Total perfect codes in Cayley graphs

Sanming Zhou

Department of Mathematics and Statistics
The University of Melbourne

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VAC29, La Trobe University

perfect codes

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Let V be an alphabet of size q, and let $1 \leq t \leq n$ be integers.

Denote by V^n the set of words of length n over V.

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If a perfect t-code exists, then a certain polynomial of degree t has t distinct zeros among $1, 2, \ldots, n$.

This was proved by Lloyd when q is a prime power and Lenstra (1972) for all q.

perfect codes

Definition

A binary code of length $n = 2^r - 1$, $r \geq 2$, with an $r \times (2^r - 1)$ parity checking matrix H is called a binary Hamming code if the columns of H are all nonzero vectors of $\mathbb{F}_2^r$.

That is, the codewords of a binary Hamming code are precisely vectors $\mathbf{v} \in \mathbb{F}_2^n$ such that $H\mathbf{v} = \mathbf{0}$.

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Hamming codes are linear, i.e. subspaces of linear spaces.

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Theorem

(Tietäväinen 1973, Leontiev & Zinoviev 1973)
When q is a prime power, the only linear perfect t-codes are

(a) *the binary Hamming codes ($t = 1$),*

(b) *the trivial perfect codes ($n = t$, or $n = 2t + 1$ and $q = 2$), and*

(c) *the two Golay codes ($(n, q, t) = (11, 3, 2)$ or $(23, 2, 3)$).*

perfect codes in graphs

Definition

(N. Biggs 1973)
Let G be a graph. A subset C of $V(G)$ is a perfect t-code in G if for any $v \in V(G)$ there exists exactly one vertex in C whose distance to v is at most t.

That is, the subsets

$$N_t(u) := \{v \in V(G) : d(u, v) \leq t\},\ u \in C,$$

form a partition of $V(G)$.

A perfect 1-code is called a perfect code.

A perfect t-code in G must be an independent set of G.

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Question

When does a graph admit a perfect t-code or perfect code?

perfect codes in distance-transitive graphs

Theorem

(N. Biggs 1973)
Let G be a distance-transitive graph of diameter k.

If G has a perfect t-code, then $1 + v_1(\lambda) + \cdots + v_t(\lambda)$ divides $1 + v_1(\lambda) + \cdots + v_k(\lambda)$ in $\mathbb{Q}(\lambda)$,

where $v_1(\lambda), \ldots, v_k(\lambda)$ are certain polynomials defined in terms of the intersection matrix of G.

This generalises LLoyd's necessary condition.

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This generalises LLoyd's necessary condition.

Theorem
(Biggs 1973)
A d-regular graph of order n admits a perfect code only when

(a) *$d + 1$ divides n, and*

(b) *-1 is an eigenvalue of the adjacency matrix of the graph.*

perfect codes in graphs

Existence of perfect t-codes has been studied for

- hypercubes (Hamming codes)

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- antipodal distance-transitive graphs (antipodal: $d(u, v) = d(u, w) = k$ (diameter) $\Rightarrow v = w$ or $d(v, w) = k$)

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- odd graphs (Hammond & Smith 1975)
- ...

terminology used by graph theorists

- efficient dominating set = perfect 1-code (Bange, Barkauskas and Slater 1987)
- perfect t-dominating set = perfect t-code

perfect codes in odd graphs

Definition

The odd graph $O(k)$ is defined to have vertices the k-subsets of $[2k+1]$ such that two vertices are adjacent iff they are disjoint.

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Theorem

(Hammond & Smith 1975)
A set of k-subsets of $[2k+1]$ is a perfect code in $O(k)$ if and only if it is a 1-$(2k+1, k, k-1)$ design.

perfect codes in a family of distance-regular graphs

Perfect 1-codes in an infinite class of distance-regular graphs were constructed by Cameron, Thas and Payne (1976) using generalized hexagons.

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A generalized hexagon of order (s, t) is a point-line geometry whose incidence graph has girth 12 and diameter 6, and such that each line (point) is incident with exactly $s + 1$ points ($t + 1$ lines).

perfect codes in direct products of cycles

Definition

The direct product $G \times H$ is defined to have vertex set $V(G) \times V(H)$ such that $(u_1, v_1) \sim (u_2, v_2)$ if and only if $u_1 \sim u_2$ in G and $v_1 \sim v_2$ in H.

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Theorem

(Žerovnik 2008)
The direct product $\times_{i=1}^{n} C_{\ell_i}$, $\ell_i \geq 2t+2$, *contains a perfect* t*-code if and only if every* ℓ_i *is a multiple of* $t^n + (t+1)^n$.

Partial results were obtained earlier by Klavžar, Špacapan and Žerovnik (Adv. Appl. Math. 2006).

perfect codes in product graphs

Perfect codes have been studied for

- direct product (Klavžar, Špacapan and Žerovnik 06)
- Cartesian product (M. Mollard 11)
- lexicographic product (D. T. Taylor 09)

E-chains in Cayley graphs

Definition

Let X be a group and $S \subseteq X - \{1\}$ be such that $S^{-1} = S$.

The Cayley graph $\mathrm{Cay}(X, S)$ is defined to have vertex set X in which $x \sim y$ iff $x^{-1}y \in S$.

Definition

(Dejter & Serra 2003)
A family of graphs

$$\{G_1, G_2, \ldots, G_i, G_{i+1}, \ldots\}$$

is called an E-chain if each G_i is an induced subgraph of G_{i+1} and each G_i has an perfect code.

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Example

$\{Q_{2^r-1} : r \geq 2\}$ is an E-chain.

E-chains in Cayley graphs

Dejter and Serra (2003) gave a methodology for constructing E-chains of Cayley graphs.

They used this method to construct E-chains of Cayley graphs on symmetric groups, including the family of star graphs (Arumugam and Kala 1996).

perfect codes in Cayley graphs

Theorem

(J. Lee 2001)

Let C be a normal subset of a finite group X (i.e. $xC = Cx$ for all $x \in X$). Then the following are equivalent:

(a) *C is a perfect code in a connected Cayley graph $\mathrm{Cay}(X, S)$;*

(b) *there exists a covering $\mathrm{Cay}(X, S) \to K_{|S|+1}$ such that $p^{-1}(v) = C$ for some vertex v of $K_{|S|+1}$;*

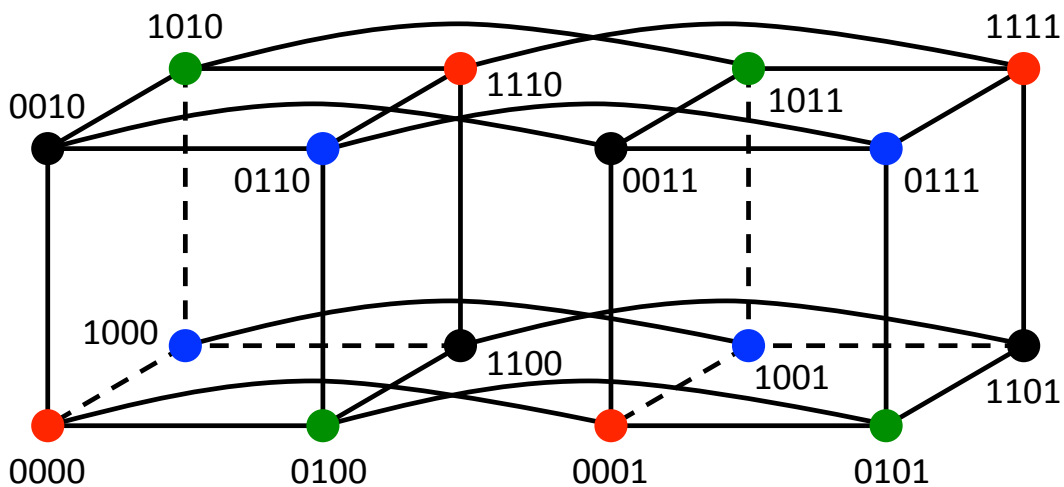
(c) *$|C|(|S| + 1) = |X|$ and $C \cap [C(S_0^2 - \{1\})] = \emptyset$, where $S_0 = S \cup \{1\}$.*

total perfect codes

Definition

A subset C of $V(G)$ is called a total perfect t-code in G if for every $u \in V(G)$ there is exactly one $v \in C$ such that $d(u, v) \leq t$.

A total perfect 1-code is called a total perfect code (Klostermeyer & Goldwasser 2006).



Total perfect codes in Q_4.

total perfect codes

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The direct product of any number of simple graphs has a total perfect code iff each factor has a total perfect code (Abay-Asmerom et al 2008).

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The direct product of any number of simple graphs has a total perfect code iff each factor has a total perfect code (Abay-Asmerom et al 2008).

The Cartesian product $C_m \Box C_n$ has a total perfect code iff both m and n are multiples of 4 (Dejter 2008).

total perfect codes in Cayley graphs

Definition

A graph H is called a pseudocover of a graph G if there exists a surjective mapping $p : V(H) \to V(G)$ such that $H[p^{-1}(v)]$ is a matching for all $v \in V(G)$ and deleting all such matching from H results in a cover of G with p the covering projection.

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Theorem

(Z 2011) Let C be a normal subset of a group X. Then the following are equivalent:

(a) *$C \subseteq X$ is a total perfect code of $\mathrm{Cay}(X, S)$;*

(b) *There exists a pseudocovering $p : \mathrm{Cay}(X, S) \to K_{|S|}$ such that Cs is a vertex-fibre of p for at least one element $s \in S$;*

(c) *C satisfies*

$$|C||S| = |X|, \;\; C \cap [C(S^2 - \{1\})] = \emptyset.$$

total perfect codes in Cayley graphs

Corollary

Let C be a normal subgroup of X. The following are equivalent:

(a) *C is a total perfect code of $\mathrm{Cay}(X, S)$;*

(b) *$\mathrm{Cay}(X, S)$ is a C-pseudocovering graph of $K_{|S|}$;*

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$$|X : C| = |S|, \quad C \cap S^2 = \{1\}.$$

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Definition

A pseudocovering $p : H \to G$ is called an A-pseudocovering if A is a subgroup of $\mathrm{Aut}(H)$ and there exists an isomorphism $h : G \to H_{\mathcal{P}_A}$ such that the quotient map $H \to H_{\mathcal{P}_A}$ is the composition of p and h, where $H_{\mathcal{P}_A}$ is the quotient graph of H with respect to the partition $\mathcal{P}_A$ of $V(H)$ into A-orbits.

Corollary

Let X be an abelian group. Then $C \subset X$ is a total perfect code in $\mathrm{Cay}(X, S)$ iff

$$|C||S| = |X| \text{ and } (C - C) \cap (S + S) = \{0\}.$$

In particular, a subgroup $C \leq X$ is a total perfect code iff $|C||S| = |X|$ and $C \cap (S + S) = \{0\}$.

an example

Example

The circulant $\mathrm{Cay}(\mathbb{Z}_n, S)$ has a total perfect code which is a subgroup of $\mathbb{Z}_n$ if and only if $|S| = m$ is a divisor of n and for any $s, s' \in S$, $s + s'$ is not a multiple of m unless $s + s' \equiv 0 \bmod n$.

In this case $C = \{[km] : k \in \mathbb{Z}\}$ is such a total perfect code.

If $n \geq 3$ is odd, then no $\mathrm{Cay}(\mathbb{Z}_n, S)$ has a total perfect code which is a subgroup of $\mathbb{Z}_n$.

$\mathrm{Cay}(\mathbb{Z}_{18}, \{[1], [9], [17]\})$ admits $C = \{[0], [3], [6], [9], [12], [15]\}$ as a total perfect code because $|S|$ divides 18 and the only common element of C and $S + S = \{[0], [2], [8], [10], [16]\}$ is $[0]$.

a family of Cayley graphs having total perfect codes

Theorem

(Z 2011) Suppose G is a connected graph such that $\mathrm{Aut}(G)$ contains a vertex-transitive abelian subgroup. Then G has total perfect codes iff its degree d is of the form $d = 2^n$ for some $n \geq 1$.

We give an explicit construction of total perfect codes in G when $d = 2^n$.

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Corollary

(van Wee 1988) Q_d has a total perfect code iff $d = 2^n$ for some $n \geq 1$. Moreover, if $d = 2^n$, then for every $n \times d$ matrix M over $\mathrm{GF}(2)$ with rank n and pairwise distinct columns, the null space of M is a total perfect code.

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Conjecture

(I. Gorodezky 2007) If $d = 2^n$, $n \geq 3$, then every minimum dominating set of Q_d is a total perfect code in Q_d.

work in progress

- Existence and construction of total perfect codes in specific families of Cayley graphs

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- Cartesian product of complete graphs (of non-uniform sizes)
- Cartesian product of cycles
- Direct product of cycles

two useful survey papers

J. H. van Lint, A survey of perfect codes, *Rocky Mountain J. Math.* 5 (1975), 199–224.

O. Heden, A survey of perfect codes, *Advances Math. Commun.* 2 (2008), 223–247.