Fuzzy Causal Networks: General Model, Inference, and Convergence

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***Abstract*—In this paper, we first propose a general framework for fuzzy causal networks (FCNs). Then, we study the dynamics and convergence of such general FCNs. We prove that any general FCN with constant weight matrix converges to a limit cycle or a static state, or the trajectory of the FCN is not repetitive. We also prove that under certain conditions a discrete state general FCN converges to its limit cycle or static state in $O(n)$ steps, where n is the number of vertices of the FCN. This is in striking contrast with the exponential running time 2^n, which is accepted widely for classic FCNs.**

***Index Terms*—Fuzzy causal network (FCN), fuzzy cognitive map, fuzzy system, inference, intelligent system.**

I. Introduction

A. Fuzzy Causal Networks

A *fuzzy causal network* (FCN) is a dynamic system $\mho$ whose topological structure is a directed graph. Each vertex of $\mho$ represents a concept whose state varies with (discrete) time, and each arc of $\mho$ indicates a causal relationship from the tail to the head of the arc. The states of vertices are quantified as real numbers, which specify the fuzzy event occurring to some degree at discrete times. At any time t, when some vertices receive a series of external stimuli [10], [24], the vertex states of such a dynamic network are updated at time $t+1$. This process is iterated until a final equilibrium state is reached [3], [4].

FCNs are evolved from fuzzy cognitive maps (FCM) [3], and they have wide applications [1]–[3], [5], [6] in knowledge representation and inference. In fact, many applications of FCNs in quite different areas have been found, including geographic information systems [8], [18]–[20], fault detection [12], [16], policy analysis [17], chemical engineering [7], etc. In recent years, FCNs have received considerable research interest due to their power for decision-support and causal discovery in an environment of uncertainty and incomplete information [1], [10], [24]. One of the main objectives in studying FCNs is to understand their dynamic properties and causal inference processes. For basic concepts on FCNs and FCMs, the reader is referred to [1]–[11], [24], [25]. For recent theoretic development of FCNs, the reader is referred to, for example, [9], [10], and [21]–[25].

B. Contributions of This Paper

In this paper, *we will focus on theoretic aspects of FCNs*. The objectives are to introduce a general model for FCNs and to study causal inference and convergence of such generalized FCNs. The major contributions of the paper and their significance are as follows.

First, we propose a general framework for FCNs. This generalized model puts the study of FCNs on a solid foundation, and enables us to apply FCNs to a wider spectrum of real-world applications. Moreover, it helps us to understand better the dynamics and causal inferences of FCNs. Under our framework we define a *general FCN* as a 5-tuple (Ω, W, h, g, f), where Ω is a directed graph, W is the weight matrix, and f, g and h are certain functions governing respectively the state-transition, input aggregation and strength of one vertex influencing another. For different applications, we may need to use different functions f, g and h. We will discern conditions to be satisfied by these functions. In the special case where h is a bilinear function and g is a linear function, a general FCN is an FCN in the usual sense; see Section II for details. The matrix W may depend on time t, though in most applications it is a constant matrix.

Second, we study the causal inference and convergence of general FCNs. We show that, if W is a constant matrix, then either the FCN converges to a limit cycle or a static state, or the trajectory of the FCN is nonrepetitive. In particular, this implies that a general FCN with discrete states and constant weight matrix always converges to a limit cycle or a static state.

Third, we study the speed of convergence of a general FCN to its limit cycle or static state. We prove that, when the initial condition is kept in force during the whole inference process, a general discrete state FCN with W constant and nonnegative always converges to a static state but not a limit cycle, and moreover it converges in at most $(m+1)n$ steps, where n is the number of vertices of Ω and $m+2$ is the number of states that can be taken by vertices of the FCN. As a consequence, a general binary FCN with W constant and nonnegative converges in at most n steps. This is a significant improvement of the exponential bound 2^n, which has been widely accepted in the community of FCNs. For a general continuous state FCN with constant and nonnegative weight matrix, we prove that, if the initial condition is kept in force, then either the FCN converges to a limit cycle or static state, or its trajectory converges to a limit point in the state space.

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We would like to emphasize that the results above are proved for general FCNs. Of course, they apply in particular to FCNs in the usual sense. We notice that, even for such classic FCNs, some of these results have not been proved rigorously in the literature, although they have been used widely. In fact, one of the motivations for this article is to clarify a few fundamental issues on classic FCNs. The reader is invited to read Section IV for remarks and discussions on the results obtained in this paper.

II. General Fuzzy Causal Networks

A. A General Framework for FCN

In this subsection, we will introduce a general framework for FCNs. Roughly speaking, an FCN is a discrete dynamic system whose topological structure is a directed graph $\Omega = (V, A)$, where V is the set of vertices and A the set of directed arcs of Ω. As usual we assume throughout the paper that Ω contains no loops and multiple arcs, where a *loop* is an arc from a vertex to itself and *multiple arcs* are distinct arcs with the same initial and terminal vertices. Each vertex of Ω represents a concept associated with a fuzzy event whose presence varies with (discrete) time $t \in \{0, 1, 2, \ldots,\}$. For each vertex $v \in V$, the extent of presence at time t of the fuzzy event associated with v is measured by a time-dependent real variable $x_v(t)$, called the *state* of v at t. After normalization when necessary we can assume that all states are between 0 and 1 all the time, that is, $x_v(t) \in [0, 1]$ for all $v \in V$ and any time t. In the following we will use S to denote the set of states allowed to take by vertices of Ω. We will also use $n = |V|$ to denote the number of vertices of Ω. The state of the FCN Ω at time t is then represented by the n-dimensional vector

$$\mathbf{x}_\Omega(t) = (x_v(t))_{v \in V} \tag{1}$$

which we call the *state vector* of Ω at t. Such state vectors are members of the *state space*

$$S^n = \{\mathbf{x} = (x_v)_{v \in V} | x_v \in S\} \tag{2}$$

of Ω, which is the set of n-dimensional vectors with coordinates in S. Naturally, we may distinguish FCNs with continuous states from those with discrete states. If $x_v(t)$ can take any real number in the interval [0,1], that is, $S = [0, 1]$, then Ω has *continuous* states, and in this case the state space of Ω is the n-cube $S^n = [0, 1]^n$. If $x_v(t)$ can take only finitely many values in [0,1], then Ω has *discrete* states. In the latter case, we may assume without loss of generality that the state set of Ω is

$$\begin{aligned} &S = \{s_0, s_1, \ldots, s_m, s_{m+1}\} \\ &(0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = 1) \end{aligned} \tag{3}$$

for some $s_0, s_1, \ldots, s_m, s_{m+1}$ and integer $m \geq 0$. To be precise we will call Ω an $(m+2)$-*state FCN* in this case. An extremely important example of discrete states is the case where $m = 0$ and $s_0 = 0$, $s_1 = 1$. In this special case $x_v(t) \in \{0, 1\}$ for each $v \in V$ and any t, so the states are *binary* and we have the *binary state space* $\{0, 1\}^n$, which is the (discrete) n-cube. For both continuous and discrete cases, if $x_v(t) > 0$ then the vertex v is said to be *active* at t; and if $x_v(t) = 0$ then v is said to be *inactive* at t.

Each arc (u, v) of Ω represents a causal relationship from the tail u to the head v, and this usually indicates that there is an influence of u on v. The strength of this influence is measured by a real number w_{uv}, called the *weight* of the arc (u, v). The influence can be positive or negative, and this is reflected by the sign of w_{uv}: If $w_{uv} > 0$ then the influence of u on v is positive, and if $w_{uv} < 0$ then it is negative. Alternatively, we may say that the influence of u on v increases or decreases, respectively, the degree of presence of the fuzzy event associated with v. Without loss of generality we may assume $w_{uv} \in [-1, 1]$, for all arcs $(u, v) \in A$, after normalization when necessary.[1] If there is no arc from u to v, then u has no any influence on v at any time; in this case, we define $w_{uv} = 0$. In particular, since we assume the loop (v, v) is not an arc of Ω for any vertex $v \in V$, we have $w_{vv} = 0$ and v has no influence on itself. Thus, associated with Ω is its *weight matrix*

$$W_\Omega = (w_{uv})_{u,v \in V} \tag{4}$$

(also called adjacency matrix in the literature), which is an $n \times n$ matrix with entries in $[-1, 1]$. Note that all diagonal entries of W_Ω are 0 since $w_{vv} = 0$ for $v \in V$. We may adopt the usual convention[2] that, if (u, v) is an arc of Ω, then the weight $w_{uv} \neq 0$ and u may have influence on v. With this convention the topological structure of Ω is determined by the weight matrix W_Ω: there is an arc of Ω from u to v if and only if the (u, v)-entry w_{uv} of W_Ω is nonzero. However, the converse of this statement is not true in general because the knowledge of connections between vertices does not provide us enough information about the weights of arcs. Usually weight matrices are built up by consulting experts, and various ways have been suggested in the literature to increase their reliability, see for example the discussion in [3], [5], and [21].

In the literature, the weights w_{uv} are usually assumed to be constants. In this case W_Ω is said to be a *constant matrix*. However, in a lot of applications they can vary with time t. In this case, we will write $w_{uv}(t)$ in place of w_{uv} to emphasize this time-depending nature, so that the weight matrix of Ω at time t becomes

$$W_\Omega(t) = (w_{uv}(t))_{u,v \in V}. \tag{5}$$

We should point out that, in this case the convention above about constant weights will not apply since for an arc (u, v) the weight $w_{uv}(t)$ can be zero at some times and nonzero at other times. In a lot of applications, the tail of an arc has only positive influence on the head, that is, $w_{uv}(t) \in [0, 1]$ for all arcs $(u, v) \in A$ and any time t. In this case we say that $W_\Omega(t)$ is *nonnegative*. FCNs with nonnegative weight matrices have been studied extensively in the literature.

The weight $w_{uv}(t)$ gives rise to the strength of influence of u on v when the fuzzy event associated with u happens *definitely*

[1]In fact, if all weights w_{uv} are in an interval $[-a, a]$, for some $a > 0$, then by using the linear transformation $w_{uv} \mapsto w'_{uv} = w_{uv}/a$ we get another metric of weights such that all $w'_{uv} \in [-1, 1]$.

[2]This has been used in the literature but not stated explicitly. As a matter of fact, if $w_{uv} = 0$ holds for some arcs (u, v) of Ω, then we delete all such arcs from Ω to get a new FCN. The study of Ω is equivalent to the study of this new FCN since they have the same dynamics and inference in view of the condition (8). So assuming $w_{uv} \neq 0$ for all arcs (u, v) will not sacrifice generality.

at time t, that is, $x_u(t) = 1$. As mentioned earlier, $x_u(t)$ measures the degree of occurrence of this fuzzy even at t. Therefore, in general the *strength* of influence of u on v at time t, denoted by $y_{uv}(t)$ in the following, will depend not only on $w_{uv}(t)$ but also on $x_u(t)$. In other words

$$y_{uv}(t) = h\left(x_u(t), w_{uv}(t)\right) \tag{6}$$

is a function of $x_u(t)$ and $w_{uv}(t)$, which can be thought as an abstract potential function. Thus, we have the $n \times n$ *strength matrix* of Ω at time t

$$Y_\Omega(t) = \left(y_{uv}(t)\right)_{u,v \in V}. \tag{7}$$

After normalization when necessary, we can always assume that all the strengths $y_{uv}(t)$ are between -1 and 1, so that $h = h(x, w)$ is a function from the Cartesian product $S \times [-1, 1]$ to $[-1, 1]$. Of course, this function must satisfy the obvious *boundary condition*

$$h\left(x_u(t), 0\right) = h\left(0, w_{uv}(t)\right) = 0 \tag{8}$$

which means that the strength of the influence of u on v is equal to 0 if the weight $w_{uv}(t) = 0$ (that is, there is no arc from u to v or there is no any influence of u on v at time t, even if the fuzzy event at u happens definitely) or the state value $x_u(t) = 0$ (that is, the fuzzy event at u does not happen definitely). In particular, since $w_{vv} = 0$ for each v, the diagonal entries $y_{vv}(t)$ of the strength matrix $Y_\Omega(t)$ are all equal to 0. Moreover, by the practical meaning of $x_u(t)$, if $w_{uv}(t) > 0$, then the strength $y_{uv}(t)$ should be nonnegative and increase with $x_u(t)$; and if $w_{uv}(t) < 0$ then it should be nonpositive and decrease with $x_u(t)$. So $h(x, w) \geq 0$ or $h(x, w) \leq 0$ according to whether $w > 0$ or $w < 0$. In the former case $h(x, w)$ increases with x, and in the latter case $h(x, w)$ decreases with x. In other words, the absolute $|h(x, w)|$ increases with x for fixed w. Also, the increase of the weight $w_{uv}(t)$ will result in the increase of $y_{uv}(t)$, regardless of the sign of $w_{uv}(t)$. So $h(x, w)$ increases with w for fixed x. These are conditions for h to represent legally the strength of influence. We now give a formal definition of a strength function.

Definition 2.1: A two-variable function $h : S \times [-1, 1] \to [-1, 1]$ is called a *strength function* if it satisfies the following conditions.

a) $h(x, 0) = h(0, w) = 0$;
b) $h(x, w) \geq 0$ if $w > 0$, and $h(x, w) \leq 0$ if $w < 0$;
c) $|h(x, w)|$ is monotonically increasing with x for any fixed w;
d) $h(x, w)$ is monotonically increasing with w for any fixed x.

So, the strength $y_{uv}(t)$ of influence of the vertex u on vertex v is given by a strength function h, as shown in (6). Note that the formula (6) applies to all pairs u, v of vertices (even if (u, v) is not an arc) because of the boundary condition (8) (or a) in the definition above). Let $[-1, 1]^{n,n}$ denote the space of $n \times n$ matrices with all entries in $[-1, 1]$. Then we have $W_\Omega(t), Y_\Omega(t) \in [-1, 1]^{n,n}$. By slight abuse of notation, we may think h as the function

$$\begin{aligned} h : S^n \times [-1, 1]^{n,n} &\to [-1, 1]^{n,n} \\ \left(\mathbf{x}_\Omega(t), W_\Omega(t)\right) &\mapsto Y_\Omega(t) \end{aligned} \tag{9}$$

governed by (6).

The dynamics of Ω is as follows. First, an initial condition

$$\mathbf{x}_\Omega(0) = \mathbf{x}_0 = (x_v)_{v \in V} \tag{10}$$

is set at time 0, and this specifies the initial states x_v of vertices v of Ω and the initial set

$$V_0 = \{v \in V | x_v > 0\} \tag{11}$$

of active vertices, where $\mathbf{x}_0 = (x_v)_{v \in V}$ is an n-dimensional vector in the state space S^n. The initial state x_v of each vertex v is set to specific values based on the belief of experts of the corresponding concept. At any time t each vertex v of Ω receives a number of inputs (stimuli) from other vertices, and these inputs are aggregated in some way which depends on the nature of the FCN. So the *aggregation* of such inputs, denoted by $z_v(t)$, is given by a function of strengths $y_{uv}(t)$, $u \in V - \{v\}$, acting on v. That is

$$z_v(t) = g\left(\left(y_{uv}(t)\right)_{u \in V - \{v\}}\right) \tag{12}$$

for some $(n-1)$-variable function g. Such a function g must satisfy the condition

$$g(0, \ldots, 0) = 0 \tag{13}$$

which means that the aggregated effect on v vanishes if all the strengths $y_{uv}(t)$ are 0, for $u \in V - \{v\}$. Also, the value of g should not change if we swap the positions of any two variables, that is, g is invariant under permutations of variables. More precisely, for any permutation π of the $n-1$ variables $\alpha, \beta, \ldots, \gamma$ of g, we have $g(\alpha, \beta, \ldots, \gamma) = g(\pi(\alpha), \pi(\beta), \ldots, \pi(\gamma))$. Formally, such a function g is called a *symmetric function*. (For example, $g(\alpha, \beta, \ldots, \gamma) = \alpha + \beta + \cdots + \gamma$ is a symmetric function.) This symmetry requirement follows from the fact that the aggregation should be the same no matter which strength $y_{uv}(t)$ is the first variable, which strength is the second variable, and so on. Note that, when the strength $y_{uv}(t)$ is increased for some $u \in V - \{v\}$, the aggegated input $z_v(t)$ must increase as well. Therefore, the function g is *monotonically increasing*, that is, $g(\alpha, \beta, \ldots, \gamma) \geq g(\alpha', \beta', \ldots, \gamma')$ if $\alpha \geq \alpha'$, $\beta \geq \beta', \ldots, \gamma \geq \gamma'$. In the following, we use $\mathbb{R}$ to denote the set of real numbers.

Definition 2.2: An $(n-1)$-variable function $g : [-1, 1]^{n-1} \to \mathbb{R}$ is called an *aggregation function* if it is symmetric, monotonically increasing and satisfies $g(0, \ldots, 0) = 0$.

Thus, the aggregation $z_v(t)$ of inputs received by a vertex v at time t is given by an aggregation function g as in (12). In the following, we will write

$$\mathbf{z}_\Omega(t) = \left(z_v(t)\right)_{v \in V}$$

and call it the *aggregated input vector* of Ω at t. Symbolically, we may take g as the function

$$\begin{aligned} g : [-1, 1]^{n,n} &\to \mathbb{R}^n \\ Y_\Omega(t) &\mapsto \mathbf{z}_\Omega(t) \end{aligned} \tag{14}$$

defined by (12).

Stimulated by $\mathbf{z}_\Omega(t)$, the state $\mathbf{x}_\Omega(t+1)$ of Ω at $t+1$ will be updated automatically. This state transition is governed by a function

$$f : \mathbb{R}^n \to S^n$$
$$\mathbf{z}_\Omega(t) \mapsto \mathbf{x}_\Omega(t+1) \tag{15}$$

which transforms the aggregated input $\mathbf{z}_\Omega(t)$ of Ω at t into the next state of Ω. In other words

$$\mathbf{x}_\Omega(t+1) = f(\mathbf{z}_\Omega(t)). \tag{16}$$

By using this state-transition function, together with the functions g and h, once the initial condition (10) is set, the state of Ω at any time will be determined recursively by the following symbolic formula:

$$\mathbf{x}_\Omega(t+1) = f(g(h(\mathbf{x}_\Omega(t), W_\Omega(t)))). \tag{17}$$

Usually, the function $f = (f_v)_{v\in V}$ is acting *coordinate-wise*, which means that its v-coordinate function $f_v : \mathbb{R} \to S$ depends only on the v-coordinate $z_v(t)$ of the aggregated input vector $\mathbf{z}_\Omega(t)$. That is, $f_v = f_v(z_v(t))$ is a function of the aggregation $z_v(t)$ and, hence, (16) gives rise to

$$x_v(t+1) = f_v(z_v(t)) \tag{18}$$

for all $v \in V$. In other words, the v-coordinate function f_v transforms the aggregated input received by v into the next state $x_v(t+1)$ of v. Setting $F_v = f_v \circ g \circ h$ to be the composition of h, g and f_v (acting from right to left), (18) is equivalent to

$$x_v(t+1) = F_v\left((x_u(t), w_{uv}(t))_{u\in V-\{v\}}\right). \tag{19}$$

Since the state of v at $t+1$ measures the extent of occurrence of the fuzzy event associated with the concept v at this time, its value $x_v(t+1)$ should be larger if v receives stronger aggregated input. This means that each f_v must be monotonically increasing, that is, $f_v(\alpha) \geq f_v(\alpha')$ whenever $\alpha \geq \alpha'$.

Definition 2.3: A function $f = (f_v)_{v\in V}$ acting coordinate-wise is called a *state-transition function* if each $f_v : \mathbb{R} \to S$ is monotonically increasing.

With the previous notation, we now give the formal definition of a general FCN.

Definition 2.4: An FCN is a five-tuple $(\Omega, W_\Omega(t), h, g, f)$, where Ω is a directed graph without loops and multiple arcs, $W_\Omega(t)$ is the weight matrix of Ω at discrete time $t \in \{0, 1, 2, \ldots\}$, h is a strength function for Ω, g is an aggregation function for Ω and f is a state-transition function for Ω. Usually we say briefly that Ω is an FCN.

The function h tells us how the strength $y_{uv}(t)$ of influence of u on v at t is calculated by using the state $x_u(t)$ and the weight $w_{uv}(t)$. Once this is done for each pair u, v of vertices, we then get the strength matrix $Y_\Omega(t)$ of Ω at t. The function g tells us how to aggregate all the strengths received by a vertex v at t into the aggregated input $z_v(t)$. From this aggregation, we then get the aggregated input vector $\mathbf{z}_\Omega(t) = (z_v(t))_{v\in V}$ at t. Finally, the function f tells us how these aggregated inputs stimulate the FCN and cause the transition of state of Ω at the next time $t+1$. This transition is given by (15), or equivalently by (16) and (18). Also, the *images* of f_v for $v \in V$, namely $\mathtt{image}(f_v) = \{f_v(x) | x \in \mathbb{R}\}$, and the state set S of Ω determine each other. That is

$$S = \bigcup_{v\in V} \mathtt{image}(f_v)$$

and the image of f_v is a subset of S. Thus, if $\mathtt{image}(f_v) = S$ for all $v \in V$, where S is the finite set in (3), then Ω has discrete states; and if $\mathtt{image}(f_v) = [0, 1]$ for all $v \in V$, then Ω has continuous states. All $W_\Omega(t)$, h, g, f are components of the FCN Ω, and altogether they make the inference and dynamics of Ω possible. Note that, as mentioned earlier, in the case where $W_\Omega(t) = W_\Omega$ is a constant matrix, the topological structure of Ω is determined completely by W_Ω. Thus, in this case we can define Ω equivalently as the quadruple (W_Ω, h, g, f).

We emphasize that, for different applications, we may need to choose different functions f, g, and h for Ω to serve for our purposes. For example, in the study of certainty fuzzy cognitive maps (in which vertices are neurons and states stand for activations), Tsadiras and Margaritis [21], [22] defined the certainty neuron transfer function as follows:

$$x_v(t+1) = f_M(x_v(t), z_v(t)) - d_v x_v(t)$$

where $d_v \in [0, 1]$ is the decay factor for the neuron v, $z_v(t) = \sum_{u\in V-\{v\}} x_u(t) w_{uv}(t)$ and f_M is given by

$$f_M(\alpha, \beta) = \begin{cases} \alpha + \beta - \alpha\beta, & \alpha, \beta \geq 0 \\ \alpha + \beta + \alpha\beta, & \alpha, \beta < 0 \\ (\alpha + \beta)/(1 - \max(|\alpha|, |\beta|), & \text{else.} \end{cases}$$

Under our notation this is equivalent to saying that the function F_v is chosen in such a way that $F_v((x_u(t), w_{uv}(t))_{u\in V-\{v\}}) = f_M(x_v(t), z_v(t)) - d_v x_v(t)$.

For general $(m+2)$-state FCNs with state set S in (3), we suggest to use the following generalized threshold function f_v for each $v \in V$:

$$f_v(\alpha) = \begin{cases} 0, & \text{if } \alpha < T_v^0 \\ s_1, & \text{if } T_v^0 \leq \alpha < T_v^1 \\ \vdots & \\ s_m, & \text{if } T_v^{m-1} \leq \alpha < T_v^m \\ 1, & \text{if } \alpha \geq T_v^m \end{cases}$$

where $T_v^0 < T_v^1 < \cdots < T_v^m$ are thresholds set for v.

B. Classic Fuzzy Causal Networks

In the literature, since the early dates of the study of FCMs, researchers have been using the bilinear function

$$y_{uv}(t) = x_u(t) w_{uv}(t) \tag{20}$$

of $x_u(t)$ and w_{uv} to represent the strength of u influencing v at time t, and the linear function

$$z_v(t) = \sum_{u\in V-\{v\}} y_{uv}(t) \tag{21}$$

of $y_{uv}(t)$ to represent the aggregated input on v at t. In other words, in almost all studies of FCNs researchers use

$$h(x, w) = xw$$
$$g(\alpha, \beta, \ldots, \gamma) = \alpha + \beta + \cdots + \gamma$$

as the strength and aggregation functions, respectively. We call an FCN with strength and aggregation functions defined this way a *classic FCN*. One can prove that indeed this h is a strength function, satisfying the conditions of Definition 2.1, and this g is an aggregation function in terms of Definition 2.2. The specific $z_v(t)$ defined in (21) gives the sum of strengths $y_{uv}(t)$ acting on v, and is called the *total input* received by v at time t in the literature. Note that, by (20) and the definition of weights $w_{uv}(t)$, those vertices u which are either inactive at t or has no arc to v contribute nothing to the summation in (21). This is consistent with our discussion in Section II-A for general FCN. With h and g as above the aggregated input vector (also called *input vector*) $\mathbf{z}_\Omega(t)$ of Ω at time t is given by the matrix product

$$\mathbf{z}_\Omega(t) = \mathbf{x}_\Omega(t) \cdot W_\Omega(t). \tag{22}$$

More explicitly, we have

$$z_v(t) = \sum_{u \in V - \{v\}} x_u(t) w_{uv}(t)$$

for each $v \in V$. So for classic FCNs the recursive formula (17) becomes

$$\mathbf{x}_\Omega(t+1) = f\left(\mathbf{x}_\Omega(t) \cdot W_\Omega(t)\right).$$

Also, we have the following linear relationship among the vertex states $x_v(t)$, the weight matrix $W_\Omega(t)$ and the strength matrix $Y_\Omega(t)$ at time t:

$$Y_\Omega(t) = X_\Omega(t) \cdot W_\Omega(t) \tag{23}$$

where $X_\Omega(t) = \mathtt{diag}(x_v(t))_{v \in V}$ is the (diagonal) matrix with diagonal entries $x_v(t)$ and all other entries 0.

We point out that, in the theory of classic FCNs, the above choices of strength and total input (strength and aggregated functions) apply uniformly to both continuous and discrete states. However, for these two cases different state-transition functions $f = (f_v)_{v \in V}$ should be used. For the continuous case, Kosko [6] suggested to use the function $f = (f_{T_v})_{v \in V}$ such that each f_{T_v} is a bounded signal function, or the sigmoid function

$$f_{T_v}(\alpha) = \frac{1}{1 + e^{-c(\alpha - T_v)}} \tag{24}$$

for some special FCNs (called simple FCNs), where T_v is a threshold for v set beforehand. In the case of binary states, the coordinate function f_v is usually chosen to be the following threshold function

$$f_v(\alpha) = \begin{cases} 0, & \text{if } \alpha < T_v \\ 1, & \text{if } \alpha \geq T_v \end{cases}. \tag{25}$$

III. Causal Inference and Convergence

A. Trajectory and Inference

The most important goal of studying FCNs is to understand their dynamics and causal inferences. As we have seen in (17), the inference process of an FCN Ω is determined by the initial condition (10), or in other words by the set V_0 of active vertices at time $t = 0$ together with the states $x_v(0) = x_v$ (for $v \in V_0$), both are set initially. So understanding the inference will help us to answer a lot of "what-if" type questions such as "what if $\mathbf{x}_\Omega(0)$ happens and keeps in force during the whole inference process ?" and so on. As mentioned earlier, the state of Ω is updated with time t by using the formula (17), and this generates the following *state sequence*:

$$\mathbf{x}_0 = \mathbf{x}_\Omega(0), \mathbf{x}_\Omega(1), \ldots, \mathbf{x}_\Omega(t), \ldots \tag{26}$$

From a geometric point of view, we may think states $\mathbf{x}_\Omega(t)$ as points of the state space S^n, and state transitions as motions of points with time governed by (17). Then the previous sequence can be interpreted as *trajectory* of Ω. To a large extent the study of FCNs is meant to understand the behavior of this trajectory, especially its limit behavior. In this subsection, we discuss this issue for general FCNs.

Clearly, we have the following two disjoint and exhaustive possibilities.

a) $\mathbf{x}_\Omega(t) \neq \mathbf{x}_\Omega(t')$, for any times t, t' with $t \neq t'$.
b) There exists $t_1 \geq 1$ such that $\mathbf{x}_\Omega(t_1)$ coincides with one of the preceding states, that is, $\mathbf{x}_\Omega(t_1) = \mathbf{x}_\Omega(t_0)$ for some t_0 with $0 \leq t_0 < t_1$.

In case a) the trajectory (26) contains no any repeated terms; whilst in case b), $\mathbf{x}_\Omega(t_1)$ and $\mathbf{x}_\Omega(t_0)$ are *repeated terms*. The following theorem tells us what is happening in case b). The result in this theorem has been accepted widely, but never proved rigorously, in the literature of classic FCNs. We now prove that it is true for *any general* FCN, as long as its weight matrix is a constant matrix.

Theorem 3.1: Let $(\Omega, W_\Omega, h, g, f)$ be an FCN with constant weight matrix W_Ω. Suppose that (26) contains repeated terms (that is, case (b) above occurs), and let t^* *be the smallest* t_1 such that $\mathbf{x}_\Omega(t_1) = \mathbf{x}_\Omega(t_0)$ for some t_0 with $0 \leq t_0 < t_1$. Then

$$\mathbf{x}_\Omega(t) = \mathbf{x}_\Omega(t') \tag{27}$$

for any t, $t' \geq t_0$ with $t \equiv t' \pmod{(t^* - t_0)}$, and $\mathbf{x}_\Omega(t_0), \mathbf{x}_\Omega(t_0 + 1), \ldots, \mathbf{x}_\Omega(t^* - 1)$ are pairwise distinct. In other words, starting from time t_0, the state vectors of Ω will repeat periodically with period $t^* - t_0$.

Proof: By its definition, t^* is the smallest t_1 such that there exists $t_0 < t_1$ satisfying $\mathbf{x}_\Omega(t_1) = \mathbf{x}_\Omega(t_0)$. From this, it follows that the terms

$$\mathbf{x}_\Omega(t_0), \mathbf{x}_\Omega(t_0 + 1), \ldots, \mathbf{x}_\Omega(t^* - 1)$$

are pairwise distinct, for otherwise repetition would occur no later than time $t^* - 1$, violating the choice of t^*. In the following, we will prove by induction that

$$\mathbf{x}_\Omega(t^* + t) = \mathbf{x}_\Omega(t_0 + t) \tag{28}$$

for any $t \geq 0$. Of course this is true for $t = 0$ since $\mathbf{x}_\Omega(t^*) = \mathbf{x}_\Omega(t_0)$ by our assumption. The equation $\mathbf{x}_\Omega(t^*) = \mathbf{x}_\Omega(t_0)$ is equivalent to saying that $x_v(t^*) = x_v(t_0)$ for all $v \in V$. So from (6) and by noting that W_Ω is a constant matrix we have $y_{uv}(t^*) = y_{uv}(t_0)$ for all u, $v \in V$. By (12) this implies $z_v(t^*) = z_v(t_0)$ for all $v \in V$. From (18) this in turn

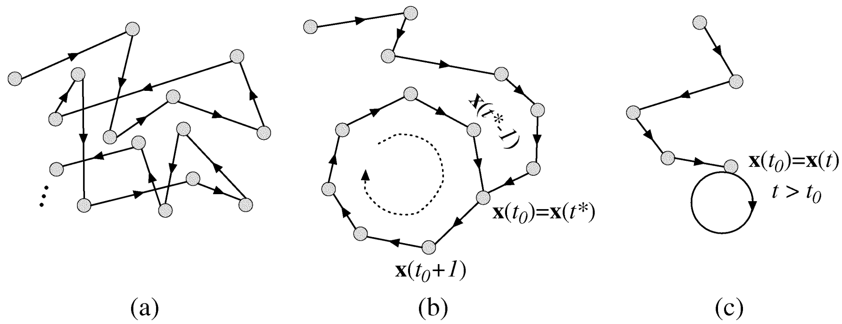

Fig. 1. Trajectory of Ω: (a) chaos and semi-chaos; (b) limit cycle; (c) static state.

implies $x_v(t^*+1) = x_v(t_0+1)$, and hence $\mathbf{x}_\Omega(t^*+1) = \mathbf{x}_\Omega(t_0+1)$. In general, suppose (28) is true for some $t \geq 0$, then we have $x_v(t^*+t) = x_v(t_0+t)$ for $v \in V$, and hence $y_{uv}(t^*+t) = y_{uv}(t_0+t)$ for all $u, v \in V$ by (6). From (12) we then have $z_v(t^*+t) = z_v(t_0+t)$ for $v \in V$. This together with (18) implies $x_v(t^*+t+1) = x_v(t_0+t+1)$, that is, $\mathbf{x}_\Omega(t^*+t+1) = \mathbf{x}_\Omega(t_0+t+1)$. So by induction (28) is true for all $t \geq 0$. However, (28) is equivalent to saying that $\mathbf{x}_\Omega(t) = \mathbf{x}_\Omega(t')$ for any $t, t' \geq t_0$ with $t \equiv t' \pmod{(t^*-t_0)}$. Hence, the proof is complete. $\square$

Theorem 3.1 tells us that, if b) occurs, then the subsequence of the state sequence (26) from $\mathbf{x}_\Omega(t_0)$ onwards will be

$$\mathbf{x}_\Omega(t_0), \mathbf{x}_\Omega(t_0+1), \ldots, \mathbf{x}_\Omega(t^*-1), \mathbf{x}_\Omega(t_0)$$
$$\mathbf{x}_\Omega(t_0+1), \ldots, \mathbf{x}_\Omega(t^*-1), \ldots$$

In particular, in the case where the period $t^* - t_0 = 1$, we have $\mathbf{x}_\Omega(t) = \mathbf{x}_\Omega(t_0)$ for all $t \geq t_0$ and, hence, the states of Ω will not undergo any change from t_0 onwards.

Definition 3.2: If there exist (the smallest) t^* and $t_0 \leq t^*-1$ such that (27) holds for any $t, t' \geq t_0$ with $t \equiv t' \pmod{(t^*-t_0)}$, then Ω is said to converge to the *limit cycle*

$$(\mathbf{x}_\Omega(t_0), \mathbf{x}_\Omega(t_0+1), \ldots, \mathbf{x}_\Omega(t^*-1), \mathbf{x}_\Omega(t_0))$$

with *period* $t^* - t_0$. In the particular case where $t^* - t_0 = 1$, this limit cycle degenerates to a cycle of length one (that is, a loop), and Ω is said to converge to the *static state* $\mathbf{x}_\Omega(t_0)$.

Thus, a static state can be regarded as a degenerated limit cycle with length one. From the aforementioned geometric viewpoint, it can be also taken as a *fixed point* of the dynamic system (17). In view of Definition 3.2, Theorem 3.1 can be restated as follows.

Theorem 3.3: Let $(\Omega, W_\Omega, h, g, f)$ be an FCN with constant weight matrix W_Ω. Then either the state sequence (26) contains no repeated terms, or Ω converges to a limit cycle, or Ω converges to a static state.

The three possibilities are illustrated in Fig. 1. When the first possibility occurs, the trajectory (26) is usually thought to behave in a chaotic manner. Nevertheless, it may not be in total disorder,[3] and further research is needed in order to understand better this "semi-chaos". We should emphasize that, without the assumption that the weight matrix is a constant matrix, the results of Theorems 3.1 and 3.3 are not guaranteed. More explicitly, in the case where the weight matrix $W_\Omega(t)$ depends on t, even if there exist $0 \leq t_0 < t_1$ such that $\mathbf{x}_\Omega(t_1) = \mathbf{x}_\Omega(t_0)$, in general we cannot draw the conclusion that the FCN converges to a limit cycle or a static state. This can be seen from the proof of Theorem 3.1, where the induction required that each $w_{uv}(t)$ does not change with time t for otherwise $x_v(t^*) = x_v(t_0)$ would not imply $y_{uv}(t^*) = y_{uv}(t_0)$ for all $u, v \in V$, and so forth.

[3]For example, in Lemma 3.6 in the next subsection we will show that (26) is increasing under certain conditions.

We should also emphasize that, in the case of continuous states, the FCN Ω may not converge (even if $W_\Omega(t)$ is a constant matrix); in other words, the possibility (a) before Theorem 3.1 may occur in this case. On the other hand, for discrete state FCN the possibility a) before Theorem 3.1 will never occur, and hence the FCN will converge definitely. We present this together with a result about the speed of convergence in the following theorem. In the special case of classic FCNs with binary states, this result has been a folklore in the literature. Note that, as explained above, the result of this theorem is guaranteed only when the weight matrix is a constant matrix.

Theorem 3.4: Let $(\Omega, W_\Omega, h, g, f)$ be a discrete state FCN with $m+2$ states and constant weight matrix W_Ω. Then Ω must converge to a limit cycle or static state. Moreover, it converges in at most $(m+2)^n$ steps. In particular, any binary FCN with constant weight matrix converges to a limit cycle or static state in at most 2^n steps.

Proof: Since there are only $m+2$ states, there are $(m+2)^n$ possibilities for the state vectors $\mathbf{x}_\Omega(t)$, $t \geq 0$. Hence, the infinite state sequence (26) must contain repeated terms, and repetition occurs no later than $(m+2)^n$. Thus, by Theorem 3.1, Ω must converge to a limit cycle or static state. Also, we have $t^* \leq (m+2)^n$, and hence Ω converges in at most $(m+2)^n$ steps. In particular, in the case where Ω is a binary FCN, we have $m = 0$ and so Ω converges to a limit cycle or static state in at most 2^n steps. $\square$

B. Convergence and Speed

In a lot of applications, we would like to keep all vertices $v \in V_0$ active during the whole process and see the impact of this initial condition (10). In other words, the vertices v in V_0 will not change their states $x_v(>0)$ and, thus, the initial condition will be kept "in force" in the whole process. This is the "what-if" question asked at the beginning of the previous subsection. Note that, under our general framework for FCNs, this is equivalent to the following inference process: reset the v-coordinate function f_v to be the constant function $f_v(x) = x_v$ for $v \in V_0$, and leave the system running automatically (without extra force) according to (16).

For any FCN, we may ask the following fundamental questions.

Question 3.5:

a) Under what circumstances will Ω converge?
b) If Ω does converge, how fast it converges to the limit cycle or static state?

For continuous state FCNs, these questions are very hard to answer in general. We will give partial answers in Theorem 3.7, which is the first main result of this subsection. For discrete state Ω, Question 3.5a) was answered by Theorem 3.4, which says that Ω will always converge as long as the weight matrix is a constant matrix. The same theorem also gives an exponential bound $(m+2)^n$ for the number of steps required. However, this bound is very crude and impracticable, especially for FCNs with large size n. Our second main result in this subsection, Theorem 3.9, shows that in fact any discrete state FCN with weight matrix constant and nonnegative converges very fast, namely in less than $(m+1)n$ steps. This answers Question 3.5b) and is in striking contrast to the exponential bound above.

At any time $t \geq 0$, the vertices of Ω fall into two categories, active and inactive, according to their states. We will use V_t to denote the set of active vertices of Ω at t, that is

$$V_t = \{v \in V | v \text{ is active at } t\} = \{v \in V |\, x_v(t) > 0\}. \quad (29)$$

Then, V is partitioned as $V = V_t \cup (V - V_t)$ at any time t, and we have the following sequence of active-vertex sets of Ω:

$$V_0, V_1, \ldots, V_t, \ldots \quad (30)$$

Recall that the initial active-vertex set V_0 has been defined in (11).

Let us prove first the following lemma, which will be used in the proofs of Theorems 3.7 and 3.9. It shows that, keeping the initial condition in force all the time, the state sequence (26) will be increasing if the weight matrix W_Ω is constant and nonnegative. For any $\mathbf{a} = (a_v)_{v\in V}$, $\mathbf{b} = (b_v)_{v\in V} \in [0,1]^n$, we write $\mathbf{a} \geq \mathbf{b}$ if $a_v \geq b_v$ for all $v \in V$, and $\mathbf{a} > \mathbf{b}$ if $\mathbf{a} \geq \mathbf{b}$ and in addition $a_v > b_v$ for at least one v.

Lemma 3.6: Let $(\Omega, W_\Omega, h, g, f)$ be an FCN with the weight matrix W_Ω constant and nonnegative. Suppose that the initial active vertices in V_0 are kept active with states unchanged in the whole inference process. Then

$$\mathbf{x}_\Omega(t) \leq \mathbf{x}_\Omega(t+1) \quad (31)$$

for all $t \geq 0$. Moreover, we have

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t \subseteq V_{t+1} \subseteq \cdots. \quad (32)$$

In other words, both state sequence and the sequence of active-vertex sets are increasing with time t.

Proof: Since W_Ω is nonnegative, by (6) and Definition 2.1 we have $h(x,w) \geq 0$ and h is increasing with x. We first prove (31) by induction on t. Since we assume that the vertices in V_0 are always kept active with their states unchanged with time, we have $x_v(t) = x_v(0)$ for $v \in V_0$ and $t \geq 0$. In particular, $x_v(1) = x_v(0)$ for $v \in V_0$. For $u \in V - V_0$, we have $x_u(0) = 0$ by our way of setting initial condition and, hence, $x_u(0) \leq x_u(1)$ is true trivially. In other words, (31) is true for $t = 0$. Suppose inductively that $\mathbf{x}_\Omega(t-1) \leq \mathbf{x}_\Omega(t)$ for some $t \geq 1$, that is, $x_u(t-1) \leq x_u(t)$ for all vertices u of V. Then, since h is monotonically increasing with the first variable, we have

$$y_{uv}(t-1) = h\left(x_u(t-1), w_{uv}\right) \leq h\left(x_u(t), w_{uv}\right) = y_{uv}(t)$$

for any u, $v \in V$. So, by the monotonicity of g, we get

$$\begin{aligned} z_v(t-1) &= g\left(\left(y_{uv}(t-1)\right)_{u\in V-\{v\}}\right) \\ &\leq g\left(\left(y_{uv}(t)\right)_{u\in V-\{v\}}\right) = z_v(t) \end{aligned}$$

for each $v \in V$. But f_v is increasing as well, so by (18), we have

$$x_v(t) = f_v\left(z_v(t-1)\right) \leq f_v\left(z_v(t)\right) = x_v(t+1)$$

for any $v \in V$, and hence $\mathbf{x}_\Omega(t) \leq \mathbf{x}_\Omega(t+1)$. By induction the proof of (31) for all $t \geq 0$ is complete. From (31), it follows that, if $x_u(t) > 0$, then $x_u(t+1) > 0$. That is, if a vertex u is active at time t, then it must be active at the next time $t+1$. Thus, we have $V_t \subseteq V_{t+1}$ for all $t \geq 0$ and (32) is proved. □

From the previous proof, one can see that the result of Lemma 3.6 is true also if the weight matrix $W_\Omega(t)$ is nonnegative and nondecreasing with t (that is, $w_{uv}(t+1) \geq w_{uv}(t)$ for any u, $v \in V$ and t), but not necessarily constant. This is due to the fact that the strength function h is increasing with its second variable w as well. Note that Lemma 3.6 applies to both continuous and discrete states. For continuous case, it leads to the following theorem. Recall that (26) is a sequence of points of $[0,1]^n$. So, we can talk about its convergence and limit in the usual sense.

Theorem 3.7: Let $(\Omega, W_\Omega, h, g, f)$ be a continuous state FCN with the weight matrix W_Ω constant and nonnegative. Suppose that the initial active vertices in V_0 are kept active with states unchanged in the whole inference process. Then either Ω converges to a limit cycle or static state, or the state sequence (26) converges to a limit $\mathbf{x}^* \in [0,1]^n$. Moreover, $\mathbf{x}^* \neq \mathbf{x}_\Omega(t)$ for any $t \geq 0$.

Proof: We have proved in Theorems 3.1 and 3.3 that, if (26) contains repeated terms, then Ω converges to a limit cycle or static state. So it remains to show that, if (26) contains no repeated terms, then it must converge and its limit lies in $[0,1]^n$. In fact, in this case (26) is strictly increasing by Lemma 3.6. So, for each $v \in V$, the sequence

$$x_v(0), x_v(1), \ldots, x_v(t), \ldots \quad (33)$$

is increasing and bounded above by 1. Hence, by a basic result in calculus, we know that (33) converges and its limit x_v^* satisfies $0 \leq x_v^* \leq 1$. Thus, the state sequence (26) (as sequence of points in $[0,1]^n$) converges to the limit $\mathbf{x}^* = (x_v^*)_{v\in V} \in [0,1]^n$. Moreover, we have $\mathbf{x}^* \neq \mathbf{x}_\Omega(t)$ for any t, since otherwise we would have $\mathbf{x}_\Omega(t') = \mathbf{x}^*$ for $t' \geq t$ by the monotonicity of (33) and, hence, (26) has repeated terms, a contradiction. This completes the proof. □

In the third possibility of the previous theorem, we know the trend of the trajectory (26), although the FCN does not converge to a limit cycle or static state.

Now let us turn to discrete state FCNs. Let Ω be an $(m+2)$-state FCN with state set S in (3). For any $\mathbf{a}=(s_i,s_j,\ldots,s_k)\in S^n$ and $\mathbf{b}=(s_p,s_q,\ldots,s_r)\in S^n$, we define

$$d(\mathbf{a},\mathbf{b})=|i-p|+|j-q|+\cdots+|k-r|.$$

In the case where $\mathbf{a}\leq\mathbf{b}$, $d(\mathbf{a},\mathbf{b})$ gives the number of "moves" needed to "jump" from $\mathbf{a}$ to $\mathbf{b}$. Here, by one "move" we mean replacing one coordinate s_ℓ in the vector by $s_{\ell+1}$ and leaving the remaining coordinates unchanged. For example, it takes one move from $(s_0,s_0,\ldots,s_0)$ to $(s_1,s_0,\ldots,s_0)$, two moves from $(s_0,s_0,\ldots,s_0)$ to $(s_1,s_1,\ldots,s_0)$ and three moves from $(s_0,s_0,\ldots,s_0)$ to $(s_2,s_1,\ldots,s_0)$. To prove our next theorem, we will need the following lemma whose proof is routine and, hence, omitted.

Lemma 3.8: Let S be as defined in (3). For any $\mathbf{a}_0,\mathbf{a}_1,\ldots,\mathbf{a}_N\in S^n$ with $\mathbf{a}_0\leq\mathbf{a}_1\leq\cdots\leq\mathbf{a}_N$, we have

$$d(\mathbf{a}_0,\mathbf{a}_N)=\sum_{t=0}^{N-1}d(\mathbf{a}_t,\mathbf{a}_{t+1}).\tag{34}$$

Theorem 3.9: Let (Ω,W_Ω,h,g,f) be an $(m+2)$-state FCN with state set S given in (3) and with weight matrix W_Ω constant and nonnegative. Suppose that the initial active vertices in V_0 are kept active with states unchanged in the whole inference process. Then Ω converges to a static state but not a limit cycle, and it converges in at most $(m+1)(n-|V_0|)$ steps.

Proof: Since W_Ω is nonnegative, by Lemma 3.6 we have $\mathbf{x}_\Omega(t)\leq\mathbf{x}_\Omega(t+1)$ for $t\geq 0$. Since Ω has $m+2$ states, by Theorem 3.4 Ω must converge to a limit cycle or static state. Let t^* be the earliest time such that $\mathbf{x}_\Omega(t^*)=\mathbf{x}_\Omega(t_0)$ for some t_0 with $0\leq t_0<t^*$, as in Theorem 3.1. Since $\mathbf{x}_\Omega(t_0)<\cdots<\mathbf{x}_\Omega(t^*-1)\leq\mathbf{x}_\Omega(t^*)$ by the monotonicity of the state sequence, we must have $t_0=t^*-1$ and, hence, Ω converges to the static state $\mathbf{x}_\Omega(t_0)$, but not a limit cycle. Thus, $\mathbf{x}_\Omega(t)=\mathbf{x}_\Omega(t^*-1)$ for all $t\geq t^*-1$, and by the definition of t^* we have

$$\mathbf{x}_\Omega(0)<\mathbf{x}_\Omega(1)<\cdots<\cdots<\mathbf{x}_\Omega(t^*-1).\tag{35}$$

Our assumption about initial condition implies that, for $v\in V_0$, the v-coordinates $x_v(t)$ of $\mathbf{x}_\Omega(t)$ $(0\leq t\leq t^*-1)$ are all the same as the initial state x_v of v. So only the rest $n-|V_0|$ coordinates $x_u(t)$, for $u\in V-V_0$, can change with t. Furthermore, for any fixed t with $0\leq t\leq t^*-2$, we have $x_u(t)\leq x_u(t+1)$ and by (35) the inequality "$<$" holds for at least one u. Thus, setting $\mathbf{x}_\Omega(t)=(s_{i_t},s_{j_t},\ldots,s_{k_t})$ for each t with $0\leq t\leq t^*-1$, we have $s_{i_t}\leq s_{i_{t+1}}$, $s_{j_t}\leq s_{j_{t+1}},\ldots,s_{k_t}\leq s_{k_{t+1}}$ with inequality "$<$" appearing at least once. Hence $i_t\leq i_{t+1}$, $j_t\leq j_{t+1},\ldots,k_t\leq k_{t+1}$ with inequality occurring at least once. So we have $d(\mathbf{x}_\Omega(t),\mathbf{x}_\Omega(t+1))=(i_{t+1}-i_t)+(j_{t+1}-j_t)+\cdots+(k_{t+1}-k_t)\geq 1$.

Denote by $\mathbf{x}$ the unique member of S^n with v-coordinates x_v, for $v\in V_0$, and all other coordinates $1(=s_{m+1})$. Then $\mathbf{x}_\Omega(t^*-1)\leq\mathbf{x}$. Note that $\mathbf{x}_\Omega(0)$ is the unique member of S^n with v-coordinates x_v, for $v\in V_0$, and all other coordinates $0(=s_0)$. So we have $d(\mathbf{x}_\Omega(0),\mathbf{x})=(m+1)(n-|V_0|)$. On the other hand, because of the monotonicity of the sequence (35), by Lemma 3.8 we have

$$d(\mathbf{x}_\Omega(0),\mathbf{x})=\sum_{t=0}^{t^*-2}d(\mathbf{x}_\Omega(t),\mathbf{x}_\Omega(t+1))+d(\mathbf{x}_\Omega(t+1),\mathbf{x})\geq(t^*-1).$$

Thus, $t^*-1\leq(m+1)(n-|V_0|)$, implying that Ω converges to the static state $\mathbf{x}_\Omega(t_0)$ in at most $(m+1)(n-|V_0|)$ steps. □

Note that Theorem 3.9 applies to *any* discrete state FCN with general strength function h, aggregation function g, and state-transition function f with $\mathtt{image}(f_v)=S$ for each $v\in V$. (Such an f is not necessarily a threshold function.) For the binary case, we get the following corollary, which shows that Ω converges in less than n steps if the weight matrix is constant and nonnegative. This improves significantly the widely accepted bound 2^n. Also, this corollary applies to not only classic but also general binary FCNs.

Corollary 3.10: Let (Ω,W_Ω,h,g,f) be a binary FCN with W_Ω constant and nonnegative. Suppose that the initial active vertices in V_0 are kept active with states unchanged in the whole inference process. Then Ω converges to a static state but not a limit cycle, and it converges in at most $n-|V_0|$ steps.

We conclude this section by pointing out that, for binary FCNs, the state sequence (26) and the sequence (30) of active-vertex sets determine each other. This is because in this case there is only one state, namely 1, for active vertices, and hence knowing active vertices is equivalent to knowing the 1-valued coordinates of the state vector at any time. This property can be taken as a characteristic of binary FCNs since it is not possessed by nonbinary FCNs in general. In fact, for nonbinary FCNs, (26) determines (30), but not conversely. Thus, it may happen that, say, $V_t=V_{t+1}$ but $\mathbf{x}_\Omega(t)\neq\mathbf{x}_\Omega(t+1)$ for some t.

IV. Concluding Remarks

In this paper, we proposed a general framework for fuzzy causal networks. This enables us to apply the theory of FCNs to many real-world application problems that are not covered by classic FCNs. We then analyzed the dynamics and convergence of general FCNs. We proved that, under certain general conditions, a general FCN converges to a limit cycle or a static state, or the trajectory of the FCN is nonrepetitive. For a discrete state general FCN, the last possibility cannot appear. We also proved that under certain conditions a discrete state general FCN converges to its limit cycle or static state in $O(n)$ steps, where n is the number of vertices. This is in striking contrast with the widely accepted exponential running time 2^n.

We emphasize that all the results obtained in Section III, namely Theorems 3.1, 3.3, 3.4, 3.7, 3.9, and Corollary 3.10, are valid for *any* strength function h, *any* aggregation function g and *any* state-transition function f. This universality for h, g, f is meant wide applications of the results to different FCNs. As pointed out in the paragraph after Theorem 3.3, the results in Theorems 3.1, 3.3 and 3.4 are not guaranteed if W_Ω is not a constant matrix. For general FCNs Ω with general weight matrix W_Ω and general functions h, g, f, it is very difficult to identify whether Ω converges and, if it converges, how fast it converges.

We solved these problems in Theorems 3.7 and 3.9 under the assumption that W_Ω is constant and nonnegative and that the initial active vertices are kept active. (For a lot of practical applications the weight matrices are indeed constant and nonnegative.) Without these conditions the results in Theorems 3.7 and 3.9 are not guaranteed. All these limitations suggest that *it is inadequate to take convergence as granted and use it unconditionally*.

Besides its significance in applications, general FCNs introduced in this paper are of interest from a mathematical point of view. Challenging problems (e.g. the convergence problem) arise from this general model, and they deserve further research.

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