Quotient FCMs—A Decomposition Theory for Fuzzy Cognitive Maps

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***Abstract*—In this paper, we introduce a decomposition theory for fuzzy cognitive maps (FCMs). First, we partition the set of vertices of an FCM into blocks according to an equivalence relation, and by regarding these blocks as vertices we construct a quotient FCM. Second, each block induces a natural sectional FCM of the original FCM, which inherits the topological structure as well as the inference from the original FCM. In this way, we decompose the original FCM into a quotient FCM and some sectional FCMs. As a result, the analysis of the original FCM is reduced to the analysis of the quotient and sectional FCMs, which are often much smaller in size and complexity. Such a reduction is important in analyzing large-scale FCMs. We also propose a causal algebra in the quotient FCM, which indicates that the effect that one vertex influences another in the quotient depends on the weights and states of the vertices along directed paths from the former to the latter. To illustrate the process involved, we apply our decomposition theory to university management networks. Finally, we discuss possible approaches to partitioning an FCM and major concerns in constructing quotient FCMs. The results represented in this paper provide an effective framework for calculating and simplifying causal inference patterns in complicated real-world applications.**

***Index Terms*—Artificial intelligence, fuzzy causal network, fuzzy cognitive map, quotient networks.**

I. Introduction

A FUZZY cognitive map (FCM) is a fuzzy digraph with feedback [11], [12] that describes the causal relationships between concepts [13]. There are three kinds of elements in an FCM, namely the concepts, the causal relationships between concepts and the effects one concept influences another concept. These elements are represented by vertices, directed arcs and numerical values (called weights) associated with the arcs, respectively. Each vertex has a state. Most papers in the literature consider the two-state cases where the state of a vertex is either 1 or 0, corresponding to active or inactive respectively. The weight of a directed arc measures the strength of the effect of the initial vertex on the terminal vertex of the arc. This effect is valid only when the initial vertex is active. In other words, if the initial vertex is inactive at some time, it is thought to have no effect on the terminal vertex, even though such an effect is very strong when the initial vertex is active. The state space of an FCM is determined initially by an initial condition and then propagated automatically through the vertex function relative to a threshold until a static pattern is reached [17]. A causal inference is achieved when the FCM reaches a stable limit cycle or fixed point [12], [13]. Recently, the theory of FCM has found many applications in politics, economics, medicine, military, social relation and information system, *et al.* [5], [8], [21], [22], [25].

To deal with very complicated real-world applications, many researchers have made modifications to the concept of FCM; see, e.g., [10], [24]. Most recently, Liu and Miao have proposed a dynamic cognitive network (DCN) [17], [18]. The DCN allows each vertex to select its own state value according to the requirements of the system. The value set can be a binary set, a fuzzy set, or a continuous interval, so the DCN can describe the strength of causality between two causal concepts as well as the degree of causal concepts. It is also able to model dynamic cognitive processes.

Simplifying and calculating causal inference patterns in FCMs have been a major research activity. Liu *et al.* have made some significant attempts [14], [16]–[18]. They investigated extensively the inference properties of FCM, carried out formal analysis of the causal inference mechanism of FCM. These results provide a feasible and effective framework for the analysis and design of FCM in large-scale real-world applications. Despite these good results in the literature, the research on very complicated and large-scale FCMs is insufficient and many problems remain. Therefore, studying "huge" FCMs has become one of the most arduous as well as imperative tasks. In general FCMs are extremely difficult to analyze due to their large sizes and complex interconnections.

To make the analysis of FCMs feasible, we introduce a decomposition theory in this paper. We first partition the vertices of an FCM into several blocks according to an equivalence relation on the set of vertices of the FCM. Then we construct a quotient FCM relative to this partition. Topologically, the vertices of the quotient FCM are the blocks of the partition, and one block is joined to another block in the quotient by a directed arc if and only if there is at least one directed arc of the original FCM from a vertex in the first block to a vertex in the second block. Finally we define some rules regarding the vertex state

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and the strength one vertex influences another in the quotient FCM. In each block of the partition the original FCM induces a sectional FCM, which inherits the topological structure as well as the inference from the original FCM. The quotient FCM tells us the causal relationships and the effects among various blocks of the partition, and hence provides us the "global" information; whereas each sectional FCM gives us "local" information. Thus, in some sense we "decompose" the original FCM into the "product" of the quotient FCM and sectional FCMs. In this way, the analysis of a large FCM can be reduced to that of the quotient and sectional FCMs, which are usually much smaller in size and complexity.

This paper is organized as follows. In Section II, we give some basic definitions and preliminary results. In Section III, we discuss in detail the decomposition theory previously sketched. In Section IV, we propose a causal algebra in quotient FCMs. In Section V, we use an example to illustrate the application of our decomposition theory. Section VI discusses possible ways of partitioning an FCM and major considerations in constructing quotient FCMs. In Section VII, we give a short summary.

II. Background

A. Definitions and Preliminaries

The topological structure of an FCM is a digraph $\mho = (V, E)$, where $V = V(\mho)$ is the set of vertices of $\mho$ and $E = E(\mho)$ the set of arcs (directed edges) of $\mho$. (The reader is referred to [3] and [20] for terminology and notation on graphs and digraphs). As usual we use v_i, $i = 1, 2 \ldots, n$, to denote the vertices of $\mho$, where $n = |V|$ is the number of vertices of $\mho$. If there is an arc from v_i to v_j, then we use the ordered pair (v_i, v_j) to denote this directed arc. Each vertex v_i stands for a concept of the FCM, and the directed arc (v_i, v_j) means that the v_i has some influence on v_j. The strength (weight) of such an influence is usually given by a real number e_{ij} which, after normalization if necessary, can be assumed between -1 and 1. If $e_{ij} > 0$, then v_i has a positive influence on v_j; if $e_{ij} < 0$, then v_i has a negative influence on v_j; and if $e_{ij} = 0$, then v_i has no influence on v_j. So the strengths of influence between adjacent vertices can be viewed as a function

$$\begin{aligned} w_\mho\colon E(\mho) &\longrightarrow [-1, 1] \\ (v_i, v_j) &\mapsto e_{ij} \end{aligned} \tag{1}$$

defined on the arcs of $\mho$. At each vertex v_i there is a state space. We use $x_i(t)$ to denote the state of vertex v_i at time t; and for simplicity we assume that the state takes only binary values 1 and 0 (the case where the state space is not binary can be dealt with similarly). The state of the FCM at time t can then be represented conveniently by the vector function

$$\phi_\mho(t) = (x_1(t), \ldots, x_n(t)). \tag{2}$$

Now, we can formally define an FCM as follows.

Definition 2.1: Let $\mho = (V, E)$, $w_\mho\colon E(\mho) \longrightarrow [-1, 1]$ and $\phi_\mho(t)$ be as before. We call the triple $(\mho, w_\mho, \phi_\mho)$ an FCM on $\mho$.

The connectivity of the FCM is represented by the adjacency matrix

$$W_\mho = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & e_{ij} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \tag{3}$$

where the ith row of W lists the values e_{ik} $(k = 1, \ldots, n)$ emitting from v_i, and the jth column lists the values e_{kj} $(k = 1, \ldots, n)$ directing to v_j. So, the total input received by v_i at time t is given by

$$\mu_i = \mu_\mho(v_i) = \sum_{k=1}^{n} e_{ki} \cdot x_k(t). \tag{4}$$

Moreover, we have

$$\phi_\mho(t) \times W_\mho = (\mu_1, \ldots, \mu_n)$$

by the definition of matrix production.

The state $\phi_\mho(t)$ of $\mho$ is determined by an initial condition and given thresholds T_i at vertices v_i, $1 \le i \le n$. When $\phi_\mho(t)$ receives a series of external input sequences, its next state $\phi_\mho(t+1)$ will be updated by the following formula:

$$\phi_\mho(t+1) = f_T(\phi_\mho(t) \times W_\mho) \tag{5}$$

where

$$\begin{aligned} T &= (T_1, \ldots, T_n) \\ f_T(\phi_\mho(t) \times W_\mho) &= (f_{T_1}(\mu_1), \ldots, f_{T_n}(\mu_n)) \end{aligned}$$

with $f_{T_i}(\mu_i)$ the vertex function at v_i defined as follows.

Definition 2.2 [17]: Given a threshold T_i for the ith vertex v_i, the *vertex function* f_{T_i} of v_i is defined by

$$f_{T_i}(\mu_i) = \begin{cases} 1, & \text{if } \mu_i \ge T_i \\ 0, & \text{if } \mu_i < T_i. \end{cases} \tag{6}$$

B. Causal Algebra in FCMs

Extending the idea of Liu and Satur [15], in a recent paper [19] the first two authors proposed a causal algebra which provides an object-oriented framework based on different context levels for interrogating the FCM structure to reveal different conceptual views. Here, we discuss briefly this causal algebra since it is essential for the analysis and design of the quotient FCM in this paper. We first give the definition of a directed path.

Definition 2.3: In an FCM $\mho$ a *directed path* $P(u, v)$ from a vertex u to a vertex v is a sequence $v_1, v_2, \ldots, v_r$ of distinct vertices of $\mho$ such that (v_i, v_{i+1}) is an arc of $\mho$ for each $i = 1, \ldots, r-1$, where $v_1 = u$ and $v_r = v$. The integer $r-1$ is called the length of the path $P(u, v)$. In the particular case where $r = 2$, $P(u, v)$ reduces to the arc (u, v).

Definition 2.4: Let u, v be distinct vertices of $\mho$, and $P = P(u, v)$ be a directed path from u to v. We define the *effect* u influences v at discrete time t via P as

$$I_P(t) = \prod_{(y, z) \in E(P)} x_y(t) \cdot e_{yz} \tag{7}$$

where $E(P)$ is the set of directed arcs on P, $x_y(t)$ is the state of vertex y at time t, and e_{yz} is the weight associated with the directed arc (y, z) on the path P.

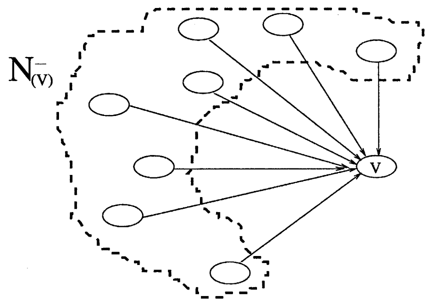

Fig. 1. Vertex v and its in-neighborhood $N^-(v)$.

From this definition, it follows that $I_P(t) \neq 0$ holds only when all vertices on P are active at time t; and, in this case, $I_P(t)$ is equal to the product of the weights of arcs on P. This fact coincides with our intuition.

In order to calculate the total effect with which a vertex u influences another vertex v at discrete time t, we should consider all directed paths from u to v. So we need the following definition.

Definition 2.5:

a) For any vertex v of $\mho$, we define

$$N^-(v) = \{u \mid u \in V,\, (u,\, v) \in E\}$$

that is, $N^-(v)$ is the set of vertices u of $\mho$ such that there is an arc from u to v. We call $N^-(v)$ the *in-neighborhood* of the vertex v.

b) For any two vertices $u,\, v$ of $\mho$, we define $\mathbf{P}(u,\, v)$ to be the set of all directed paths $P(u,\, v)$ from u to v, where $P(u,\, v)$ is as in Definition 2.3. For any $u_i \in N^-(v)$, let $P(u,\, u_i,\, v)$ denote a directed path from u to v which passes u_i, and define $\mathbf{P}(u,\, u_i,\, v)$ to be the set of all such $P(u,\, u_i,\, v)$.

We illustrate the concept of in-neighborhood by Fig. 1. In the case where $u_i = u$, which occurs only when $(u,\, v)$ is an arc of $\mho$, $P(u,\, u_i,\, v)$ is taken as the directed path $u,\, v$ of length 1, that is, the arc $(u,\, v)$; in this case this arc is the unique member of $\mathbf{P}(u,\, u_i,\, v)$. In general case, if there is no any directed path in $\mho$ from u to v via u_i, then of course $\mathbf{P}(u,\, u_i,\, v)$ is an empty set. Similarly, if there is no directed path from u to v, then $\mathbf{P}(u,\, v)$ is an empty set. From the definition, it follows that

$$\mathbf{P}(u,\, v) = \bigcup_{u_i \in N^-(v)} \mathbf{P}(u,\, u_i,\, v). \tag{8}$$

This simple observation is crucial to the following definition of the total effect vertex u has on vertex v at discrete time t via all directed paths from u to v.

Definition 2.6: For any two vertices $u,\, v$ of $\mho$, we define the *total effect* u has on v via all directed paths from u to v as

$$T_{(u,\,v)}(t) = \sum_{u_i \in N^-(v)} \max_{P \in \mathbf{P}(u,\, u_i,\, v)} I_P(t). \tag{9}$$

Also, we define $I^*_{(u,\,v)}(t)$ and $I_{*(u,\,v)}(t)$ to be the *strongest* and *weakest effect* u influences v at time t via all directed paths from u to v, respectively. In other words, we define

$$I^*_{(u,\,v)}(t) = \max_{P \in \mathbf{P}(u,\, v)} I_P(t) \tag{10}$$

$$I_{*(u,\,v)}(t) = \min_{P \in \mathbf{P}(u,\, v)} I_P(t). \tag{11}$$

In the particular case where there is a unique directed path $P = P(u,\, v)$ from u to v, it is clear from the definition that

$$T_{(u,\,v)}(t) = I^*_{(u,\,v)}(t) = I_{*(u,\,v)}(t) = I_P(t). \tag{12}$$

From (8), one can see that in the definition of total effect all directed paths from u to v are under our consideration. We should point out that, even if $(v,\, u)$ is an arc of $\mho$, it will make no contribution to the effect of u on v since it is in the opposite direction. On the other hand, if $(u,\, v)$ is an arc of $\mho$, then it does make contribution to the effect of u on v if u is active. In fact, in this case, we have $u \in N^-(v)$ and $\mathbf{P}(u,\, u,\, v)$ contains only one member, namely the directed path $Q = (u,\, v)$ of length 1. Hence $\max_{P\in\mathbf{P}(u,\,u,\,v)} I_P(t) = I_Q(t) = x_u(t)e_{uv}$, and this is contributed to (9). (Of course, in this case other directed paths from u to v, if existed, make contribution to the effect of u on v in an "indirect" way). Note that the values of $I_P(t), T_{(u,\,v)}(t), I^*_{(u,\,v)}(t)$ and $I_{*(u,\,v)}(t)$ all depend on the states $x_i(t)$ of vertices v_i on some paths from u to v, and in turn such x_i depends on the thresholds T_j at vertices $v_j,\, 1 \leq j \leq n$. Thus, when $\phi_\mho(t)$ receives a series of external input sequences, its next state $\phi_\mho(t+1)$ as well as the corresponding values $I_P(t+1)$, $T_{(u,\,v)}(t+1)$, $I^*_{(u,\,v)}(t+1)$ and $I_{*(u,\,v)}(t+1)$ will be updated automatically by (7), (9)–(11), respectively. We illustrate the discussion above by the simple example shown in Fig. 2, where the FCM has nine vertices. The input vertex v_1 (government's investment) is where the external stimulus can affect the system, and the directed paths from v_1 to v_9 (research quality) are the channels that carry the causal effect. The adjacency matrix of the FCM is

$$W_\mho = \begin{pmatrix} 0 & 0.6 & 0.7 & 0.5 & 0.7 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0.8 \\ -0.1 & 0 & 0 & 0 & 0 & 0.5 & 0.8 & 0.4 & 0 \\ -0.1 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0.5 & 0.4 \\ -0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 \\ 0.9 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \end{pmatrix}. \tag{13}$$

First, let us calculate the causal inference pattern. In order to do this, we can simply keep v_1 active during the inference cycle and indicate this as $\boxed{\mathbf{1}}$. Now, it is started with the government investment policy

$$\phi_\mho(0) = \left(\boxed{\mathbf{1}}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right).$$

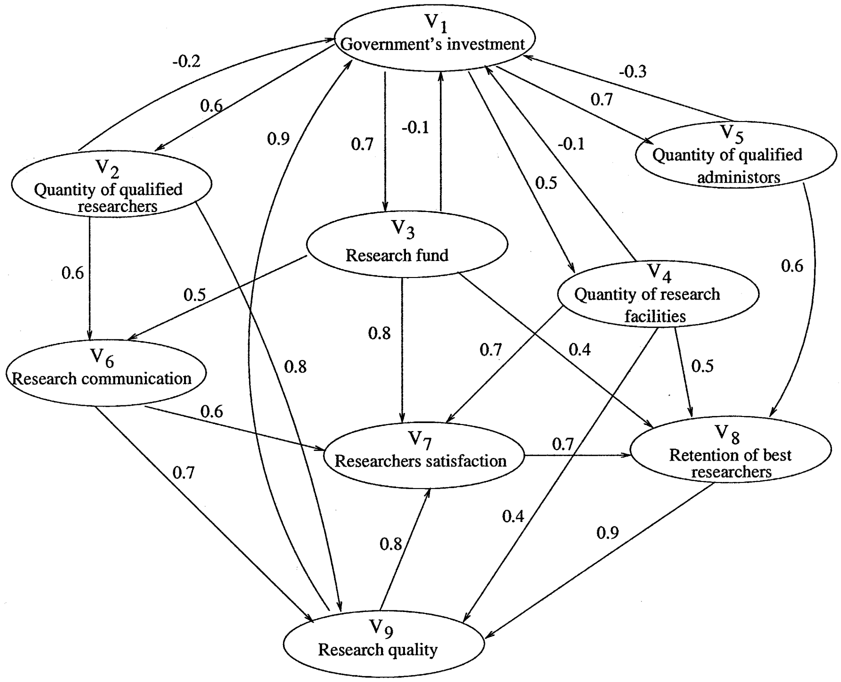

Fig. 2. FCM based on the goal "what influence does the government's investment to a university have on its research quality?"

In the following, we take the thresholds at nine vertices as $T_i = 0\ (i = 1, 2, 3, 4, 5)$, $T_6 = 1$, $T_j = 2\ (j = 7, 8, 9)$, respectively. Then

$$\begin{aligned}
\phi_{\mho}(0) \times W_{\mho} &= (0\ 0.6\ 0.7\ 0.5\ 0.7\ 0\ 0\ 0\ 0)\\
\phi_{\mho}(1) &= f_T(\phi_{\mho}(0) \times W_{\mho})\\
&= \left(\boxed{1}\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\right)\\
\phi_{\mho}(1) \times W_{\mho} &= (0\ 0.6\ 0.7\ 0.5\ 0.7\ 1.1\ 1.5\ 1.5\ 1.2)\\
\phi_{\mho}(2) &= f_T(\phi_{\mho}(1) \times W_{\mho})\\
&= \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\right)\\
\phi_{\mho}(2) \times W_{\mho} &= (0\ 0.6\ 0.7\ 0.5\ 0.7\ 1.1\ 2.1\ 1.5\ 1.9)\\
\phi_{\mho}(3) &= f_T(\phi_{\mho}(2) \times W_{\mho})\\
&= \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\right)\\
\phi_{\mho}(3) \times W_{\mho} &= (0\ 0.6\ 0.7\ 0.5\ 0.7\ 1.1\ 2.1\ 2.2\ 1.9)\\
\phi_{\mho}(4) &= f_T(\phi_{\mho}(3) \times W_{\mho})\\
&= \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\right)\\
\phi_{\mho}(4) \times W_{\mho} &= (0\ 0.6\ 0.7\ 0.5\ 0.7\ 1.1\ 2.1\ 2.2\ 2.8)\\
\phi_{\mho}(5) &= f_T(\phi_{\mho}(4) \times W_{\mho})\\
&= \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\right)\\
\phi_{\mho}(5) \times W_{\mho} &= (0\ 0.6\ 0.7\ 0.5\ 0.7\ 1.1\ 2.9\ 2.2\ 2.8)\\
\phi_{\mho}(6) &= f_T(\phi_{\mho}(5) \times W_{\mho})\\
&= \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\right).
\end{aligned}$$

Therefore, the state $\phi_{\mho}(t)$ for this FCM at time t is determined by an initial condition and the thresholds given above. When $\phi(t)$ receives a series of external input sequences, i.e.,

$$\phi_{\mho}(0) = \left(\boxed{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right)$$

its next states $\phi(t+1)$ will be updated automatically by the following processes until the fixed point $\phi_{\mho}(5)$ is reached:

$$\phi_{\mho}(0) \to \phi_{\mho}(1) \to \phi_{\mho}(2) \to \phi_{\mho}(3) \to \phi_{\mho}(4) \to \phi_{\mho}(5).$$

That is

$$\begin{aligned}
&\left(\boxed{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right)\\
&\quad \to \left(\boxed{1}\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\right) \to \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\right)\\
&\quad \to \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\right) \to \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\right)\\
&\quad \to \left(\boxed{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\right).
\end{aligned} \tag{14}$$

Second, according to Definition 2.5, we obtain $N^-(v_9) = \{v_2, v_4, v_6, v_8\}$. From Fig. 2, it is easy to find that, there is one directed path from v_1 to v_9 via v_2 and another directed path from v_1 to v_9 via v_4; there are two directed paths from v_1 to v_9 via v_6 and six directed paths from v_1 to v_9 via v_8. We can then simulate the effect that the government's investment to a university (v_1) has on the university's research quality (v_9) at different time t, which are shown in Table I.

Finally, from Table I, it is easy to find that the strongest and weakest effect v_1 influences v_9 via all directed paths from v_1 to v_9, for times $t = 0, 1, \ldots, 5$, are as follows:

$$\begin{aligned}
I^*_{(v_1, v_9)}(t) &= \max_{P \in \mathbf{P}(v_1, v_9)} I_P(t) = I_{(v_1, v_2, v_9)}(t)\\
&= 0,\ 0.48,\ 0.48\ 0.48\ 0.48,\ 0.48\\
I_{*(v_1, v_9)}(t) &= \min_{P \in \mathbf{P}(v_1, v_9)} I_P(t) = I_{(v_1, v_3, v_6, v_7, v_8, v_9)}(t)\\
&= 0,\ 0,\ 0,\ 0,\ 0.1323,\ 0.1323.
\end{aligned}$$

TABLE I
EFFECTS $I_P(t)$ VERTEX v_1 INFLUENCES VERTEX v_9 AT DIFFERENT TIME t VIA ALL DIRECTED PATHS $P(v_1, v_9)$

$P = P(v_1, v_9)$	$I_P(t)$					
	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
(v_1, v_2, v_9)	0	0.48	0.48	0.48	0.48	0.48
(v_1, v_4, v_9)	0	0.2	0.2	0.2	0.2	0.2
(v_1, v_2, v_6, v_9)	0	0	0.252	0.252	0.252	0.252
(v_1, v_3, v_6, v_9)	0	0	0.245	0.245	0.245	0.245
(v_1, v_3, v_8, v_9)	0	0	0	0	0.252	0.252
(v_1, v_4, v_8, v_9)	0	0	0	0	0.225	0.225
(v_1, v_5, v_8, v_9)	0	0	0	0	0.378	0.378
$(v_1, v_3, v_7, v_8, v_9)$	0	0	0	0	0.3528	0.3528
$(v_1, v_2, v_6, v_7, v_8, v_9)$	0	0	0	0	0.13608	0.13608
$(v_1, v_3, v_6, v_7, v_8, v_9)$	0	0	0	0	0.1323	0.1323

Furthermore, from Table I, we can obtain that, for $t = 0, 1, \ldots, 5$

$$\max_{P\in\mathbf{P}(v_1, v_2, v_9)} I_P(t) = 0, 0.48, 0.48, 0.48, 0.48, 0.48$$
$$\max_{P\in\mathbf{P}(v_1, v_4, v_9)} I_P(t) = 0, 0.2, 0.2, 0.2, 0.2, 0.2$$
$$\max_{P\in\mathbf{P}(v_1, v_6, v_9)} I_P(t) = 0, 0, 0.252, 0.252, 0.252, 0.252$$
$$\max_{P\in\mathbf{P}(v_1, v_8, v_9)} I_P(t) = 0, 0, 0, 0, 0.378, 0.378.$$

According to (9), we know that the total effect v_1 influences v_9 via all directed paths from v_1 to v_9 is

$$T_{(v_1, v_9)}(t) = \sum_{v_i\in N^-(v_9)} \max_{P\in\mathbf{P}(v_1, v_i, v_9)} I_P(t)$$

that is

$$T_{(v_1, v_9)}(t) = \max_{P\in\mathbf{P}(v_1, v_2, v_9)} I_P(t) + \max_{P\in\mathbf{P}(v_1, v_4, v_9)} I_P(t) + \max_{P\in\mathbf{P}(v_1, v_6, v_9)} I_P(t) + \max_{P\in\mathbf{P}(v_1, v_8, v_9)} I_P(t).$$

Therefore, $T_{(v_1, v_9)}(t)$ at times $t = 0, 1, 2, 3, 4, 5$ are 0, 0.68, 0.932, 0.932, 1.31, 1.31, respectively. The aforementioned results can be explained in detail as follows.

1) The causal inference pattern indicates that the persistent vertex v_1 (government's investment) makes v_2 (quantity of qualified researchers), v_3 (research fund), v_4 (quantity of research faculities) and v_5 (quantity of qualified administrators) active. Next step, the vertex v_6 (research communication) is activated. Then, the next two vertices v_7 (researchers satisfaction) and v_8 (retention of best researchers) are activated. Finally, v_9 (research quality) is activated.
2) The values of $I^*_{(v_1, v_9)}(t)$ at different times indicate that the strongest effect that v_1 has on v_9 via different directed paths is the effect $I_{(v_1, v_2, v_9)}(t)$ from v_1 to v_9 via v_2 (quantity of qualified researchers). This means that if government's investment can make the university employ enough qualified researchers, then these qualified researchers will produce the strongest as well as most directly impact on the research quality.

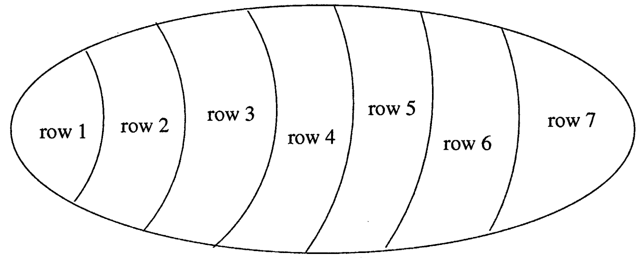

Fig. 3. Partitioning a class of students into seven groups.

3) The values of $I_{*(v_1, v_9)}(t)$ at different times indicate that the weakest effect that v_1 has on v_9 via different directed paths is the effect $I_{(v_1, v_3, v_6, v_7, v_8, v_9)}(t)$ from v_1 to v_9 via v_3 (research fund), v_6 (research communication), v_7 (research satisfaction) and v_8 (retention of best researchers).
4) The values of $T_{(v_1, v_9)}(t)$ at different times indicate that the total effect with which v_1 influences v_9 via all possible paths from v_1 to v_9 equals the sum of the strongest effects that v_1 has on v_9 via the directed paths through the in-neighbor v_i $(i = 2, 4, 6, 8)$ of v_9.

Note that the final stable state may be quite different if we choose different thresholds at the nine vertices of Fig. 2. For example, if we set $T_i = 0.75$ for each v_i, then v_9 will not be activated even if the initial condition remains the same, and as a result v_1 will have no influence on v_9.

C. Partition of the Vertices of an FCM

In order to construct a quotient FCM, we need to partition the set of vertices of the original FCM. For this purpose, we first introduce the concept of equivalence relation on a set.

Definition 2.7: Given two sets S and T, a *binary relation* ρ between S and T is a subset of the Cartesian product $S \times T$. If $(x, y) \in \rho$, then we say that x and y have the binary relation ρ, and we denote this fact by $x\rho y$. In particular, for a set S, a binary relation on $S \times S$ (the set of ordered pairs of elements of S) is called a binary relation on S.

Definition 2.8: Let ρ be a binary relation on a set S. Then, ρ is *reflexive* $\Longleftrightarrow (\forall x)(x \in S \Rightarrow (x, x) \in \rho)$.

ρ is *symmetric* $\Longleftrightarrow (\forall x)(\forall y)(x \in S, y \in S, (x, y) \in \rho \Rightarrow (y, x) \in \rho)$.

ρ is *transitive* $\Longleftrightarrow (\forall x)(\forall y)(\forall z)(x \in S, y \in S, z \in S, (x, y) \in \rho, (y, z) \in \rho \Rightarrow (x, z) \in \rho)$.

A binary relation ρ which is reflexive, symmetric and transitive is called an *equivalence relation*.

We can illustrate an important feature of the equivalence relation by a simple example as shown in Fig. 3. Let S be the set of students in a class. Define an equivalence relation ρ on S by "$x\rho y \Leftrightarrow x$ sits in the same row as y in the class." If the students sit along seven rows and if the students sitting in the same row are grouped together, then S can be divided into seven groups in such a way that every student in the class belongs to one and only one group. Similar result can be extended to any equivalence relation on any set. For this we need the following definition which is basic to constructing our quotient FCMs.

Definition 2.9: A *partition* of a set S is a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of nonempty subsets $B_1, \ldots, B_k$ of S satisfying the following conditions:

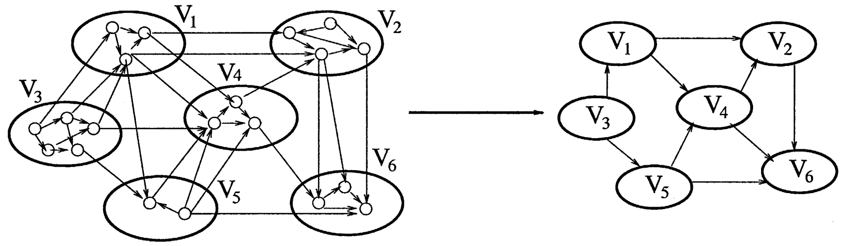

Fig. 4. Quotient FCM relative to a partition.

a) $B_1 \cup B_2 \cup \cdots \cup B_k = S$;
b) $B_i \cap B_j = \emptyset$ whenever $i \neq j$.

In this case, each B_i is called a *block* of the partition $\mathcal{B}$.

It is well known (see, e.g., [9]) that any equivalence relation ρ on a set S induces a partition $\mathcal{B}$ of S such that two elements of S are in the same block of $\mathcal{B}$ if and only if they are equivalent under ρ. Conversely, any partition $\mathcal{B}$ of S induces an equivalence relation ρ on S in which two elements of S have relation ρ if and only if they are in the same block of $\mathcal{B}$.

III. Quotient FCM

FCM is powerful and flexible in representing structured knowledge and in combining knowledge from different human experts. However, in real-world applications, the FCM usually contains a large number of vertices with very complicated connections among them. As a consequence, it is either impossible or difficult to analyze such FCMs directly. To make the analysis feasible, we construct a quotient FCM relative to a partition of vertices. In this section, we introduce the construction of the quotient FCM as well as sectional FCMs.

A. Topological Structure

We first discuss the topological structures of the quotient and sectional FCMs.

Definition 3.1: Suppose $\mho = (V, E)$ is an FCM with n vertices, and $\mathcal{B} = \{V_1, \ldots, V_k\}$ is a partition of the vertex set V. Define a new FCM with vertices the blocks of $\mathcal{B}$ such that (V_i, V_j) is an arc if and only if there exists at least one arc of $\mho$ initiating at a vertex in V_i and terminating at a vertex in V_j. We call this new FCM the *quotient fuzzy cognitive map* of $\mho$ relative to the partition $\mathcal{B}$, or simply a quotient FCM, and denote it by $\mho_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$.

We point out that, in a quotient FCM, just as in a usual FCM, pointing back arcs are allowed. That is, there may exist blocks V_i, V_j such that both (V_i, V_j) and (V_j, V_i) are arcs of the quotient $\mho_{\mathcal{B}}$. This will cause no problem in our analysis of quotient FCMs. Also, we should point out that the way of partitioning an FCM should match with (and is often determined by) our interest and purpose, see the discussion in Section VI-B and at the end of Section V. In a sense, the quotient FCM tells us "globe" information about the original FCM. However, it does not contain "local" information about interaction among vertices within the same block. Such information is contained in the corresponding sectional FCM, defined as follows. The analysis of the original FCM is thus reduced to the analysis of the quotient FCM and sectional FCMs.

Definition 3.2: Let $\mho$ and $\mathcal{B}$ be as in Definition 3.1. For each $V_i \in \mathcal{B}$, let E_i denote the set of arcs of $\mho$ with both end-vertices in V_i. We call $\mho_i = (V_i, E_i)$ the *sectional fuzzy cognitive map* of $\mho$ on V_i, or simply the sectional FCM on V_i. The state of a vertex and the strength with which one vertex in V_i influences another vertex in V_i are defined to be the same as that in $\mho$.

The definitions above can be illustrated by Fig. 4, where the left part is an FCM which has been partitioned into six sectional FCMs (blocks). We regard the set of vertices of each sectional FCM as a new vertex, and construct the quotient FCM based on these new vertices. Such a quotient is shown in the right part of Fig. 4.

The following theorem tells us the relationship between the original FCM and its sectional FCMs together with "intersectional" connections.

Theorem 3.1: Suppose $\mho = (V, E)$ is an FCM, and $\mathcal{B} = \{V_1, \ldots, V_k\}$ is a partition of the vertex set V. Let $\mho_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ be the quotient FCM relative to the partition $\mathcal{B}$. Then

$$\mho = \left(\bigcup_{i=1}^{k} \mho_i \right) \cup \left(\bigcup_{j=1}^{k} \bigcup_{\substack{i \neq j \\ i=1}}^{k} B(\mho_j, \mho_i) \right)$$

where $\mho_i = (V_i, E_i)$ is the sectional FCM of $\mho$ induced on V_i and

$$B(\mho_i, \mho_j) = \begin{cases} \{(v_i, v_j) | v_i \in V_i,\, v_j \in V_j\}, & \text{if } i \neq j \\ \emptyset, & \text{if } i = j. \end{cases}$$

The proof of this theorem is similar to that of [17, Th. 2] and is omitted. Furthermore, from the definition of the sectional FCM and the remark at the end of Section II-C we have the following properties:

$$V(\mho_i) \cap V(\mho_j) = \emptyset, \qquad i \neq j$$
$$E(\mho_i) \cap E(\mho_j) = \emptyset, \qquad i \neq j.$$

B. States of Vertices in the Quotient

In the aforementioned discussion, we just define the topological structure of the quotient FCM $\mho_{\mathcal{B}}$. This is not enough since we need to define the states of the vertices (blocks) and the strength each vertex (block) influences each of its neighboring vertices (blocks). As before, for the sake of simplicity and without loss of generality, we focus only on binary vertex states. The case of real-value states can be dealt with similarly. Let us first discuss the vertex state of $\mho_{\mathcal{B}}$.

Definition 3.3: Let $\mho$ and $\mathcal{B}$ be as in Definition 3.1, and let $\mho_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ be the quotient FCM relative to $\mathcal{B}$. Let $x_i^{\mathcal{B}}(t) = x_i^{\mathcal{B}}(\mho_{\mathcal{B}})$ stand for the state of the vertex V_i of $\mho_{\mathcal{B}}$ at time t, and let

$$\phi_{\mho_{\mathcal{B}}}(t) = (x_1^{\mathcal{B}}(t), \ldots, x_k^{\mathcal{B}}(t)) \tag{15}$$

stand for the state vector of $\mho_{\mathcal{B}}$ at time t. We define $x_i^{\mathcal{B}}(t)$ such that $x_i^{\mathcal{B}}(t) = 1$ if and only if there is at least one vertex $v_j \in V_i$ with $x_j(t) = 1$, and for such a v_j there exists at least one arc (v_j, v_ℓ) from v_j to another block V_ℓ $(1 \le \ell \le k, \ell \ne i)$, where as before $x_j(t)$ is the state of v_j at time t. In other words, we define

$$x_i^{\mathcal{B}}(t) = \max_{\substack{1 \le \ell \le k, \ell \ne i \\ v_j \in V_i, (v_j, v_\ell) \in B(\mho_i, \mho_\ell)}} x_j(t) \tag{16}$$

for each $i = 1, \ldots, k$.

This definition indicates that the state value $x_i^{\mathcal{B}}(t)$ of vertex V_i at time t is determined by the state values of those vertices of V_i having at least one out-going directed arc to other blocks.

C. Weights of Arcs in the Quotient

To complete the construction of the quotient FCM we also need to define the weight f_{ij} of the arc (V_i, V_j) of $\mho_{\mathcal{B}}$. For this purpose, we use the techniques for aggregating fuzzy subsets and a number of useful methods in the literature (e.g., [6], [7], [27], and [28]). Among these methods, the t-norm and t-conorm, which generate the intersection and union operations, are the two basic classes of aggregation operators. Averaging operators are useful for global evaluation of an action as lying between the worst and the best local ratings or the conflicting goals [28]. In the following, we use the ordered-weighted averaging (OWA) operators introduced in [27].

Definition 3.4: Let $\mho$ and $\mathcal{B}$ be as in Definition 3.1, and $\mho_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ be the quotient FCM relative to $\mathcal{B}$. Suppose there are ℓ arcs from V_i to V_j in $\mho$, whose weights are $e_{i_1j_1}, \ldots, e_{i_lj_\ell}$. Then, the strength f_{ij} the vertex V_i influences the vertex V_j in $\mho_{\mathcal{B}}$ is a function

$$\begin{aligned} f\colon \mathbf{R}^\ell &\longrightarrow \mathbf{R} \\ (e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}) &\mapsto f_{ij} \end{aligned}$$

of $e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}$, and is defined by

$$f_{ij} = f(e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}) = \sum_{k=1}^{\ell} \xi_k b_k \tag{17}$$

where

$$\xi_k = \frac{e_{i_kj_k}}{\sum_{k=1}^{\ell} e_{i_kj_k}} \tag{18}$$

and b_k is the kth largest element of the multi-set $\{e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}\}$.

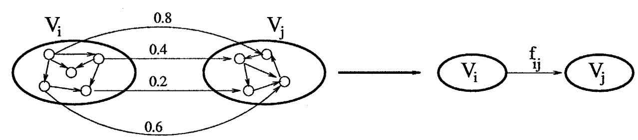

Fig. 5. Strength of an arc in the quotient FCM.

Obviously, these ξ_ks satisfy

$$\sum_{k=1}^{\ell} \xi_k = 1.$$

In addition, if all weights $e_{i_kj_k}$ are nonnegative, we have also

$$\xi_k \in [0, 1], \qquad 1 \le k \le \ell.$$

From the previous definition, one can prove that $-1 \le f_{ij} \le 1$. Hence, we have the following weight function for the quotient FCM $\mho_{\mathcal{B}}$:

$$\begin{aligned} w_{\mho_{\mathcal{B}}}\colon E_{\mathcal{B}} &\to [-1, 1] \\ (V_i, V_j) &\mapsto f_{ij}. \end{aligned} \tag{19}$$

This definition can be illustrated by Fig. 5. Suppose in the left-hand side of Fig. 5 there are four arcs from V_i to V_j with values

$$e_{i_1j_1} = 0.2,\ e_{i_2j_2} = 0.6,\ e_{i_3j_3} = 0.8,\ e_{i_4j_4} = 0.4.$$

Then, we obtain

$$\begin{aligned} \xi_1 &= \frac{0.2}{2} = 0.1 \quad & \xi_2 &= \frac{0.6}{2} = 0.3 \\ \xi_3 &= \frac{0.8}{2} = 0.4 \quad & \xi_4 &= \frac{0.4}{2} = 0.2. \end{aligned}$$

Thus, according to Definition 3.4, we have

$$\begin{aligned} f_{ij} &= f(0.2, 0.6, 0.8, 0.4) \\ &= 0.8 \cdot 0.1 + 0.6 \cdot 0.3 + 0.4 \cdot 0.4 + 0.2 \cdot 0.2 = 0.46. \end{aligned}$$

In general, from Definition 3.4, we have the following properties a)–d) for the strengths f_{ij} previously defined.

a) Boundary: Let m and M be the minimal and maximal value of $\{e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}\}$, respectively. Then

$$m \le f(e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}) \le M.$$

b) Commutativity: Let $\{d_{i_1j_1}, \ldots, d_{i_\ell j_\ell}\}$ be any permutation of $\{e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}\}$. Then

$$f(e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}) = f(d_{i_1j_1}, \ldots, d_{i_\ell j_\ell}).$$

c) Monotonicity: Let $\{d_{i_1j_1}, \ldots, d_{i_\ell j_\ell}\}$ and $\{e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}\}$ be two collections of aggregations such that $d_{i_kj_k} \ge e_{i_kj_k}$ for each $k = 1, \ldots, \ell$. Then

$$f(d_{i_1j_1}, \ldots, d_{i_\ell j_\ell}) \ge f(e_{i_1j_1}, \ldots, e_{i_\ell j_\ell}).$$

d) Idempotency: If $e_{i_k j_k} = e_{ij} = \text{constant}$ for all $k = 1, \ldots, \ell$, then

$$f(e_{i_1 j_1}, \ldots, e_{i_\ell j_\ell}) = e_{ij}.$$

We should emphasize that the strength f_{ij} defined in (17) relies on the particular choice of ξ_ks. In our definition above, we chose ξ_ks as given by (18). We may also choose different $\{\xi_1, \ldots, \xi_\ell\}$ satisfying $\sum_{k=1}^{\ell} \xi_k = 1$, and this will result in different weight function $w_{\mho_\mathcal{B}}$ and (17) will give rise to a different set of aggregation values. The following are three important special cases of the aggregation procedure.

i) f^*: In this case, we choose $(\xi_1, \ldots, \xi_\ell) = (1, \ldots, 0)$. Then

$$f^*(e_{i_1 j_1}, \ldots, e_{i_\ell j_\ell}) = \max\{e_{i_1 j_1}, \ldots, e_{i_\ell j_\ell}\} = M.$$

ii) f_*: In this case, we choose $(\xi_1, \ldots, \xi_\ell) = (0, \ldots, 1)$. Then

$$f_*(e_{i_1 j_1}, \ldots, e_{i_\ell j_\ell}) = \min\{e_{i_1 j_1}, \ldots, e_{i_\ell j_\ell}\} = m.$$

iii) f_A: In this case, we choose $(\xi_1, \ldots, \xi_\ell) = (1/\ell, \ldots, 1/\ell)$. Then

$$f_A(e_{i_1 j_1}, \ldots, e_{i_\ell j_\ell}) = \frac{1}{\ell} \cdot \sum_{k=1}^{\ell} e_{i_k j_k}.$$

D. Connectedness of the Quotient FCM

We conclude this section by proving the following property (Theorem 3.2) regarding the (weak) connectedness of the quotient FCM, which is important since it ensures that the FCM is not separated into several irrelevant pieces. First, we give the following definition.

Definition 3.5: A network is called (weak) *connected* if any two vertices in the network can be joined by an undirected path of the network, where an undirected path is a sequence of arcs, regardless of their directions, such that any two consecutive arcs in the sequence share one common vertex and that no vertex along these arcs appears more than once.

Without loss of generality, we may always assume that the original FCM under consideration is connected.[1] A fundamental problem relating to our theory of quotient FCM is as follows: If the original FCM is connected, whether the quotient FCM is connected as well? The answer to this question is affirmative, as we prove in the following theorem.

Theorem 3.2: Suppose $\mho = (V, E)$ is an FCM, and $\mathcal{B} = \{V_1, \ldots, V_k\}$ is a partition of the vertex set V of $\mho$. If $\mho$ is connected, then the quotient FCM $\mho_\mathcal{B} = (V_\mathcal{B}, E_\mathcal{B})$ is connected as well.

Proof: Suppose to the contrary that $\mho_\mathcal{B}$ is not connected. Then we can find two vertices in $\mho_\mathcal{B}$, say V_1, V_2, such that there is no path of $\mho_\mathcal{B}$ joining them. From this and the definition of the quotient FCM, it follows that any vertex in V_1 is not joined by a path of $\mho$ to any vertex in V_2. This contradicts with our assumption that $\mho$ is connected. Therefore, $\mho_\mathcal{B}$ must be connected. $\square$

[1]The reason lies in that, if an FCM $\mho$ is not connected, then the connected components of $\mho$ are irrelevant with each other. Since these irrelevant components induce irrelevant sectional FCMs, we can study them separately.

IV. Causal Algebra in Quotient FCMs

As mentioned before, FCM is a kind of inference network. Designing an FCM structure for a real-world application problem is only the first step. The main purpose for constructing the FCM is to calculate the inference pattern, so that we can provide a decision-support for scientists, decision-makers and policy proponents. Therefore, for a quotient FCM, calculating accurately the inference pattern is also a fundamental and important issue. In this section, we provide a causal algebra for quotient FCM by extending the causal algebra of the original FCM into the quotient FCM. First, we extend some definitions relative to the former to that of the latter.

Let $(\mho, w_\mho, \phi_\mho)$ be an FCM as in Definition 2.1, where $\mho = (V, E)$, $w_\mho$ is as in (1) and $\phi_\mho(t)$ is as in (2). Let $\mathcal{B} = \{V_1, \ldots, V_k\}$ be a partition of the vertex set V of $\mho$, and let the triple $(\mho_\mathcal{B}, w_{\mho_\mathcal{B}}, \phi_{\mho_\mathcal{B}})$ be the quotient FCM of $\mho$ relative to $\mathcal{B}$, where $\mho_\mathcal{B}$ is as defined in Section III-A, $\phi_{\mho_\mathcal{B}}$ is given by (15) in Definition 3.3, and $w_{\mho_\mathcal{B}}$ is as in (19) with f_{ij} defined in Definition 3.4. Then, the adjacency matrix of the quotient FCM can be expressed as

$$W_{\mho_\mathcal{B}} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & f_{ij} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}. \tag{20}$$

In order to decide how the next state $\phi_{\mho_\mathcal{B}}(t+1)$ of $\mho_\mathcal{B}$ be updated automatically, we define the threshold $T_i^\mathcal{B}$ $(i = 1, \ldots, k)$ at vertex V_i for the quotient FCM as follows.

Definition 4.1: Let $(\mho, w_\mho, \phi_\mho)$ be an FCM as in Definition 2.1. Let $\mathcal{B} = \{V_1, \ldots, V_k\}$ be a partition of the vertex set V of $\mho$, and let $(\mho_\mathcal{B}, w_{\mho_\mathcal{B}}, \phi_{\mho_\mathcal{B}})$ be the quotient FCM relative to $\mathcal{B}$. We define the threshold $T_i^\mathcal{B}$ at vertex V_i as the maximum of all the thresholds T_j of those $v_j \in V_i$ such that there is at least one arc (v_j, v_ℓ) from v_j to some vertex v_ℓ in another block V_ℓ $(1 \le \ell \le k,\ \ell \ne i)$. In other words, the threshold $T_i^\mathcal{B}$ at vertex V_i in the quotient FCM is defined by

$$T_i^\mathcal{B} = \max_{\substack{1 \le \ell \le k, \ell \ne i \\ v_j \in V_i,\ (v_j, v_\ell) \in B(\mho_i, \mho_\ell)}} T_j. \tag{21}$$

We denote by

$$T^\mathcal{B} = (T_1^\mathcal{B}, \ldots, T_k^\mathcal{B})$$

the vector of such thresholds.

The total input μ_i received by V_i is given by the formula

$$\mu_i = \mu_{\mho_\mathcal{B}}(V_i) = \sum_{j=1}^{k} f_{ji} \cdot x_j^\mathcal{B}(t). \tag{22}$$

From this and from (20) and (15), it follows that

$$\phi_{\mho_\mathcal{B}}(t) \times W_{\mho_\mathcal{B}} = (\mu_1, \ldots, \mu_k).$$

Once we have computed μ_i for each i, we then use the following definition to determine the vertex function $f_{T_i^\mathcal{B}}(\mu_i)$ at V_i.

Definition 4.2: Given the threshold $T_i^{\mathcal{B}}$ of V_i as defined in (21), the *vertex function* $f_{T_i^{\mathcal{B}}}$ of V_i is defined by

$$f_{T_i^{\mathcal{B}}}(\mu_i) = \begin{cases} 1, & \text{if } \mu_i \geq T_i^{\mathcal{B}} \\ 0, & \text{if } \mu_i < T_i^{\mathcal{B}}. \end{cases} \tag{23}$$

It is obvious that the state $\phi_{\mho_{\mathcal{B}}}(t)$ of $\mho_{\mathcal{B}}$ at time t is determined by an initial condition and the thresholds $T_i^{\mathcal{B}}$ at vertices V_i, $1 \leq i \leq k$. When $\phi_{\mho_{\mathcal{B}}}(t)$ receives a series of external input sequences, its next state $\phi_{\mho_{\mathcal{B}}}(t+1)$ is determined according to the following formula:

$$\phi_{\mho_{\mathcal{B}}}(t+1) = f_{T^{\mathcal{B}}}(\phi_{\mho_{\mathcal{B}}}(t) \times W_{\mho_{\mathcal{B}}}) \tag{24}$$

where

$$f_{T^{\mathcal{B}}}(\phi_{\mho_{\mathcal{B}}}(t) \times W_{\mho_{\mathcal{B}}}) = (f_{T_1^{\mathcal{B}}}(\mu_1), \ldots, f_{T_k^{\mathcal{B}}}(\mu_k)).$$

The following theorem gives some properties about the causal algebra of the quotient FCM.

Theorem 4.1: Let $(\mho, w_{\mho}, \phi_{\mho})$ be an FCM. Let $\mathcal{B} = \{V_1, \ldots, V_k\}$ be a partition of the vertex set of V, let $(\mho_{\mathcal{B}}, w_{\mho_{\mathcal{B}}}, \phi_{\mho_{\mathcal{B}}})$ be the quotient FCM relative to the partition $\mathcal{B}$, and let $T_i^{\mathcal{B}}$ be the threshold at vertex V_i of the quotient FCM as in Definition 4.1. Then, the following properties hold at any time t.

a) If the vertex V_i in the quotient FCM is active, then there exists at least one vertex $v_j \in V_i$ such that v_j is active, and for such a v_j there is at least one arc (v_j, v_ℓ) directing to another block V_ℓ $(1 \leq \ell \leq k, \ell \neq i)$ of $\mathcal{B}$.

b) If any vertex v_j of V_i in the FCM is active, where v_j satisfies the condition set in a), then the vertex V_i in the quotient FCM is definitely active.

Proof:

a) Assume that the vertex V_i in the quotient FCM is active. Then $x_i^{\mathcal{B}}(t) = 1$ according to Definition 3.3. From (16), we have

$$x_i^{\mathcal{B}}(t) = \max_{\substack{1 \leq \ell \leq k, \ell \neq i \\ v_j \in V_i,\, (v_j, v_\ell) \in B(\mho_i, \mho_\ell)}} x_j(t).$$

Hence, there exists at least one vertex v_j in V_i such that $x_j(t) = 1$, where v_j satisfies the condition specified in a). It follows that this vertex v_j is also active.

b) Assume that any vertex v_j of V_i is active, that is, its state value $x_j(t) = 1$. From the expression of $x_i^{\mathcal{B}}(t)$, we can obtain immediately that $x_i^{\mathcal{B}}(t) = 1$, which means that the vertex V_i of the quotient FCM is active. $\square$

We can also extend the mechanism of the strongest, weakest and total effects that one vertex has on another vertex in the FCM to the associated quotient FCMs.

For distinct vertices U, V of the quotient FCM $\mho_{\mathcal{B}}$, let $P(U, V)$ denote a directed path of $\mho_{\mathcal{B}}$ from U to V. Let $\mathbf{P}(U, V)$ denote the set of all such directed paths $P(U, V)$ from U to V. For $P \in \mathbf{P}(U, V)$, let $I_P(t)$ denote the effect with which vertex U influences vertex V at time t via the directed path P. Then, from (7) in Definition 2.4, we can calculate $I_P(t)$ by using the formula

$$I_P(t) = \prod_{(Y, Z) \in E(P)} x_Y^{\mathcal{B}}(t) \cdot f_{YZ} \tag{25}$$

where $E(P)$ is the set of directed arcs on the directed path P, $x_Y^{\mathcal{B}}(t)$ is the state of vertex Y at time t and f_{YZ} is the weight associated with the directed arc (Y, Z) on the path P. If P consists of distinct vertices sequences $U = V_1, \ldots, V_r = V$, then this effect of U on V at time t via P can be written as

$$I_P(t) = I_{(V_1, V_2, \ldots, V_r)}(t) = \prod_{i=1}^{r-1} x_i^{\mathcal{B}}(t) \cdot f_{i\, i+1}$$

where $x_i^{\mathcal{B}}(t)$ is given by (16) and $f_{i\, i+1}$ is as defined in (17). For each vertex V of $\mho_{\mathcal{B}}$, let $N^-(V) = \{U \mid U \in V_{\mathcal{B}}, (U, V) \in E_{\mathcal{B}}\}$ be the set of vertices U of $\mho_{\mathcal{B}}$ such that there is an arc from U to V, that is, $N^-(V)$ is the in-neighborhood (as defined in Definition 2.5) of the vertex V in the quotient FCM. For $U_i \in N^-(V)$, let $P(U, U_i, V)$ denote a directed path from U to V passing the vertex U_i, and let $\mathbf{P}(U, U_i, V)$ be the set of all such paths. Let $T_{(U,V)}(t)$, $I^*_{(U,V)}(t)$ and $I_{*(U,V)}(t)$ be the total, strongest and weakest effect that U influences V via all directed paths from U to V, respectively. Then, from (9)–(11), we can compute the values of these effects by the following formulas:

$$T_{(U,V)}(t) = \sum_{U_i \in N^-(V)} \max_{P \in \mathbf{P}(U, U_i, V)} I_P(t) \tag{26}$$

$$I^*_{(U,V)}(t) = \max_{P \in \mathbf{P}(U,V)} I_P(t) \tag{27}$$

$$I_{*(U,V)}(t) = \min_{P \in \mathbf{P}(U,V)} I_P(t). \tag{28}$$

In particular, if there is a unique directed path $P = P(U, V)$ from U to V, then all these effects are equal, that is

$$T_{(U,V)}(t) = I^*_{(U,V)}(t) = I_{*(U,V)}(t) = I_{P(U,V)}(t). \tag{29}$$

Note that, in general, the values of $I_P(t)$, $T_{(U,V)}(t)$, $I^*_{(U,V)}(t)$ and $I_{*(U,V)}(t)$ at time t are determined by an initial condition and the thresholds $T_i^{\mathcal{B}}$ at vertices V_i, for $1 \leq i \leq k$. When $\phi_{\mho_{\mathcal{B}}}(t)$ receives a series of external input sequences, their next values will be updated according to (25)–(28), respectively, until the static states are reached. This is consistent with the general theory for ordinary FCMs discussed in Section II-B.

V. Example

As a demonstration we now use our decomposition theory to analyze a university management network. Obviously, the decision-makers at a university have to modify and adjust their policies from time to time in order to attract more good students as well as maintain a high retention rate. The retention rate is often thought to indicate students' satisfaction with their university program, and influences potentially the quality and reputation of the university. What actions can the university take to achieve the above goal? The decision-makers need to collect feedbacks from the staff. In this case, it is very difficult for the decision-makers to deal directly with staff individually. A natural approach would be to collect feedbacks from different units

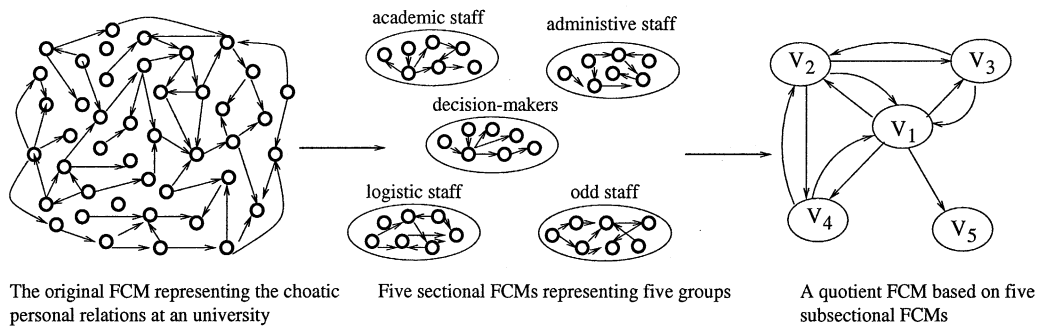

Fig. 6. University management network and its quotient and sectional FCMs.

of the university. In general, we can partition the staff into several blocks based on their official duties. Now let us simulate the university management network by using the scheme introduced in the previous two sections.

First, we define the original FCM for the decision-makers. Let v_i stand for a staff member of the university. For simplicity, we use the same notation to denote the feedback given by v_i to some policies. Let e_{ij} stand for the causal strength from staff member v_i to staff member v_j. Then, the FCM $\mho = (V, E)$ models the university management network, where $V = \{v_i | v_i$ is a staff member of the university$\}$ and $E = \{e_{ij} | e_{ij}$ is the causal strength from v_i to $v_j\}$. Note that, for brevity, here we use the same notation E for both the arc set and the entire effect of arcs. This applies also to the quotient FCM constructed in the following. We take the vertex state value x_i as 1 or 0, which means respectively that v_i has response or no response to these policies. Second, we partition the vertex set V into nonempty disjoint blocks by the following equivalence relation

"$v_i \rho v_j \Leftrightarrow v_i$ is responsible for the same official duty as v_j at the university."

Then the staff at the university can be partitioned into five blocks, i.e., decision-makers, academic staff, administrative staff, logistic staff, and odd staff. We denote them by $\mathcal{B} = \{V_1, V_2, V_3, V_4, V_5\}$, respectively. Third, we regard each $V_i (i = 1, \ldots, 5)$ as a new vertex and define a quotient FCM $\mho_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ based on these new vertices, where $V_{\mathcal{B}} = V(\mho_{\mathcal{B}}) = \{V_1, V_2, V_3, V_4, V_5\}$, and $E_{\mathcal{B}} = E(\mho_{\mathcal{B}}) = \{f_{ij} | i, j = 1, \ldots, 5\}$ stands for the entire effect of blocks V_i on blocks V_j. The vertex state values and the weights on the arcs of the quotient FCM can be calculated by Definitions 3.3 and 3.4. Finally, each block induces a sectional FCM. The analysis of a very complicated university management network is thus reduced to the analysis of a quotient network and some sectional FCMs, which is much easier to manage, see Fig. 6. Due to the limit of space, we will not go into details.

Note that, for the same university management network above, if the policy makers would like to make some policies to promote the multiculturalism in the university to attract more international students, the partition may be based on ethnic backgrounds, so that the policy-makers can get accurate feedbacks from different cultural backgrounds. In the next section, we will discuss general methods of partitioning an FCM together with related concerns.

VI. Discussion

A. Procedures for Constructing a Quotient FCM

Thus far, we have introduced in detail the decomposition theory for fuzzy cognitive maps. As we mentioned before, the theory can be used as a tool to simplify the analysis and design of fuzzy cognitive maps, which is important in practical problems. In general, to analyze a complex real-world problem, we can implement the proposed decomposition theory as follows.

1) Model the original FCM for the real-world problem.
2) Define an appropriate equivalence relation ρ on the vertex set V, and then partition V into some blocks $\mathcal{B} = \{V_1, \ldots, V_k\}$ according to this equivalence relation.
3) Regard each block V_i as a new vertex, and construct a quotient FCM based on these new vertices.
4) Calculate the vertex state values and the weights of the arcs relative to the quotient FCM. Then, analyze the causal inference of the quotient FCM, which can provide the "global" information of the original FCM.
5) Each block V_i induces a sectional FCM, which keeps the topological structure as well as the inference of the original FCM. Such sectional FCMs can provide the "local" information of the original FCM.

In applying our decomposition theory, if it is still difficult to analyze the quotient FCM and the sectional FCM due to their large sizes, we may implement the scheme recursively until a satisfactory solution is reached.

B. Partitioning an FCM

It is obvious that partitioning appropriately the vertex set of an FCM into nonempty disjoint blocks is the first step for constructing a quotient FCM of the original FCM. In doing so we must consider the interest and purpose of the particular FCM under consideration. Even for the same practical application problem, if the interest we are concerned with is different, we may need to choose a different partition procedure; and this will result in a quite different quotient FCM as well as causal inference pattern. We have encountered this at the end of the previous section. In a number of cases, it might be very efficient to partition an FCM into "clusters," that is, subsets of vertices within which there are many interconnections. In the language of graph theory, this is to say that each block induces a very "dense" subgraph, meaning that the ratio of the number of arcs to the number of vertices is large.

A lot of fuzzy events in real-world applications can be naturally partitioned into different blocks according to their intrinsic characteristics or the role they play in the applications [4], and this is the case in our example in Section V. Also, many useful techniques regarding how to partition many chaotic events into some regular blocks have been found in the literature, e.g., PROCFIN [1], production rules, k-nearest neighbor, multi-layer perception and logistic regression (e.g., [2], [6], [7], [23], and [26]). Among these techniques, a very popular approach is to prepartition a certain fuzzy set into several blocks based on a fuzzy scoring function, which represents the closest resemblance with an event. We may then apply the majority-voting rule to assessing each ambiguous case in an appropriate block [7]. The results presented in the literature are encouraging and have shown the usefulness in resolving medical classification, social relation and networks management, *et al.* However, we must point out that some events in some very complicated practical applications possess very ambiguous intrinsic characteristics, or are extremely uncertain. As a result, it is very difficult to decide which block they should belong to. In this case, it is inappropriate to construct a quotient FCM subjectively.

C. Major Concerns in Constructing Quotient FCMs

Constructing an appropriate quotient FCM is not an easy task. Many factors can lead to an inappropriate quotient FCM, which then results in inaccurate or even unreliable causal inferences. In the following, we discuss three major factors that may lead to problems.

1) Inappropriate original FCM model

   In many real-world applications, the specialist may have vague or ambiguous ideas about their expectations or goals. This may lead to an inappropriate original FCM.

2) Inappropriate partition of vertices

   As discussed in Section VI-B, sometimes intrinsic characteristics of vertices in the original FCM is not well-defined. Partitioning can be a difficult task, as there may exist several partitions that all make sense. This might result in an inaccurate and inconsistent inference pattern.

3) Inappropriate multiedges aggregation

   The aggregation procedure for the arcs of the quotient FCM plays an important role in obtaining an accurate causal inference result. Each method of aggregation has its own merits and disadvantages, and it is difficult to find a unifying approach which suits all cases. For example, although the aggregation technique introduced in this paper has a satisfactory effect on the performance in many situations, it may not be universally applicable to all applications.

VII. Conclusion

In this paper, we have proposed the theory of quotient FCMs and sectional FCMs for a fuzzy cognitive map. The main result is that if the vertex set of an FCM can be partitioned into blocks, then the analysis of the original FCM can be reduced to that of the quotient FCM and the sectional FCMs relative to the partition. Similar to the original FCM, we proposed the causal algebra in quotient FCMs and proved that the strongest, weakest and total effects that one vertex has on another vertex in the quotient FCM also depend on the weights and the states of the vertices along the directed paths from the first vertex to the second vertex. Each quotient FCM is based on different hypotheses, premises and epistemological perspectives, and deals with different aspects of real-world problems. We have presented our initial attempt to establish a decomposition theory that enables us to break a complex, large FCM into quotient FCMs. This effort is similar to the study of separation theory in Bayesian networks, which has laid a solid foundation for the application of the Bayesian networks in many areas. We believe that, in order for FCM to be of practical use, it is necessary to carefully study its structural properties.

Finally, we point out that, although we concentrated on the case of binary states in the discussion, our decomposition theory can be generalized without any difficulty to the case of real-value states, which appear in many real-world applications. Also, the theory can be generalized to the analysis of dynamic cognitive networks introduced recently in [18].

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