v)$ make contributions to $\Gamma_t^-(u)$); and the latter is the total input that v has on all out-neighbors of v (active or not). Applying (7) and (9) to the singleton case, where $S = \{v\}$, and using the definition (3) of $y_v(t)$, we obtain the following relationships, for any vertex v of $\mho = (V, A)$ active at t:

$$\begin{aligned}\Gamma_t^-(v) &= y_v(t) = \sum_{u \in N_t^-(v)} \rho_{uv}(t) \\ \Gamma_t^+(v) &= \sum_{u \in V-\{v\}} \rho_{vu}(t) = \sum_{u \in N_t^+(v)} \rho_{vu}(t) \\ &\quad + \sum_{u \in N^+(v) \cap (V-V_t)} \rho_{vu}(t).\end{aligned}$$

We can recognize the role of an active vertex $v \in \mho$ at time t by comparing the values of its in-strength and out-strength. Similar to the classification of active subsets, we may classify the vertices of V_t into the following three categories: a) $\Gamma_t^+(v) > \Gamma_t^-(v)$; b) $\Gamma_t^+(v) < \Gamma_t^-(v)$; and c) $\Gamma_t^+(v) = \Gamma_t^-(v)$.

Let S be an active subset of $\mho$. To quantify mutual influence of vertices within S, we define the *inner-strength* of S at t

$$\Gamma_t^{in}(S) = \sum_{v,v' \in S} \rho_{v,v'}(t). \tag{10}$$

Indeed, since $\rho_{v,v'}(t) = 0$ for $(v, v') \notin A$, $\Gamma_t^{in}(S) = \sum_{v,v' \in S, (v,v') \in A} \rho_{v,v'}(t)$, and hence $\Gamma_t^{in}(S)$ is the sum of strengths of all arcs with both end-vertices in S. It is not difficult to verify that

$$\begin{aligned}\partial^-(S) &= \bigcup_{v \in S} \partial^-(v) - \{(v, v') \in A \mid v, v' \in S\} \\ \partial^+(S) &= \bigcup_{v \in S} \partial^+(v) - \{(v, v') \in A \mid v, v' \in S\}.\end{aligned}$$

From these and using the definition (10) of $\Gamma_t^{in}(S)$, we have the following relationships for any active subset S of $\mho$ at time t:

$$\begin{aligned}\Gamma_t^-(S) &= \sum_{v \in S} \Gamma_t^-(v) - \Gamma_t^{in}(S) \\ \Gamma_t^+(S) &= \sum_{v \in S} \Gamma_t^+(v) - \Gamma_t^{in}(S).\end{aligned}$$

B. Dynamic Dominating Set

Let us first give the following definition of one vertex dominating another.

Definition 4.3: Let $\mho = (V, A)$ be an FCN, and let u and v be two vertices of $\mho$. We say that v dominates u at time t, or u is dominated by v at t, if the following two conditions are satisfied:

a) v is active at t, that is, $0 < x_v(t) \le 1$;
b) there exists an arc (v, u) of $\mho$ from v to u.

In other words, v dominates u if and only if $v \in V_t$ and $(v, u) \in A$.

Note that we require only that the dominating vertex v be active at t. The dominated vertex u can be active or inactive at t. If v is inactive at time t, it does not dominate any other vertex u. If v dominates u at t, the strength of this domination is given by $\rho_{vu}(t) = x_v(t) w_{vu}$, as defined in (2), which relies on the state $x_v(t)$ of v and the weight w_{vu} only. For example, in Fig. 1, if $x_v(t) = 0.7$ and v is active, the strength of v dominating u is: $0.7 \times 0.8 = 0.56$. However, if v is inactive at some other time t', that is, $x_v(t') = 0$, v is regarded as not dominating u at t'

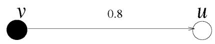

Fig. 1. Active vertex v dominating another vertex u.

even though the weight $w_{vw} = 0.8$ of the arc (v, u) remains the same.

We are now in a position to give the definition of a dynamic dominating set in an FCN.

Definition 4.4: Let $\mho = (V, A)$ be an FCN. A subset S of the vertex set of $\mho$ is called a dynamic dominating set (DDS) of $\mho$ at time $t \geq 1$ if

a) S is an active subset of $\mho$ at t;
b) each active vertex in $V - S$ at t is dominated by at least one vertex in S.

From Definitions 2.1 and 4.4, we have the following.

Theorem 4.5: For any FCN $\mho = (V, A)$ and subset S of V, the following are equivalent:

a) S is a DDS at time $t \geq 1$;
b) $S \subseteq V_t$ and S is a dominating set of the sub-FCN $\mho_t$ induced by V_t;
c) S is an active subset of $\mho$ at t such that $N_t^+(S) = V_t - S$.

From b), it follows that at any time $t \geq 1$ the FCN $\mho$ has at least one DDS, the largest being V_t, see the end of Section II-B for a related discussion.

The strength of dominance of a DDS S is measured by how strong it influences the vertices (active or not) outside S. Thus, we give the following definition.

Definition 4.6: Let S be a DDS of $\mho = (V, A)$ at time t. The strength of S dominating $\mho$ is defined to be the out-strength $\Gamma_t^+(S)$ of S, which is given by (6).

We illustrate these concepts by the following example.

Example 1: In the FCN $\mho$ shown in Fig. 2, we assume $x_i(t) = 0.8$ for $i = 3, 4$, so that v_3 and v_4 are active. One can see that v_3 dominates v_2 and v_6, and v_4 dominates v_1 and v_5. So $N^+(S) = V - S = \{v_1, v_2, v_5, v_6\}$ for $S = \{v_3, v_4\}$. Each active vertex outside S is dominated by one of the two vertices in S, although we do not know which vertices outside S are active at t. This implies that S is a DDS of $\mho$ at t. The strength of S dominating $\mho$ at t is: $\Gamma_t^+(S) = 0.8 \cdot (0.2 + 0.7) + 0.8 \cdot (0.8 + 0.5) = 1.76$.

The concept of DDS is important in many applications. For different FCNs the interpretation of domination may be different, but in general we may view vertices in a DDS of $\mho$ as the major vertices which are influential at t. For instance, in a social network, a DDS may consist of representatives (committee members) who play dominating roles in making decisions for the society, where by "dominating" we mean each nonrepresentative is accessible by at least one representative.

Usually, $\mho$ contains a number of DDS. In fact, if S is a DDS of $\mho$ at t, then every active superset S' of S in V_t (that is, $S \subseteq S' \subseteq V_t$) is also a DDS of $\mho$. In view of this, it would be desirable to find a DDS of $\mho$ with the smallest size. Such a DDS is called a *minimum DDS* of $\mho$. At the same time, in many applications it is desirable to seek a DDS with maximum possible strength. Unfortunately, the problem of finding a minimum dynamic dominating set of $\mho$ and that of finding such a set with maximum strength are both NP-complete for general FCN.

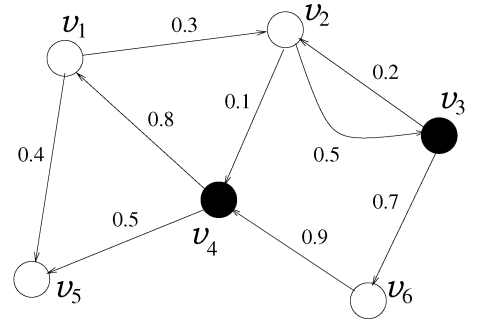

Fig. 2. Dynamic dominating set $S = \{v_3, v_4\}$ in an FCN.

This follows from the NP-completeness [1] of the problem of determining a minimum dominating set in an arbitrary directed graph. Thus, unless "$P = NP$," it would be impossible to devise a polynomial time algorithm to find a minimum DDS of $\mho$ at t or such a DDS with maximum strength.

C. Minimal Dynamic Dominating Set

On the other hand, in most applications there is no need to find a dynamic dominating set with minimum size. Instead, we are more interested in those DDS S which have no "redundancy," meaning that deleting any vertex from S will result in a nondominating set. In other words, every member in S is essential to maintaining the dominance of S. In this sense, S is minimal with respect to the property of domination. A subset S' of S is called a proper subset if $S' \neq S$.

Definition 4.7: Let $\mho = (V, A)$ be an FCN. A subset S of the vertex set of $\mho$ is called a minimal dynamic dominating set (MDDS) of $\mho$ at t if S is, but any proper subset of S is not, a DDS of $\mho$ at t.

From Theorem 4.5, we have the following corollary.

Corollary 4.8: An active set S of $\mho$ is an MDDS S of $\mho$ if and only if it is a minimal dominating set of the sub-FCN $\mho_t$ induced by V_t.

Alternatively, a DDS S is an MDDS if, for any proper subset S' of S, there exists at least one $u \in V_t - S'$ which is not dominated by any vertex of S'. Since an MDDS is a DDS, the same formula (6) can be used to calculate its strength of domination.

The concept of MDDS is very useful in modeling real-world problems. For instance, suppose that a society, e.g., the National Geographic Society, plans to send an expedition team to the Antarctic. In the first instance, the society chooses experts in the society who have the required expertise to obtain valuable information effectively. On the other hand, in order to reduce costs, the society may have to choose as *few* experts as possible. In this exercise, the society is in fact building a DDS that consists of a minimum set of currently active, leading experts in the society. This is a problem of finding a minimum DDS. Similarly, we can use MDDS to model many other real-world applications involving the placement of a minimal set of objects with maximum dominating strength to the whole FCN, such as hospitals, schools, fire stations, post offices, police stations, warehouses, service centers and so on, or even the placement of undesirable objects with minimum dominating strength to the FCN, such as toxic wastes, nuclear reactors, airports, etc.

The following theorem gives a criterion to test when a DDS is an MDDS. It is essentially the same as the result of Ore (see, e.g., [5]), although dynamics is involved in the present case. Note that condition a) that follows is satisfied if and only if $N_t^-(v) \subseteq V_t - S$, and the condition b) says that v is a unique vertex in S which dominates u.

Theorem 4.9: Let $\mho$ be an FCN, and S a DDS of $\mho$ at t. Then S is an MDDS of $\mho$ at t if and only if, for each vertex $v \in S$, one of the following conditions is satisfied:

a) $N^-(v) \subseteq V - S$;

b) there exists a vertex $u \in V_t - S$ such that $N^-(u) \cap S = \{v\}$.

Proof: Suppose that S is an MDDS of $\mho$ at t. Then, S is active at t, and for each $v \in S$, $S - \{v\}$ is not a dominating set of $\mho_t$ at t. This latter condition implies that there exists at least one vertex $u \in (V_t - S) \cup \{v\}$ not dominated by any vertex in $S - \{v\}$. If $v = u$, then this means $N_t^-(v) \subseteq V_t - S$; hence $N^-(v) \subseteq V - S$ and a) is true. If $u \in V_t - S$, then since u is dominated by S but not by $S - \{v\}$, v must be the only in-neighbor of u in S, that is, $N^-(u) \cap S = \{v\}$ and b) is true.

Conversely, suppose that S is a DDS of $\mho$ at t such that either a) or b) is true for each $v \in S$. We will prove that S must be an MDDS of $\mho$ at t. Suppose otherwise, then there exists at least one vertex $v \in S$ such that $S - \{v\}$ is also a DDS of $\mho$ at t. Thus, v is dominated by at least one vertex in $S - \{v\}$, and hence condition a) does not hold. On the other hand, since $S - \{v\}$ is a DDS of $\mho$ at t, every vertex $u \in V_t - S$ is dominated by at least one vertex in $S - \{v\}$, that is, condition b) does not hold for u. This contradiction shows that S is an MDDS of $\mho$ at t, as required. $\square$

An important problem for our domination theory is the generation of DDS and MDDS for $t \geq 1$. The following algorithm generates a DDS and an MDDS successively. Note that from every DDS we can obtain at least one MDDS just by deleting the "redundant" vertices one by one. This idea is used in the second part of the algorithm.

Algorithm 4.10 Input: An FCN $\mho = (V, A)$; **Output**: A DDS S and an MDDS T of $\mho$ at time $t \geq 1$.

1. Set $S = \emptyset$;
2. set $R = V_t - (S \cup N^+(S))$;
3. if $R = \emptyset$, then output S, set $T := S$ and go to Step 4; otherwise, choose $v \in R$ such that $|N_t^+(v)|$ is as large as possible, set $S := S \cup \{v\}$ and go to Step 2;
4. check whether there exists $u \in T$ such that $N_t^+(u) \subseteq N_t^+(T - \{u\})$ and $N^-(u) \cap T \neq \emptyset$;
5. if there exists such a vertex u, then set $T := T - \{u\}$ and go to Step 4; otherwise stop and output T.

Theorem 4.11: Let $\mho = (V, E)$ be an FCN. Let S and T be subsets of V generated by Algorithm 4.10. Then S is a DDS and T an MDDS of $\mho$ at t.

Proof: By the algorithm, S is an active subset of $\mho$ at time t since at each step an active vertex v is added up. This vertex v dominates the vertices in $N^+(v)$, and S dominates $N^+(S)$. Therefore, according to the rule in Step 3, when the algorithm outputs S, the set S dominates all vertices active at t and hence is a DDS of $\mho$ at t.

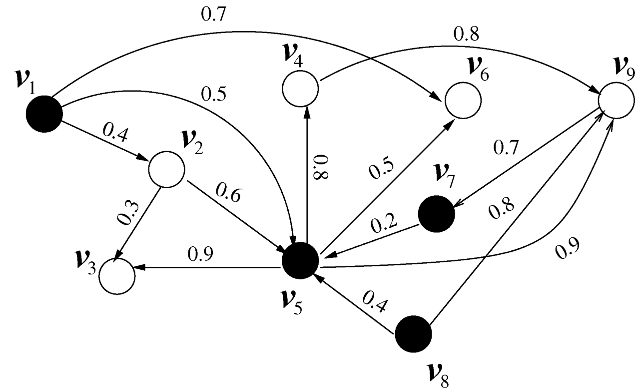

Fig. 3. Minimal dynamic dominating set $S = \{v_1, v_5, v_7, v_8\}$ in an FCN.

Now, let us consider the second part of the algorithm. If there exists $u \in T$ such that $N_t^+(u) \subseteq N_t^+(T - \{u\})$ and $N^-(u) \cap T \neq \emptyset$, then u is redundant and, by Definition 4.4, $T - \{u\}$ is also a DDS of $\mho$ at t. When the algorithm terminates, there is no redundant vertices u, and hence $T - \{u\}$ is not a DDS for any $u \in T$. In other words, the final T is an MDDS of $\mho$ at t. $\square$

Now, let us illustrate Algorithm 4.10 by the following example.

Example 2: Applying Algorithm 4.10 to the FCN $\mho$ shown in Fig. 3, we can obtain an MDDS of $\mho$ at t by the following procedure. We assume that v_1, v_5, v_7, v_8 are active at t with $x_1(t) = 0.8$ and $x_i(t) = 0.6$ for $i = 5, 7, 8$, and all other vertices are inactive at t. Setting $S = \emptyset$ initially, we have $N^+(S) = \emptyset$ and $R = V_t \neq \emptyset$. So we choose, say $v_1 \in R$, and add it to S. Thus, in the next round we have $S = \{v_1\}$ and, hence, $R = \{v_7, v_8\} \neq \emptyset$ as $N_t^+(v_1) = \{v_5\}$. Now we choose, say v_7, and the current S then becomes $S = \{v_1, v_7\}$. For this S, we have $R = \{v_8\} \neq \emptyset$, and so finally we put v_8 into S, so that the next S is $\{v_1, v_7, v_8\}$ and the next R becomes empty. So we output $S = \{v_1, v_7, v_8\}$ which is a DDS of $\mho$ at t. Setting $T = S$, one can check that, for any $u \in T$, $N_t^+(u)$ is not a subset of $N_t^+(T - \{u\})$. So $T = \{v_1, v_7, v_8\}$ is an MDDS of $\mho$ at t. The strength of T dominating $\mho$ is $\Gamma_t^+(T) = 0.8 \cdot (0.7 + 0.5 + 0.4) + 0.6 \cdot 0.2 + 0.6 \cdot (0.4 + 0.8) = 2.12$.

D. Dynamic Dominating Process

A very important feature of FCN is its dynamic behaviors, which motivates us to introduce the concept of *dynamic dominating process*. Let us begin with the following example. In a limited-liability company, the share holders with large shares at time t form a dominating set that plays a dominating role in the operation, capital turnover and decision making of the company. As the time goes on, the turbulent stock market frequently changes the members in the so-called major players club, which means that someone who belongs to the DDS at t may be driven out of this subset at time $t + 1$, and at the same time there may be new comers joining the DDS. This dynamic nature of DDS is true and reflects many real scenarios. In general, since the vertex states are updated automatically when the vertices receive a series of external input stimuli, a DDS of $\mho$ at t may become a non-DDS at $t+1$. On the other hand, a non-DDS at t may become a DDS at $t+1$. Therefore, in general we may describe the domination in an FCN by a dynamic dominating process that consists of a series of DDS at different $t \geq 0$. The starting point of this process is the minimal dominating set of $\mho$ used to set the initial condition. Formally, we give the following definition of this process.

Definition 4.12: In an FCN, $\mho = (V, A)$, a dynamic dominating process (DDP) is a sequence

$$(S_t : t = 0, 1, 2, \ldots,) = (S_0, S_1, S_2, \ldots)$$

where S_0 is the minimal dominating set of $\mho$ used to set the initial condition, as in Section III-A, and for $t \geq 1$, S_t is a DDS of $\mho$ at t generated by using a given set (possibly empty) of rules and is usually connected with S_{t-1}.

When studying this dynamic dominating process, we can regard S_t as an instantaneous dominating set; that is, S_t can be regarded as transversal of the process at t. Understanding this kind of *instantaneous* dominating set and its properties will enable us to study the behavior of this dynamic dominating process and its development trend with t. This will help us predict more effectively the future from the current events or make reasonable decisions in complex, dynamic situations.

The rules governing the generation of the DDP are set according to the nature and goal of the problem, and they determine to a certain degree the transition from S_{t-1} to S_t. Thus, setting rules properly for a given FCN is a very important issue in the study of DDP. The rules used can be deterministic, stochastic, or a combination of them. In the following, we propose a few rules for generating DDP. The first one is the following basic deterministic rule that arises from the requirement of irredundancy at any time.

Rule 4.13: Choose S_t to be an MDDS of $\mho$ for $t \geq 1$.

We call a DDP generated by using this rule a minimal dynamic dominating process.

Definition 4.14: A dynamic dominating process $(S_t : t = 0, 1, 2, \ldots,)$ of an FCN $\mho$ is called a minimal dynamic dominating process (MDDP) if S_t is an MDDS of $\mho$ for $t \geq 1$.

Rule 4.13 can be used in combination with other rules, which we will give in the following, to generate DDP $(S_t : t = 0, 1, 2, \ldots)$. Note first that, in Rule 4.13, S_t is irrelevant to the previous S_{t-1}. This may, however, not be true in many applications. For example, an MDDS can be interpreted as the board of trustees of an organization. From time to time the board needs to be updated to reflect the up-to-date personnel status and the operation of the organization. In each change, the board may retain as many trustees as necessary to maintain the stability in the organization. This suggests the following rule, which is important and useful.

Rule 4.15: Choose S_t to be an MDDS of $\mho$ at each time $t \geq 1$ such that it has the maximum overlap with S_{t-1}, that is, $|S_{t-1} \cap S_t|$ is maximized.

The next rule, based on a similar idea, is to retain all vertices of S_{t-1} that are active at t. Note that in this way S_t may not be an MDDS of $\mho$, in other words, the DDP generated by this rule may not be an MDDP.

Rule 4.16: Choose S_t to be a DDS of $\mho$ at each time $t \geq 1$ such that $S_{t-1} \cap V_t \subseteq S_t$.

Naturally, one may choose an MDDS each time with the maximum possible strength. This leads to the following rule.

Rule 4.17: Choose S_t to be an MDDS of $\mho$ at each time $t \geq 1$ such that it has the maximum strength $\Gamma_t^+(S_t)$.

In the previous rules, the choice of S_t is not unique. Usually there are many candidates from which S_t can be chosen. We may choose S_t randomly among all legal candidates, and this involves the usage of stochastic rule. A basic rule of this kind is the following one, in which by "uniformly at random" we mean each legal candidate has the same probability to be chosen. For instance, if there are eight legal candidates at t, then each of them has the same chance (1/8) of being chosen as S_t.

Rule 4.18: Choose S_t uniformly at random from all MDDS of $\mho$ at $t \geq 1$; in general choose S_t uniformly at random from all legal candidates that have been generated by a set of deterministic rules.

Depending on the set of rules used, we have different types of DDP and MDDP. For instance, if we choose S_t uniformly at random from all MDDS S of $\mho$ with $|S_{t-1} \cap S|$ as large as possible (combination of Rules 4.15 and 4.18), then we get a "random maximum overlapping" MDDP $(S_t : t = 0, 1, 2, \ldots,)$. In practice we may generate DDP and MDDP by using a given order of priorities. For example, in forming a committee we may set the first priority as being an MDDS, the second one as having maximum strength, and the third one as having maximum inner-strength (see (10) for definition), and so on. So we first choose MDDS of $\mho$ at t, and among them we use the ones with maximum possible strength. If this determines S_t uniquely, we are done; otherwise, among the MDDS with maximum strength, we choose S_t with maximum inner-strength. If S_t can be determined uniquely, we are done; otherwise, check the next priority. Continue this procedure until S_t can be determined, or all priorities have been used but we still have a bunch of candidates remaining. In the latter case we may resort to the stochastic rule described in Rule 4.18.

There are many problems in dealing with DDP and MDDP, which will be the subjects of our future research. In this paper, however, we present the following theorem in a simple FCN. (An FCN is simple if it contains no directed cycles, see the paragraph following Theorem 3.3.)

Theorem 4.19: Let $\mho = (V, A)$ be a simple FCN. Then there exists a DDP $(S_t : t = 0, 1, 2, \ldots,)$ of $\mho$ such that S_t is an independent set of $\mho$ for each $t \geq 0$.

Proof: Since $\mho$ is simple, by Theorem 3.4(a) there exists an $S_0 \subseteq V$ such that S_0 is a dominating as well as independent set of $\mho$. To start the process, we choose such an S_0 as the initial condition. For $t \geq 1$, $\mho_t$ is also a simple FCN. (In general, any subgraph of an acyclic, directed graph is necessarily acyclic.) Thus, again by using Theorem 3.4(a), we can find $S_t \subseteq V_t$ such that it is an independent dominating set of $\mho_t$. Since $\mho_t$ is an induced directed subgraph of $\mho$, S_t must be an independent set of $\mho$ also. Thus, $(S_t : t = 0, 1, 2, \ldots,)$ is a DDP of $\mho$ with desired property. $\square$

V. EFFECTIVE DYNAMIC DOMINATING SET

FCNs are tools for decision support and causal discovery [13]. Complex real-world applications require FCN to have the capability to learn, to reason, to adjust, and to react in the way that is consistent with the way we carry out our daily routines. In the literature, this goal is usually approached by estimating impact of an initial condition on the whole FCN and by collecting feedback [6]. Feedback enables the system to adjust (adapt) itself in response to the changing environment and to the information about the given goals and actual outcomes. In Section III, we proposed a new inference procedure for setting initial conditions. In this section, we will focus on DDS with *extended feedback*, and introduce effective dynamic dominating set. Moreover, we will reveal an interesting phenomenon, which we call *revolution* of vertex states.

A. Extended Feedback

Let us first review briefly the definitions of feedback and extended feedback. Roughly speaking any directed cycle $(v_0, v_1, \ldots, v_{r-1}, v_0)$ defines a piece of *feedback* for v_0, meaning that v_0 receives feedback via consecutive effects of v_0 on v_1, v_1 on $v_2, \ldots, v_{r-2}$ on v_{r-1}, and finally v_{r-1} on v_0. This is the feedback received by one particular vertex. In studying the feedback received by a subset S of V, it is reasonable to count not only cycles but also paths with both starting and terminating vertices in S. This leads to the concept of extended feedback, which improves the inference and representation capabilities of FCNs.

Definition 5.1: Let $\mho = (V, A)$ be an FCN, and S a subset of V. Let $P = (v_0, v_1, \ldots, v_r)$ be a directed path or cycle (in the case where $v_0 = v_r$) of $\mho$. We call P a piece of *extended feedback* of $\mho$ for S if $r \geq 2$, $v_0, v_r \in S$ and $v_i \in V - S$ for $i = 1, 2, \ldots, r-1$.

We require $r \geq 2$, since otherwise P will reduce to an arc within S and obtain no feedback from outside S. In the special case where $S = \{u\}$ is a singleton, we must have $v_0 = v_r = u$; hence P is a directed cycle and defines a piece of feedback for v_0 in the usual sense. For $P = (v_0, v_1, \ldots, v_r)$ as in the previous definition, the *indirect effect* of v_0 on v_r at t via P is defined [13] as

$$I_P(t) = I_{(v_0, v_1, \ldots, v_r)}(t) = \prod_{i=1}^{r} \rho_{v_{i-1}v_i}(t).$$

Note that only when all v_i are active at t, for $i = 0, 1, \ldots, r-1$, is there a nonzero effect $I_P(t)$ of v_0 on v_r via P, since in the opposite case one of the factors $\rho_{v_{i-1}v_i}(t)$ is equal to 0 and hence $I_P(t) = 0$.

B. Effective Dynamic Dominating Set

In this section, we will study dynamic dominating sets S with extended feedback. We first prove the following theorem, which provides a necessary and sufficient condition for the existence of a piece of extended feedback for S with nonzero effect.

Theorem 5.2: Let $\mho = (V, A)$ be an FCN and S a DDS of $\mho$ at t. There exists a piece P of extended feedback for S with nonzero effect $I_P(t)$ if and only if $\partial_t^-(S) \neq \emptyset$.

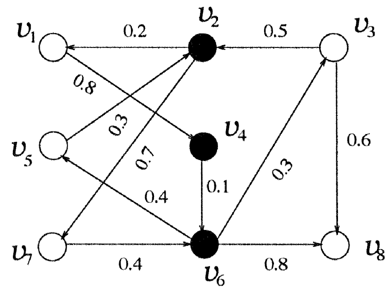

Fig. 4. Effective dynamic dominating set $S = \{v_2, v_4, v_6\}$ of an FCN.

Proof: Suppose there exists a piece P of extended feedback for S at t with $I_P(t) \neq 0$. Let $P = (v_0, v_1, \ldots, v_r)$. Then $v_0, v_r \in S$ and $v_1, v_2, \ldots, v_{r-1} \in V - S$. Since $I_P(t)$ is nonzero, from the discussion above, all v_i must be active at t for $i = 0, 1, \ldots, r-1$. In particular, we have $v_1 \in V_t - S$. Thus v_1 is dominated by some vertex u in S at t, because S is a DDS of $\mho$ at t. From this we have $(v_1, u) \in \partial_t^-(S)$ and $\partial_t^-(S) \neq \emptyset$.

Conversely, suppose $\partial_t^-(S) \neq \emptyset$ and $(v, u) \in \partial_t^-(S)$. Then, $v \in S$ and $u \in V_t - S$ by the definition of $\partial_t^-(S)$. On the other hand, since S is a DDS of $\mho$ at t, there exists $v' \in S$ which dominates u, that is, $(v', u) \in A$. (It may happen that $v' = v$, but the proof goes the same way.) Consequently, (v', u, v) is a path or cycle (if $v' = v$) of length 2. Note that v, v', u are all active at t, and that $\rho_{v'u}(t) \neq 0$ and $\rho_{uv}(t) \neq 0$ by the convention made in Section II-A (see footnote 2). Now we have $I_{(v',u,v)}(t) = \rho_{v'u}(t)\rho_{uv}(t) \neq 0$, and the path (v', u, v) defines a piece of extended feedback for S at t with nonzero effect. □

Definition 5.3: Let $\mho = (V, A)$ be an FCN and S be a DDS of $\mho$ at t. If $\partial_t^-(S) \neq \emptyset$, then we call S an effective dynamic dominating set (EDDS) of $\mho$ at t.

In view of Theorem 5.2, an EDDS can be defined equivalently as a DDS with extended feedback of nonzero effect. We can imagine an EDDS as a group of decision makers who, as a whole, receive feedback from those outside the group. Since an EDDS is required to be a DDS, of course its strength of domination is given by $\Gamma_t^+(S)$, as in Definition 4.6. The condition $\partial_t^-(S) \neq \emptyset$ above is equivalent to the existence of an active vertex u in $V - S$ such that u dominates a vertex in S at time t; that is, not only does S dominate $V - S$, but also S receives feedback from $V - S$ as well. For example, in the FCN shown in Fig. 4, $S = \{v_2, v_4, v_6\}$ is an EDDS, if we assume that all the vertices in the FCN are active at t. In fact, from Definition 4.4 one can see that S is a DDS, because v_1, v_7 are dominated by v_2 and v_3, v_5, v_8 are dominated by v_6. Note that S receives feedback from $V - S$ via (v_1, v_4) or (v_3, v_2). So S is an EDDS of $\mho$ at t by Definition 5.3. Assuming $x_i(t) = 0.9$ for $i = 2, 4, 6$, the strength of S dominating the FCN is then $\Gamma_t^+(S) = 2.25$.

In the case where all weights w_{uv} are nonnegative for $u, v \in V$, we say that the weight matrix $W_\mho$ is *nonnegative*. In this case we can prove that, for a DDS S of $\mho$ at t, $\partial_t^-(S) \neq \emptyset$ if and only if $\Gamma_t^-(S) > 0$. Note that the "only if" part is not guaranteed if both positive and negative weights are presented on arcs of $\mho$.

Theorem 5.4: Let $\mho = (V, A)$ be an FCN with nonnegative weight matrix. Then a DDS S of $\mho$ at t is an EDDS of $\mho$ at t if and only if $\Gamma_t^-(S) > 0$.

In the following, we will discuss the relationship between DDS and EDDS. We prove first the following theorem.

Theorem 5.5: Let $\mho = (V, A)$ be an FCN and S a DDS of $\mho$ at t. Then one and only one of the following is true.

a) S is an EDDS of $\mho$, or $S \cup \{u\}$ is an EDDS of $\mho$ at t for some $u \in V_t - S$.
b) $N^+(u) \subseteq V - V_t$ for all $u \in V_t - S$.

Proof: Suppose b) is not true. Then there exists $u \in V_t - S$ such that $N^+(u) \cap V_t \neq \emptyset$; that is, (u, v) is an arc of $\mho_t$ for some $v \in V_t$. By definition, if $v \in S$, then S is an EDDS of $\mho$ at t; if $v \in V_t - S$, then $S \cup \{v\}$ is an EDDS of $\mho$ at t. Thus, a) is true if b) is not. On the other hand, if b) is true, then there is no arc from $V_t - S$ to S or from $V_t - (S \cup \{u\})$ to $S \cup \{u\}$ for any $u \in V_t - S$; hence neither S nor $S \cup \{u\}$ is an EDDS of $\mho$. Thus, a) and b) cannot occur at the same time. □

We point out that, for an arbitrary FCN $\mho$, there may exist no EDDS of $\mho$ at some time t. The following theorem provides necessary and sufficient conditions for the existence of an EDDS at t.

Theorem 5.6: For any FCN $\mho = (V, A)$, the following conditions are equivalent.

a) There exists an EDDS of $\mho$ at t.
b) There exists a vertex u of $\mho$ active at t such that $N_t^+(u) \neq \emptyset$ and $N_t^-(u) \neq \emptyset$.
c) The sub-FCN $\mho_t$ induced by V_t contains a directed path or cycle of length 2.

Proof: Suppose a) is true and S is an EDDS of $\mho$ at time t. Then by definition S is a DDS at t, and there exists an arc (u, v) in $\mho$ such that $u \in V - S$, $v \in S$ and u is active. Since S dominates $\mho$, u is dominated by a vertex v' in S. (It may happen that v' coincides with v.) So we have $v \in N_t^+(u)$, $v' \in N_t^-(u)$; hence, both sets are nonempty and b) is true.

Suppose b) is true, and let, say, $v \in N_t^+(u)$ and $v' \in N_t^-(u)$. Then (v', u, v) is a path or cycle (if $v' = v$) in $\mho_t$ with a length of 2. This means that c) is true.

Finally, suppose c) is true, and let (v', u, v) be a path or cycle (if $v' = v$) in $\mho_t$ with length of 2. Then $V_t - \{u\}$ is an EDDS, thus a) is true. This completes the proof. □

Let us define $\ell_\mho(t)$ to be the length of a longest directed path or cycle in $\mho_t$. Then condition c) in Theorem 5.6 is equivalent to saying that $\ell_\mho(t) \geq 2$. If this is satisfied, then by this theorem an EDDS of $\mho$ at t exists. The following theorem shows further that, if $\ell_\mho(t) \geq 3$, then we can obtain an EDDS of $\mho$ at t based on any MDDS of $\mho$ at t.

Theorem 5.7: Let $\mho = (V, A)$ be an FCN. Suppose $\ell_\mho(t) \geq 3$. Then, we can construct an EDDS of $\mho$ at t from any MDDS S of $\mho$ at t in the following way.

a) If $N^-(S) \cap (V_t - S) \neq \emptyset$, then S is an EDDS of $\mho$ at t.
b) If there exists a vertex $u \in V_t - S$ such that $N^-(u) \cap (V_t - S) \neq \emptyset$, then $S \cup \{u\}$ is an EDDS of $\mho$ at t.
c) If neither of the previous conditions is satisfied, then there exists $v \in S$ such that $(S - \{v\}) \cup U$ is an EDDS of $\mho$ at t, where $U = \{u \in V_t - S \mid N^-(u) \cap S = \{v\}\}$ is the set of vertices of $V_t - S$ having v as their unique in-neighbor in S.

Proof: Since $\ell_\mho(t) \geq 3$, by Theorem 5.6 $\mho$ has EDDS at time t. If $N^-(S) \cap (V_t - S) \neq \emptyset$, then $\partial_t^-(S) \neq \emptyset$ and hence S is an EDDS of $\mho$ at t. If $N^-(u) \cap (V_t - S) \neq \emptyset$ for some $u \in V_t - S$, say $u' \in N^-(u) \cap (V_t - S)$, then $\partial_t^-(S \cup \{u\})$ is nonempty since it contains the arc (u', u). Also,since S is a DDS of $\mho$ at t, so is $S \cup \{u\}$. Hence, $S \cup \{u\}$ is an EDDS of $\mho$ at t. In the following, we assume that the condition in neither a) nor b) is satisfied. Then, there is no any arc of $\mho$ from $V_t - S$ to S, and $V_t - S$ is an independent set of $\mho$. In other words, all vertices of $V_t - S$ have out-degree 0 in the sub-FCN $\mho_t$. From this and the assumption $\ell_\mho(t) \geq 3$, one can show that there exists a path (u, v, w) of $\mho_t$ with $u, v, w \in S$. In particular, this implies $N^-(v) \nsubseteq V - S$. Since S is an MDDS, from Theorem 4.9 it follows that the set $U = \{u \in V_t - S \mid N^-(u) \cap S = \{v\}\}$ is nonempty. Also, v is dominated by u, and by the definition of U each vertex in $V_t - S - U$ is dominated by at least one vertex in $S - \{v\}$. Hence, $(S - \{v\}) \cup U$ is a DDS of $\mho$ at t. Moreover, it is an EDDS since $\partial_t^-((S - \{v\}) \cup U)$ contains at least one arc, namely (v, w). □

Based on Theorem 5.7 we may develop an algorithm for finding an EDDS of $\mho$ at t if the condition $\ell_\mho(t) \geq 3$ is satisfied. In a lot of cases the EDDS obtained in this way may be close to "minimal"—in case a) of Theorem 5.7, S itself is an MDDS, and in b) we obtain an EDDS $S \cup \{u\}$ which has only one more vertex than the MDDS S. This is the advantage of the method. To start with we will need an MDDS of $\mho$ at t, which can be generated by using Algorithm 4.10. Note that cases a) and b) in Theorem 5.7 are not mutually exclusive.

C. Revolution: Sudden Change of Dominant Members

An *undirected path* of an FCN, $\mho = (V, A)$, is a sequence $v_0, v_1, \ldots, v_r$ of vertices of $\mho$ such that for each $i = 1, \ldots, r$ either (v_{i-1}, v_i) or (v_i, v_{i-1}) is an arc of $\mho$. A *connected component* of $\mho$ is a maximal subgraph of $\mho$ in which any two vertices are joined by an undirected path. If any two vertices of $\mho$ are joined by an undirected path, then $\mho$ has only one component; in this case $\mho$ is called *connected*. Without loss of generality *we* may always assume that the FCN $\mho$ we are dealing with is connected, because otherwise we may study each component individually (see [17, Sec. 3.4] for a more detailed explanation). In this section we will adopt this assumption. To exclude trivial cases we will also assume that $\mho$ has at least two arcs. Under these assumptions, $\mho$ contains a directed path or cycle of length at least 2. So, by a similar argument as in the proof of Theorem 5.6, one can prove that $\mho$ contains a dominating set S_0 with extended feedback. This set S_0 can be used to set the initial condition. We will also use it as the starting point of our effective dynamic dominating process, which is defined as follows.

Definition 5.8: Let $\mho = (V, A)$ be a connected FCN with at least two arcs. An effective dynamic dominating process (EDDP) is a sequence $(S_t : t = 0, 1, 2, \ldots,)$, where S_0 is a dominating set of $\mho$ with extended feedback setting the initial condition, and $S_t (t \geq 1)$ is an EDDS of $\mho$ at t whenever such an EDDS exists or a minimum DDS of $\mho$ at t otherwise.

We emphasize that, at some time t an EDDS of $\mho$ may not exist, so the second possibility for S_t in the definition above may occur. Moreover, if $\mho$ has no EDDS, then S_t is unique since $\mho$ has only one MDDS (hence one minimum DDS) by our next

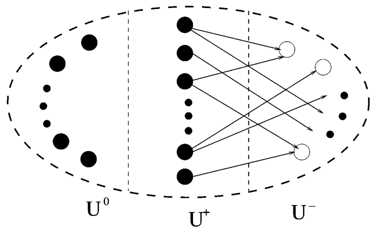

Fig. 5. Structure of the sub-FCN $\mho_t$ when a revolution occurs.

theorem. Furthermore, in this case $\mho_t$ has very special structure. For a vertex $v \in V_t$, we use

$$d_t^-(v) = \left|N_t^-(v)\right| \quad d_t^+(v) = \left|N_t^+(v)\right|$$

to denote the in-degree and out-degree of v in $\mho_t$, respectively.

Theorem 5.9: Let $\mho = (V, A)$ be a connected FCN with at least two arcs. Then $\mho$ contains no EDDS at time t if and only if the set V_t of active vertices at t can be partitioned into three parts, namely $V_t = U^0 \cup U^+ \cup U^-$, where

$$U^0 = \left\{v \in V_t \mid d_t^-(v) = d_t^+(v) = 0\right\}$$
$$U^+ = \left\{v \in V_t \mid d_t^-(v) = 0, d_t^+(v) > 0\right\}$$
$$U^- = \left\{v \in V_t \mid d_t^+(v) = 0, d_t^-(v) > 0\right\}$$

which are all independent sets of $\mho$. Thus, $\mho_t$ has the special structure depicted in Fig. 5, and all arcs of $\mho_t$ are from U^+ to U^-. Furthermore, $\mho$ has a unique MDDS, namely $U^0 \cup U^+$.

Proof: By Theorem 5.6, $\mho$ contains no EDDS at time t if and only if, for each $v \in V_t$, at least one of $d_t^-(v)$ and $d_t^+(v)$ is zero. This is equivalent to saying that V_t can be partitioned into U^0, U^+ and U^- as required. The set U^0 consists of all vertices of $\mho_t$ without either in-coming or out-going arcs. So U^0 is an independent set of $\mho_t$ and, hence, an independent set of $\mho$ since $\mho_t$ is an induced subgraph of $\mho$. The vertices of U^+ have no in-coming arcs in $\mho_t$, and hence there is no arc of $\mho$ between any two vertices of U^+. In other words, U^+ is an independent set of $\mho$. Similarly, U^- is an independent set of $\mho$. Hence all the arcs of $\mho_t$ are from U^+ to U^-, and $\mho_t$ has the structure depicted in Fig. 5. Moreover, $U^0 \cup U^+$ must be contained in any dominating set of $\mho_t$, since each vertex of $U^0 \cup U^+$ has in-degree 0 and, hence, is not dominated by any vertex in $\mho_t$. On the other hand, each vertex of U^- is dominated by at least one vertex of U^+. Hence, $U^0 \cup U^+$ is a minimal dominating set of $\mho_t$, that is, an MDDS of $\mho$ at t. Moreover, since each vertex of U^- does not dominate any vertex, any minimal dominating set of $\mho_t$ is disjoint from U^-. From this, it follows that $U^0 \cup U^+$ is the unique MDDS of $\mho$ at t. □

We can summarize the previous discussion as follows: At $t \geq 1$, if $\mho$ has no EDDS, then $\mho_t$ must have the special structure as shown in Fig. 5. In this case, the t-th term of the EDDP $(S_t : t = 0, 1, 2, \ldots,)$ must be $S_t = U^0 \cup U^+$, which is the unique MDDS of $\mho$ at t. Also, for each $v \in S_t$, the total input $y_v(t)$ [see (3)] received by v is 0 since there is no active in-coming arc to v. (There may be in-coming arcs to v from inactive vertices, but they contribute 0 to $y_v(t)$.) Thus, all vertices in S_t will become inactive at $t+1$ and are driven out of S_{t+1}. In other words, $S_{t+1} \cap S_t = \emptyset$. Intuitively, this means that all the dominant members in the set at time t (the vertices in S_t) lose their dominance all of a sudden at $t+1$. We call this interesting phenomenon a *revolution* in vertex states, which is stated formally in the theorem below.

Theorem 5.10: Let $\mho = (V, A)$ be a connected FCN with at least two arcs. Then at any time $t \geq 1$ either there exists an EDDS of $\mho$ or there is a revolution. Thus, for any EDDP $(S_t : t = 0, 1, 2, \ldots,)$ of $\mho$, either S_t is an EDDS of $\mho$ at t, or there is a revolution at t. In the latter case we have $S_t = U^0 \cup U^+$, $S_{t+1} \cap S_t = \emptyset$ and $\mho_t$ has the special structure as described previously.

Identifying revolution is an important issue because it predicts when dramatic changes of the FCN will happen. This problem is solved by Theorem 5.9, in which a necessary and sufficient condition is given in terms of the structure of $\mho_t$. A number of problems relating to vertex state revolution remains open and deserves further investigation. For example, under what conditions will revolution occur, when will the first revolution appear, what is the behavior of the FCN when a revolution occurs, and so on.

VI. Illustrative Example

In this section, we will apply the new inference procedure proposed in Section III-A to a simplified application, and compare this procedure with the conventional method of setting initial conditions. This example will also be used to illustrate the concepts of EDDS and MDDS. However, in order to keep this paper in a reasonable length we will not be able to simulate (minimal, effective) dynamic dominating processes.

Let us construct an FCN for the department of computer science at a university, as shown in Fig. 6. To achieve excellence in education and research, the decision makers of the department must develop policies and measure their impact on the department. Furthermore, they have to modify or adjust their policies from time to time based on the feedback and extended feedback collected from the staff. We have conducted two simulations to investigate the functionality of the FCN. The simulation results demonstrate the following advantages of the new inference procedure. First, the inference pattern obtained by our inference procedure is more reliable and reasonable than that obtained by the conventional inference procedure [7], [9], [12], [13], [17]. Second, the new inference procedure converges more quickly to its final stable state than the conventional inference procedure. Finally, the concept of EDDS proposed in the previous section further improves the inference and representation capabilities of the FCN.

Denote by $\mho$ the FCN above and v_i the i-th staff member of the department. Associated with v_i is a fuzzy event that varies with time. The vertex state $x_i(t) \in [0, 1]$ of v_i at time t specifies the fuzzy degree of v_i at t. We assume that the department has 12 staff members with one leading expert in each of the following areas: computer network, database and computer vision. As shown in Fig. 6, we assume that the weight matrix of $\mho$ is given by the equation shown at the bottom of the next page, where $w_{ij} \in [0, 1]$ represents the degree that v_i influences v_j on department's policies. In the following simulations, the state transition functions f_i are threshold functions with thresholds $T_i = 0.5$ $(i = 1, \ldots, 12)$ for all vertices involved.

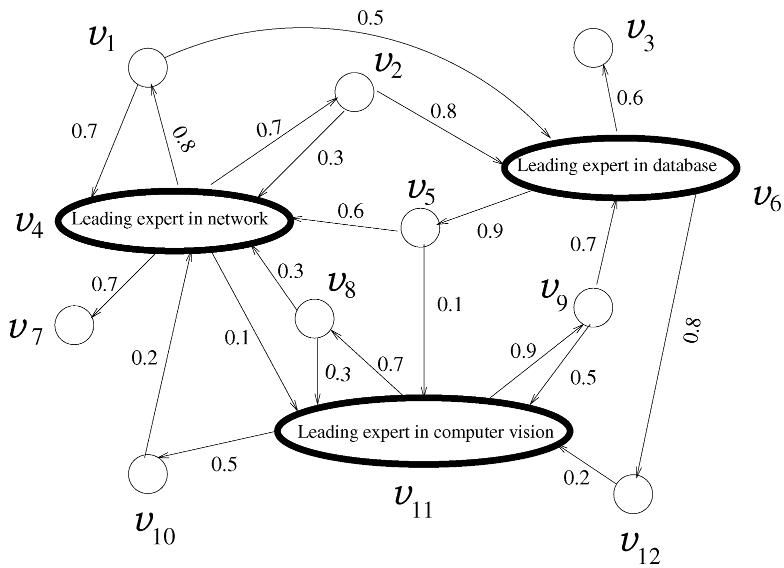

Fig. 6. FCN of a computer science department.

A. Simulation I: Setting Initial Condition at Random

In the first simulation, we choose randomly one vertex of $\mho$ and let it be active. We then estimate the impact of such an initial condition on the whole FCN by calculating its causal inference pattern. In doing so, we simply keep $x_6(t) = 1$ during the inference cycle and indicate this as $\boxed{\mathbf{1}}$, which means that only the expert v_6 is fully activated by the external input at any time t. The inference pattern changes as follows:

$$
\begin{aligned}
\mathbf{x}_\mho(0) &= \left(0,0,0,0,0,\boxed{\mathbf{1}},0,0,0,0,0,0\right)\\
\mathbf{x}_\mho(0)\cdot W_\mho &= (0,0,0.6,0,0.9,0,0,0,0,0,0,0.8)\\
\mathbf{x}_\mho(1) &= f\left(\mathbf{x}_\mho(0)\cdot W_\mho\right)\\
&= \left(0,0,1,0,1,\boxed{\mathbf{1}},0,0,0,0,0,1\right)\\
\mathbf{x}_\mho(1)\cdot W_\mho &= (0,0,0.6,0.6,0.9,0,0,0,0,0,0.3,0.8)\\
\mathbf{x}_\mho(2) &= f\left(\mathbf{x}_\mho(1)\cdot W_\mho\right)\\
&= \left(0,0,1,1,1,\boxed{\mathbf{1}},0,0,0,0,0,1\right)\\
\mathbf{x}_\mho(2)\cdot W_\mho &\\
&= (0.8,0.7,0.6,0.6,0.9,0,0.7,0,0,0,0.4,0.8)\\
\mathbf{x}_\mho(3) &= f\left(\mathbf{x}_\mho(2)\cdot W_\mho\right)\\
&= \left(1,1,1,1,1,\boxed{\mathbf{1}},1,0,0,0,0,1\right)\\
\mathbf{x}_\mho(3)\cdot W_\mho &= (0.8,0.7,0.6,1.6,0.9,1.3,\\
&\qquad 0.7,0,0,0,0.4,0.8)\\
\mathbf{x}_\mho(4) &= f\left(\mathbf{x}_\mho(3)\cdot W_\mho\right)\\
&= \left(1,1,1,1,1,\boxed{\mathbf{1}},1,0,0,0,0,1\right).
\end{aligned}
$$

Note that $\mathbf{x}_\mho(4) = \mathbf{x}_\mho(3)$, based on which we know $\mathbf{x}_\mho(t) = \mathbf{x}_\mho(3)$ for $t \geq 3$. From the expression of $\mathbf{x}_\mho(3)$ we have $x_1(3) = x_2(3) = x_3(3) = x_4(3) = x_5(3) = x_7(3) = x_{12}(3) = 1$, which means that $\{v_1, v_2, v_3, v_4, v_5, v_7, v_{12}\}$ is the set of active staff members at the final state at time $t = 3$. In other words, with the initial activation of v_6 and after *three* rounds of inference, v_4 (another leading expert in network) and $v_1, v_2, v_3, v_4, v_5, x_7, v_{12}$ (other six remaining staff members) are gradually activated directly or indirectly by the department's policies.

$$
W_\mho = \begin{pmatrix}
0 & 0 & 0 & 0.7 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0.3 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.8 & 0.7 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0.1 & 0\\
0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0\\
0 & 0 & 0.6 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0\\
0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0.5 & 0\\
0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0.9 & 0.5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0
\end{pmatrix}
$$

B. Simulation II: Setting a Dominating Set Active

Now, we calculate causal inference pattern of the same FCN in the following way. First, we choose a minimal dominating set S of $\mho$ and keep the vertices in S active. For example, we may choose $S = \{v_4, v_6, v_{11}\}$, which is a minimal dominating set of $\mho$ in Fig. 6, and set $x_4(t) = x_6(t) = x_{11}(t) = 1$ for $t \geq 0$. That is, we keep S active during the whole inference process, which is indicated by $\boxed{1}$:

$$\begin{aligned}
\mathbf{x}_{\mho}(0) &= \left(0, 0, 0, \boxed{1}, 0, \boxed{1}, 0, 0, 0, 0, \boxed{1}, 0\right) \\
\mathbf{x}_{\mho}(0) \cdot W_{\mho} &= (0.8, 0.7, 0.6, 0, 0.9, 0, 0.7, 0.7, \\
&\qquad 0.9, 0.5, 0.1, 0.8) \\
\mathbf{x}_{\mho}(1) &= f\left(\mathbf{x}_{\mho}(0) \cdot W_{\mho}\right) \\
&= \left(1, 1, 1, \boxed{1}, 1, \boxed{1}, 1, 1, 1, 1, \boxed{1}, 1\right) \\
\mathbf{x}_{\mho}(1) \cdot W_{\mho} &= (0.8, 0.7, 0.6, 2.1, 0.9, 2.0, 0.7, 0.7, \\
&\qquad 0.9, 0.5, 1.2, 0.8)\,. \\
\mathbf{x}_{\mho}(2) &= f\left(\mathbf{x}_{\mho}(1) \cdot W_{\mho}\right) \\
&= \left(1, 1, 1, \boxed{1}, 1, \boxed{1}, 1, 1, 1, 1, \boxed{1}, 1\right).
\end{aligned}$$

Since $\mathbf{x}_{\mho}(2) = \mathbf{x}_{\mho}(1)$, $\mathbf{x}_{\mho}(t) = \mathbf{x}_{\mho}(1)\ \forall\ t \geq 1$; that is, the FCN reaches its equilibrium state in only *one* step. From $\mathbf{x}_{\mho}(1)$ we have $x_i(1) = 1 (i = 1, \ldots, 12)$, which indicates that all staff members in the department are now active after only *one* inference cycle.

C. Comparison and Discussion

From the inference pattern obtained in Section VI-A, we can see that when v_6 receives an external input from the department at t, v_3, v_5, v_{12} are activated through the first inference cycle. After the second inference cycle, v_4, v_7 are activated. Finally, v_1, v_2 are activated through the third inference cycle. Thus, for the conventional inference procedure employed in Simulation I we need three inference cycles to obtain the inference pattern. In contrast, Simulation II converges quickly to its final equilibrium state (after only *one* inference cycle) and moreover the system is able to take all members' concerns into account.

In Simulation I, we notice that v_6 receives only two pieces of feedback: $(v_6, v_5, v_4, v_1, v_6)$ and $(v_6, v_5, v_4, v_2, v_6)$. This can be explained as follows: in the final equilibrium state, although the majority of staff members are gradually activated by the initial condition directly or indirectly, all other staff members are still inactive. As a result, there is no active directed path to transport the responses from these activated staff members. Therefore, the decision makers cannot receive feedback from individual staff members if the department gives an external input to one leading expert only. This is obviously undesirable in most decision-making or decision-adjusting processes.

For the second simulation, since $S_t = \{v_4, v_6, v_{11}\}$ satisfies the two conditions in Definition 5.3, it is an EDDS of $\mho$ at t. In addition, it also satisfies the conditions in Definitions 4.4 and 4.7, which means that it is an MDDS as well. In the final equilibrium state, every vertex in S_t receives feedback. Table I summarizes all the feedback received by the vertices in S_t along with the relevant information. From the table we can see that three vertices in S_t receive 28 pieces of feedback in total. We also notice that all vertices in S_t receive extended feedback. Table II summarizes all extended feedback received by these vertices and gives related information.

From Tables I and II, we can see that the three leading experts in the department form an EDDS and MDDS of $\mho$. If the decision makers give an external input to every leading expert in the MDDS (EDDS), then the remaining staff members will be activated directly or indirectly by the three leading experts. As a result, the decision makers not only can obtain feedback and extended feedback needed for policy/strategy modifications, but also can estimate their strengths reasonably well. From the extended feedback of this subset, the department is able to receive

TABLE I
FEEDBACK P CONTAINED IN $S_t = \{v_4, v_6, v_{11}\}$ AND RELATED INFORMATION

P	$I_P(t)$	Who receives P		
		v_4	v_6	v_{11}
(v_4, v_1, v_4)	0.560	√		
(v_4, v_2, v_4)	0.210	√		
$(v_4, v_1, v_6, v_5, v_4)$	0.216	√		
$(v_4, v_2, v_6, v_5, v_4)$	0.302	√		
(v_4, v_{11}, v_8, v_4)	0.021	√		
$(v_4, v_{11}, v_{10}, v_4)$	0.010	√		
$(v_4, v_{11}, v_9, v_6, v_5, v_4)$	0.034	√		
$(v_4, v_1, v_6, v_5, v_{11}, v_8, v_4)$	0.008	√		
$(v_4, v_1, v_6, v_5, v_{11}, v_{10}, v_4)$	0.004	√		
$(v_4, v_2, v_6, v_5, v_{11}, v_8, v_4)$	0.011	√		
$(v_4, v_2, v_6, v_5, v_{11}, v_{10}, v_4)$	0.005	√		
$(v_6, v_5, v_4, v_1, v_6)$	0.216		√	
$(v_6, v_5, v_4, v_2, v_6)$	0.302		√	
$(v_6, v_{12}, v_{11}, v_9, v_6)$	0.101		√	
$(v_6, v_{12}, v_{11}, v_{10}, v_4, v_1, v_6)$	0.007		√	
$(v_6, v_{12}, v_{11}, v_{10}, v_4, v_2, v_6)$	0.009		√	
$(v_6, v_{12}, v_{11}, v_8, v_4, v_1, v_6)$	0.013		√	
$(v_6, v_{12}, v_{11}, v_8, v_4, v_2, v_6)$	0.019		√	
(v_{11}, v_8, v_{11})	0.210			√
(v_{11}, v_9, v_{11})	0.450			√
$(v_{11}, v_8, v_4, v_{11})$	0.021			√
$(v_{11}, v_{10}, v_4, v_{11})$	0.010			√
$(v_{11}, v_9, v_6, v_{12}, v_{11})$	0.101			√
$(v_{11}, v_9, v_6, v_5, v_{11})$	0.057			√
$(v_{11}, v_9, v_6, v_5, v_4, v_{11})$	0.034			√
$(v_{11}, v_8, v_4, v_1, v_6, v_{12}, v_{11})$	0.013			√
$(v_{11}, v_{10}, v_4, v_1, v_6, v_{12}, v_{11})$	0.006			√
$(v_{11}, v_{10}, v_4, v_2, v_6, v_{12}, v_{11})$	0.009			√

TABLE II
EXTENDED FEEDBACK P CONTAINED IN $S_t = \{v_4, v_6, v_{11}\}$
AND RELATED INFORMATION

P	$I_P(t)$	Who receives P		
		v_4	v_6	v_{11}
(v_6, v_5, v_4)	0.54	√		
(v_{11}, v_8, v_4)	0.21	√		
(v_{11}, v_{10}, v_4)	0.10	√		
(v_4, v_1, v_6)	0.40		√	
(v_4, v_2, v_6)	0.56		√	
(v_{11}, v_9, v_6)	0.63		√	
(v_4, v_{11})	0.10			√
(v_6, v_5, v_{11})	0.09			√
(v_6, v_{12}, v_{11})	0.16			√

feedback from all staff members. This provides a good, practical basis for the department to make sensible policies or policy modifications. Furthermore, in Simulation II the department is only responsible for determining the initial condition, i.e., the MDDS or EDDS, and concerned only with feedback and extended feedback from the subset.

VII. Conclusion

In this paper we developed a dynamic domination theory for FCN. We introduced in Section II-B the concept of domination in an FCN, $\mho = (V, A)$, and presented an algorithm to generate a minimal dominating set. As the first major contribution we proposed a new inference procedure, in which we set vertices in a minimal dominating set active as the initial condition. At time $t \geq 1$, we discussed the role played by an active subset S of $\mho$ by considering the strength of influence it receives and gives. Next, we studied DDS and MDDS, and gave an algorithm to generate them simultaneously. In practice, a DDS/MDDS S can be interpreted as a set of influential vertices which plays dominating roles in the FCN at t. To measure the degree of dominance of S, we introdued the dominating strength of S, which is given by the out-strength of S and is expressed in terms of the states of vertices of S at t and the weights of arcs from S to $V - S$. To reflect the change of domination with time we introduced the concepts of DDP and MDDP, which are crucial in understanding the dynamic behavior of an FCN in applications. The starting point of a DDP/MDDP $(S_t : t = 0, 1, 2, \ldots)$ of $\mho$ is the minimal dominating set S_0 which was used to set the initial condition in the new inference procedure. For $t \geq 1$, the t-th term S_t of a DDP/MDDP is a DDS/MDDS of $\mho$ at t. We presented several rules that govern the "power transition" from S_t to S_{t+1}. The study of DDS/MDDS and DDP/MDDP represents the second major contribution of this paper. The third major contribution is the unveiling of the following important phenomenon: at each time $t \geq 1$ either $\mho$ has a DDS S_t with extended feedback or the state of $\mho$ is undergoing a dramatic change. In the first possibility, S_t is called an EDDS. It is the second possibility that is most interesting: In this case, all active vertices at t that dominate $\mho$ suddenly become inactive at $t + 1$ and be driven out of the "major players club"; that is, a *revolution* has occurred. In this case, we recognized the special structure of the sub-FCN induced by the active vertices at t. We also gave necessary and sufficient conditions for the existence of an EDDS at t, which were used to identify the potential revolution. In addition, we studied the relationship between DDS/MDDS and EDDS. Finally, we simulated our new inference procedure and illustrated the concepts of EDDS and MDDS by using a simplified application.

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Jian Ying Zhang received the B.S. degree in mathematics from Hunan Normal University, China, the M.S. degree in mathematics from Zhengzhou University, China, and the Ph.D. degree in computer science and software engineering from The University of Melbourne, Australia, in 2004

She is currently a Postdoctoral Research Fellow at Swinburne University of Technology, Australia. Before that, she was a Lecturer or Tutor in Deakin University, RMIT University, The University of Western Australia, Wuhan Institute of Science and Technology, and Zhengzhou Food Industry College, respectively, in both Australia and China. Her recent research interest lies mainly in grid computing, service oriented computing, dynamic information modeling, fuzzy system, and networks optimization. She has published/completed more than 20 academic papers in these areas and gained two grants for her research projects.

Zhi-Qiang Liu received the M.A.Sc. degree in aerospace engineering from the Institute for aerospace studies, The University of Toronto, Toronto, ON, Canada, and the Ph.D. degree in electrical engineering from The University of Alberta, Canada.

He is a Professor in City University of Hong Kong, China. Previously, he was with the Department of Computer Science and Software Engineering, the University of Melbourne, Australia. His interests are neural-fuzzy systems, machine learning, human-media systems, media computing, computer vision, serving the community, computer vision, and computer networks.

Sanming Zhou received the Ph.D. degree (with distinction) in algebraic combinatorics from The University of Western Australia, in 2000.

Since then he has been working in the Department of Mathematics and Statistics, The University of Melbourne, where he is currently a Senior Lecturer. His research interest spans from pure to applied aspects of discrete mathematics, including algebraic combinatorics, combinatorial optimization, random graph processes and randomized algorithms, and optimization problems from theoretical computer science and telecommunication. He has published more than 40 papers in major international journals in these areas.

Dr. Zhou was awarded the 2003 Kirkman Medal from the International Institute of Combinatorics and its Applications. In 2004, he was elected Fellow of the same organization.