Symmetric graphs and transitive block designs

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October 30, 2011

Outline

- There are several ways to measure the symmetry of a graph, e.g. symmetry respect to vertices, edges, arcs, etc.

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- Information on symmetry of a graph is contained in its automorphism group.

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- The group

$$\mathrm{Aut}(\Gamma) = \{\text{automorphisms of } \Gamma\}$$

under the usual composition of permutations is called the automorphism group of Γ.

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  of $s + 1$ vertices such that α_i, α_{i+1} are adjacent and $\alpha_{i-1} \neq \alpha_{i+1}$.
- An oriented path of length s is an s-arc, but the converse is not true.

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- (G, s)-arc transitivity $\Rightarrow$ $(G, s-1)$-arc transitivity $\Rightarrow \cdots \Rightarrow$ $(G, 1)$-arc transitivity ($=$ G-symmetry)

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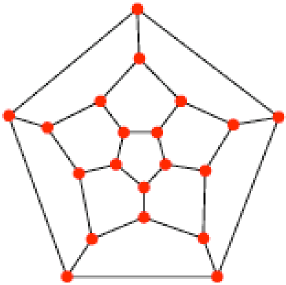
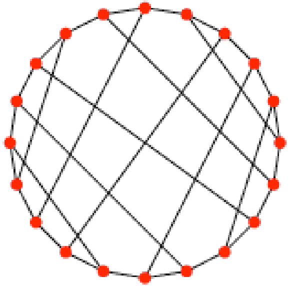
- The analogy is not true when $s \geq 3$.

- The dodecahedron graph is A_5-arc transitive.





examples

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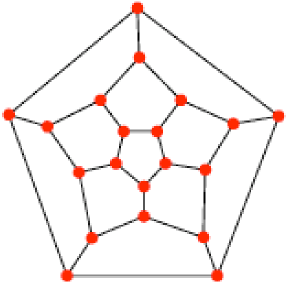

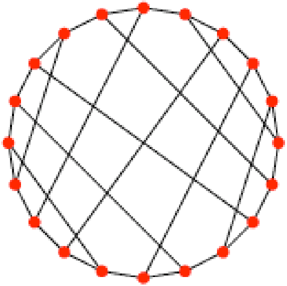

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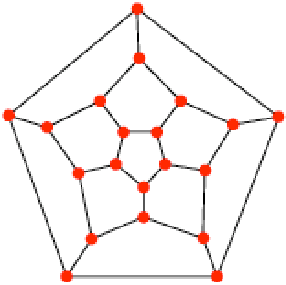
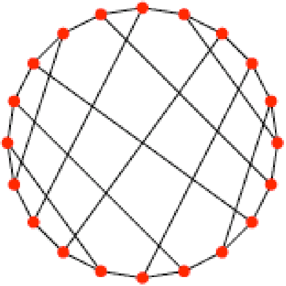
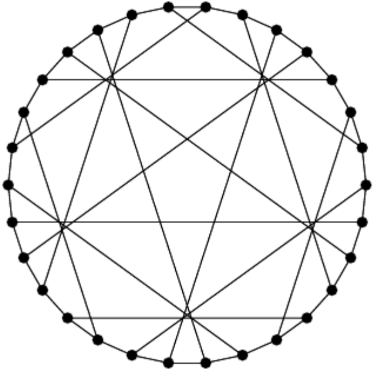
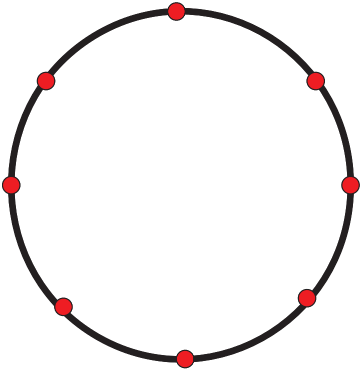
- For $n \geq 3$, $K_{n,n}$ is 3-arc transitive but not 4-arc transitive.







Tutte's 8-cage is 5-arc transitive. It is a cubic graph of girth 8 with minimum order (30 vertices).



Cycles are s-arc transitive for any $s \geq 1$.

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- A lot of work has been done on constructing 5-arc transitive graphs.
- Conder found infinitely many such graphs (for all but finitely many n, both S_n and A_n can be automorphism groups of 5-arc transitive cubic graphs).

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- A lot of work has been done on 2-arc transitive graphs.

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- We focus on the imprimitive case.

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imprimitivity and incidence geometries

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- $\Gamma_{\mathcal{B}}$: quotient graph with vertex set $\mathcal{B}$ such that $B, C \in \mathcal{B}$ are adjacent iff there is at least one edge of Γ between B and C

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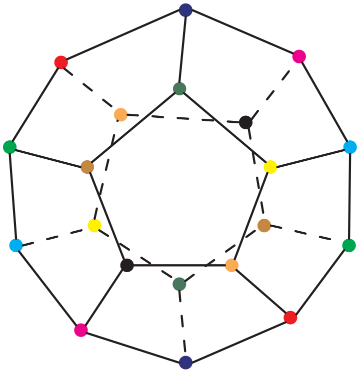
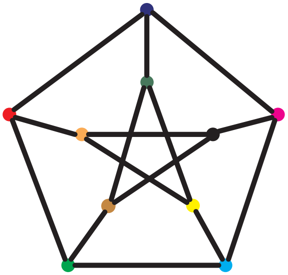
- $(\Gamma, G, \mathcal{B}) \to (\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$
- This "geometric approach" (Gardiner and Praeger 1995) is analogous to the "composition-extension" approach in group theory.

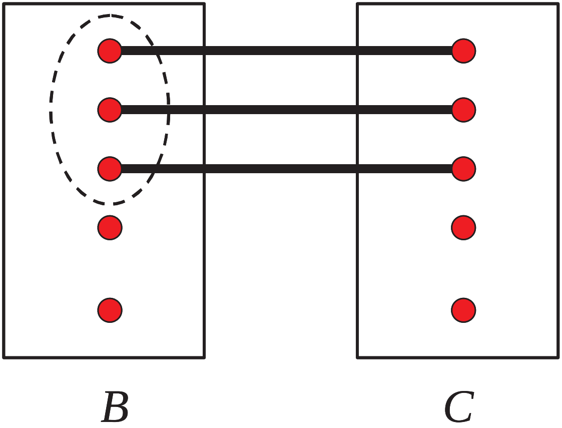
example: quotient graph





The dodecahedron is A_5-arc transitive and the partition with each part containing antipodal vertices is A_5-invariant. The quotient graph is isomorphic to Petersen graph.

example: $\Gamma[B, C]$



An illustration of the bipartite graph $\Gamma[B, C]$

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- Other invariant partitions are also interesting.

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- Answered when $k = v - 2 \geq 1$
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- Answered when $k = 2$ [Z, EJC 2008]
- When $k = v - 2$ or 2, results are given in terms of auxiliary graphs determined by $\mathcal{D}(B)$.

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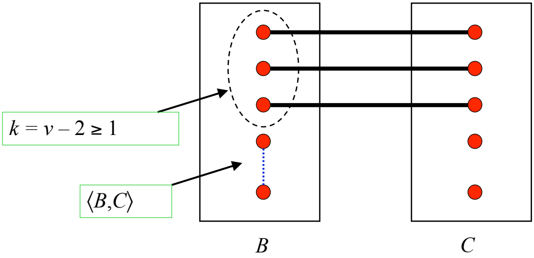

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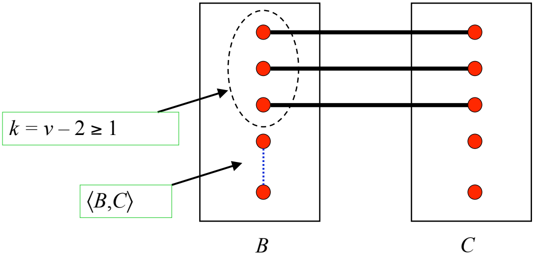

- $\Gamma^B :=$ multigraph with vertex set B and (multi)edges $\langle B, C \rangle$

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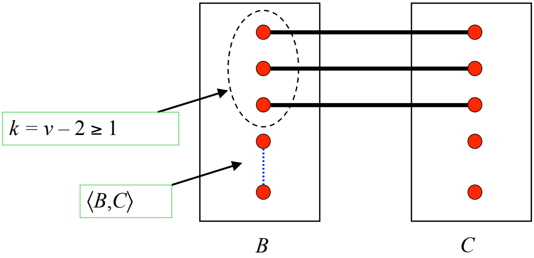

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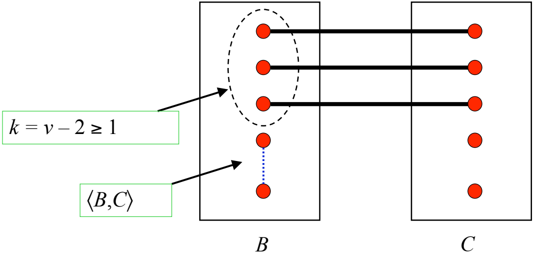

- Γ^B := multigraph with vertex set B and (multi)edges $\langle B, C \rangle$
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(IPZ, 2005) If $k = v - 2 \geq 1$, then

(a) Γ^B *is connected; or*

(b) v *is even and sim*(Γ^B) *is a perfect matching.*

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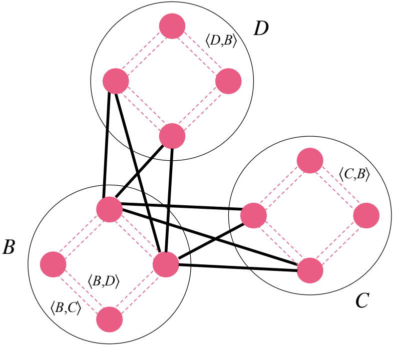

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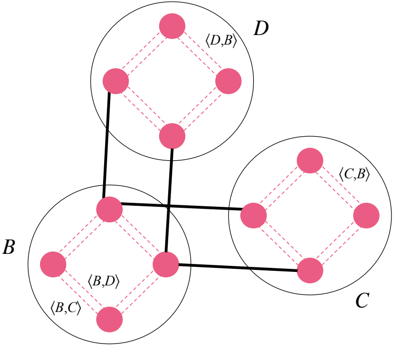

case $k = v - 2 \geq 1$ and $sim(\Gamma^B)$ is a cycle

Theorem

(Li, Praeger & Z, EJC 2010) If $k = v - 2 \geq 1$, $\Gamma_{\mathcal{B}}$ is connected, and $sim(\Gamma^B)$ is a cycle, then one of the following occurs (for a certain m):

(a) $v = 3$ *and* Γ *has degree* m;

(b) $v = 4$, $\Gamma[B, C] = K_{2,2}$, *and* Γ *is connected of degree* $4m$;

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We construct an infinite family of graphs for each case when $m = 1$.

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Corollary

There exists an infinite family of connected symmetric graphs Γ *of degree* 4 *which have a quotient graph* $\Gamma_{\mathcal{B}}$ *of degree* 4 *such that* Γ *is not a cover of* $\Gamma_{\mathcal{B}}$*.*

This is the first (infinite) family of graphs with these properties.

case $k = v - 2 \geq 1$ and $sim(\Gamma^B)$ is a cycle

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(LPZ, 2010) If $k = v - 2$ and Γ^B is connected, is v bounded by some function of the degree of $sim(\Gamma^B)$?

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(Xu & Z, BAustM 2010) If $k = v - 2 \geq 1$, $\Gamma_{\mathcal{B}}$ is connected and $sim(\Gamma^B)$ is connected with degree $d \geq 2$, then either

(a) *$sim(\Gamma^B) \cong K_v$, $v = d + 1$, $b = m(v-1)v/2$, and G_B is 2-homogeneous on B; or*

(b) *$sim(\Gamma^B) \cong K_{v/2,v/2}$, $v = 2d$, and $b = mv^2/4$.*

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Problem

Study the structure of Γ and $\Gamma_{\mathcal{B}}$ in each case.

2-arc transitive quotients (cont'd)

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Question

If $\mathcal{D}^*(B)$ is known, can we determine Γ and / or $\Gamma_{\mathcal{B}}$?

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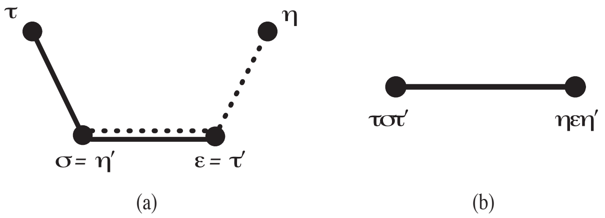

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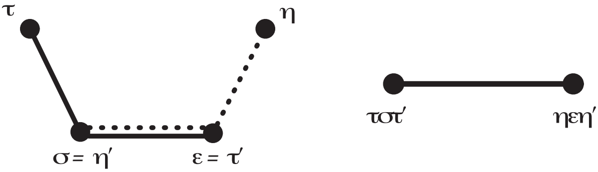

(a) (b)

This is just the 2-path graph construction but restricted to Δ.

a degenerate case: $(\lambda, r) = (1, 2)$

Theorem

(Lu & Z, JGT 2007)

(a) *Let* Σ *be* $(G, 2)$*-arc transitive with degree* ≥ 3*. Let* Δ *be a self-paired* G*-orbit on the set of 3-arcs of* Σ*.*

Then $\Gamma := \Gamma_2(\Sigma, \Delta)$ *is* G*-symmetric admitting a* G*-invariant* $\mathcal{B}$ *such that* $\Gamma_{\mathcal{B}}$ *is* $(G, 2)$*-arc transitive and not multi-covered by* Γ *and that* $(\lambda, r) = (1, 2)$*.*

(b) *Any* Γ *such that* $\Gamma_{\mathcal{B}}$ *is* $(G, 2)$*-arc transitive and not multicovered by* Γ *and* $(\lambda, r) = (1, 2)$*, is isomorphic to* $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ *for a self-paired* G*-orbit* Δ *on the set of 3-arcs of* $\Gamma_{\mathcal{B}}$*.*

a classification theorem

Theorem

(Z, DM 2009) All G-symmetric Γ such that $\Gamma_{\mathcal{B}} \cong K_{b+1}$ is $(G,2)$-arc transitive and $(\lambda, r) = (1,2)$ are classified: either each component of Γ is K_3 and G is an arbitrary 3-transitive group, or one of the following holds.

(a) *Γ is the 2-path graph of K_{b+1}, and $G = S_{b+1}$ $(b \geq 3)$, A_{b+1} $(b \geq 5)$, or M_{b+1} $(b = 10, 11, 22, 23)$.*

(b) *Γ is a second-type cross ratio graph.*

(c) *Γ is a second-type twisted cross ratio graph.*

(d) *Γ is one of two "affine graphs" associated with $\mathrm{AG}(d,2)$ (where $d \geq 2$), $b = 2^d - 1$, and either $G = \mathrm{AGL}(d,2)$, or $d = 4$ and $G = Z_2^4.A_7$.*

(e) *$G = M_{11}$, $b = 11$, and Γ is one of two graphs from M_{11}.*

(f) *$G = M_{22}$, $b = 21$, and Γ is one of two graphs from M_{22}.*

second-type cross ratio graphs

Vertices: wuy $(= yuw)$, where w, u, y are distinct points of

$$\mathrm{PG}(1, b) = \mathrm{GF}(b) \cup \{\infty\}$$

Adjacency: $wuy \sim uyz$ iff the cross ratio

$$c(u, w; y, z) = \frac{(u-y)(w-z)}{(u-z)(w-y)}$$

belongs to a certain subset of $\mathrm{GF}(b)$

Definition

(Z, EJC 2002) Let $\mathcal{D}$ be a G-point- and G-block-transitive 1-design. Let $\Omega(\sigma)$ be the set of flags of $\mathcal{D}$ with point entry σ.

A G-orbit Ω on the flags of $\mathcal{D}$ is feasible with respect to G if

(1) $|\Omega(\sigma)| \geq 3$;

(2) $L \cap N = \{\sigma\}$, for distinct $(\sigma, L), (\sigma, N) \in \Omega(\sigma)$;

(3) $G_{\sigma,L}$ is transitive on $L \setminus \{\sigma\}$, for $(\sigma, L) \in \Omega$; and

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$((\sigma, L), (\tau, N)) \in \Omega \times \Omega$ is called compatible with Ω if

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Definition

If $\Psi := ((\sigma, L), (\tau, N))^G$ is self-paired, call $\Gamma(\mathcal{D}, \Omega, \Psi) = (\Omega, \Psi)$ the G-flag graph of $\mathcal{D}$ wrt (Ω, Ψ).

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(Z, EJC 2002) The case $k = v - 1 \geq 2$ *occurs iff* $\Gamma \cong \Gamma(\mathcal{D}, \Omega, \Psi)$ *for some* $(\mathcal{D}, \Omega, \Psi)$.

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Corollary

(Z, EJC 2002) The following statements are equivalent.

(a) $(\Gamma, G, \mathcal{B})$ *is such that* $k = v - 1 \geq 2$ *and* $\Gamma_{\mathcal{B}}$ *is a complete graph.*

(b) $\Gamma \cong \Gamma(\mathcal{D}, \Omega, \Psi)$ *for a* G*-doubly point-transitive and* G*-block-transitive* 2-(v, k, λ) *design* $\mathcal{D}$ *and some* (Ω, Ψ).

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- The flag graph construction above is a special case of a general construction.
- Both versions were used to characterise / classify some families of symmetric graphs.

classification problems

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Classify all graphs $(\Gamma, G, \mathcal{B})$ *such that* $k = v - 1 \geq 2$ *and* $\Gamma_{\mathcal{B}}$ *is complete.*

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The corresponding group G is

- almost simple (that is, G has a nonabelian simple normal subgroup N such that $N \trianglelefteq G \leq \mathrm{Aut}(N)$); or
- contains a regular normal subgroup which is elementary abelian.

Theorem

(Kantor 1985) All 2-point-transitive linear spaces are known. In the almost simple case, they are:

(a) $\mathcal{D} = \mathrm{PG}(d-1,q)$, $N = \mathrm{PSL}(d,q)$, $d \geq 3$;

(b) $\mathcal{D} = U_H(q)$, $N = \mathrm{PSU}(3,q)$, $q > 2$ *a prime power;*

(c) $\mathcal{D} =$ *Ree unital* $U_R(q)$, $N = {}^2G_2(q)$ *is the Ree group,* $q = 3^{2s+1} \geq 3$;

(d) $\mathcal{D} = \mathrm{PG}(3,2)$, $N = A_7$.

a classification result

Theorem

(Giulietti, Marcugini, Pambianco & Z 2010)

Let $(\Gamma, G, \mathcal{B})$ *be such that* $k = v - 1 \geq 2$, $\Gamma_{\mathcal{B}}$ *is complete and the design involved is a nontrivial linear space. Suppose* G *is almost simple. Then one of the following occurs:*

(a) Γ *is isomorphic to one of the two graphs associated with* $\mathrm{PG}(d-1, q)$, *and* $\mathrm{PSL}(d, q) \trianglelefteq G \leq \mathrm{P\Gamma L}(d, q)$, *for some* $d \geq 3$ *and prime power* q*;*

(b) Γ *is isomorphic to a unitary graph and* $\mathrm{PGU}(3, q) \trianglelefteq G \leq \mathrm{P\Gamma U}(3, q)$, *for a prime power* $q > 2$*;*

(c) Γ *is isomorphic to one of four graphs from* $\mathrm{PG}(3, 2)$ *with order 105.*

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unitary groups and Hermitian unitals

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- $\mathrm{PGU}(3, q) = \mathrm{GU}(3, q)/Z$, where Z is the center of $\mathrm{GU}(3, q)$

unitary groups and Hermitian unitals

- $q = p^e > 2$, p a prime
- $V(3, q^2)$: 3-dimensional vector space over $\mathbb{F}_{q^2}$
- $\sigma : x \mapsto x^q = \bar{x}$ defines an automorphism of $\mathbb{F}_{q^2}$
- $\beta : V(3, q^2) \times V(3, q^2) \to \mathbb{F}_{q^2}$: nondegenerate σ-Hermitian form, i.e.
  - $\beta(\mathbf{u} + \mathbf{u}', \mathbf{v}) = \beta(\mathbf{u}, \mathbf{v}) + \beta(\mathbf{u}', \mathbf{v})$
  - $\beta(a\mathbf{u}, b\mathbf{v}) = ab^q\beta(\mathbf{u}, \mathbf{v})$
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- $\mathrm{P\Gamma U}(3, q) := \mathrm{PGU}(3, q) \rtimes \langle\psi\rangle$, where $\psi : x \mapsto x^p, x \in \mathbb{F}_{q^2}$

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- Flag $=$ incident point-line pair

Definition

$q = p^e > 2$; $r \geq 1$ a divisor of $2e$
$\lambda \in \mathbb{F}^*_{q^2}$: λ^q is in the $\langle \psi^r \rangle$-orbit on $\mathbb{F}_{q^2}$ containing λ

Unitary graph $\Gamma_{r,\lambda}(q)$:
Vertex set $=$ set of flags of $U_H(q)$

$(\langle a_1, b_1, c_1 \rangle, L_1) \sim (\langle a_2, b_2, c_2 \rangle, L_2)$ iff there exist $0 \leq i < 2e/r$ and nonisotropic $(a_0, b_0, c_0) \in V(3, q^2)$ orthogonal to both (a_1, b_1, c_1) and (a_2, b_2, c_2) such that L_1 and L_2 are given by:

$$L_1 : \begin{vmatrix} x & a_1 & a_0 + a_2 \\ y & b_1 & b_0 + b_2 \\ z & c_1 & c_0 + c_2 \end{vmatrix} = 0$$

$$L_2 : \begin{vmatrix} x & a_2 & a_0 + \lambda^{qp^{ir}} a_1 \\ y & b_2 & b_0 + \lambda^{qp^{ir}} b_1 \\ z & c_2 & c_0 + \lambda^{qp^{ir}} c_1 \end{vmatrix} = 0$$

Problem (restated)

Study the following problems for various subfamilies of symmetric graphs (e.g. $k = v - 3 \geq 1$, $k = 3$, etc.):

- *Under what circumstances is $\Gamma_{\mathcal{B}}$ $(G, 2)$-arc transitive?*
- *What can we say about Γ if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive?*
- *When does $\Gamma_{\mathcal{B}}$ inherit $(G, 2)$-arc transitivity from Γ?*

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Problem (restated)

Study the structure of Γ and $\Gamma_{\mathcal{B}}$ when $k = v - 2 \geq 1$ and $sim(\Gamma^B)$ is connected with degree ≥ 2.

Recall that $sim(\Gamma^B) \cong K_v$ or $K_{v/2,v/2}$.

Problem (restated)

Classify $(\Gamma, G, \mathcal{B})$ *such that* $k = v - 1 \geq 2$ *and* $\Gamma_{\mathcal{B}}$ *is complete.*

Major tools:

- flag graph construction
- classification of 2-transitive groups:
  - $\mathrm{PSL}(n, q) \leq G \leq \mathrm{P\Gamma L}(n, q)$;
  - $\mathrm{Sp}_{2d}(2)$ $(d \geq 2)$;
  - $\mathrm{PSU}(3, q) \leq G \leq \mathrm{PSL}(3, q^2)$;
  - $\mathrm{Suz}(q)$ $(q = 2^{2a+1} > 2)$;
  - $\mathrm{Ree}(q)$ $(q = 3^{2a+1} > 3)$;
  - ...

problems (cont'd)

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  - ...

Problem

In particular, complete the classification of Γ *such that* $k = v - 1 \geq 2$, $\Gamma_{\mathcal{B}}$ *is complete, and the design involved is a linear space (affine case remaining).*

problems (cont'd)

Problem

Investigate those Γ *such that* $\Gamma_{\mathcal{B}}$ *is* $(G, 2)$*-arc transitive and* $\mathcal{D}^*(B)$ *is symmetric.*

problems (cont'd)

Problem

Investigate those Γ such that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and $\mathcal{D}^*(B)$ is symmetric.

Problem

In particular, classify such graphs Γ such that $\mathcal{D}^*(B)$ is a linear space.