Unitary graphs, and classification of a family of symmetric graphs with complete quotients

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Outline

Definition of unitary graphs

Characterisation of unitary graphs

Classification of a family of arc-transitive graphs

Unitary groups and Hermitian unitals

- $q = p^e > 2$, p a prime
- $V(3, q^2)$: 3-dimensional vector space over $\mathbb{F}_{q^2}$
- $\sigma : x \mapsto x^q = \bar{x}$ defines an automorphism of $\mathbb{F}_{q^2}$
- $\beta : V(3, q^2) \times V(3, q^2) \to \mathbb{F}_{q^2}$: nondegenerate σ-Hermitian form, i.e.
  - $\beta(\mathbf{u} + \mathbf{u}', \mathbf{v}) = \beta(\mathbf{u}, \mathbf{v}) + \beta(\mathbf{u}', \mathbf{v})$
  - $\beta(a\mathbf{u}, b\mathbf{v}) = ab^q \beta(\mathbf{u}, \mathbf{v})$
  - $\beta(\mathbf{u}, \mathbf{v}) = \beta(\mathbf{v}, \mathbf{u})^q$

- $\mathrm{GU}(3, q)$: group of nonsingular linear transformations of $V(3, q^2)$ leaving β invariant
- $Z = \{aI : a \in \mathbb{F}_{q^2}, a^{q+1} = 1\}$: center of $\mathrm{GU}(3, q)$
- $\mathrm{PGU}(3, q) = \mathrm{GU}(3, q)/Z$
- $\psi : x \mapsto x^p, x \in \mathbb{F}_{q^2}$: Frobenius map
- $\mathrm{P\Gamma U}(3, q) := \mathrm{PGU}(3, q) \rtimes \langle\psi\rangle$
- $\mathrm{PGU}(3, q) \rtimes \langle\psi^r\rangle$, r a divisor of $2e$

- Choose an appropriate basis for $V(3, q^2)$ such that β is expressed by

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

- Thus, for $\mathbf{u}_1 = (x_1, y_1, z_1), \mathbf{u}_2 = (x_2, y_2, z_2) \in V(3, q^2)$,

$$\beta(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{u}_1 D \bar{\mathbf{u}}_2^T = -x_1 x_2^q + y_1 z_2^q + z_1 y_2^q.$$

- $\mathbf{u} = (x, y, z) \in V(3, q^2)$
- $\langle \mathbf{u} \rangle = \langle x, y, z \rangle$: 1-dim subspace of $V(3, q^2)$ spanned by $\mathbf{u}$
- $\mathbf{u}_1, \mathbf{u}_2 \in V(3, q^2)$ are orthogonal if $\beta(\mathbf{u}_1, \mathbf{u}_2) = 0$
- $\mathbf{u}$ is isotropic if it is orthogonal to itself and nonisotropic otherwise
- Set of 1-dim subspaces spanned by isotropic vectors:

$$X = \{\langle x, y, z \rangle : x, y, z \in \mathbb{F}_{q^2}, x^{q+1} = yz^q + zy^q\}$$

- Elements of X are called the absolute points
- $|X| = q^3 + 1$
- $\mathrm{PGU}(3, q)$ is 2-transitive on X

- $\mathbf{u}_1, \mathbf{u}_2$ isotropic $\Rightarrow \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ contains exactly $q+1$ absolute points
- $U_H(q)$: Hermitian unital, with point set X, a subset of X is a block (line) iff it is the set of absolute points contained in some $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$
- $U_H(q)$ is a linear space with q^3+1 points, $q^2(q^2-q+1)$ lines, $q+1$ points in each line, and q^2 lines meeting at a point (i.e. $U_H(q)$ is a 2-$(q^3+1, q+1, 1)$ design)
- A linear space is an incidence structure of points and lines such that any point is incident with at least two lines, any line with at least two points, and any two points are incident with exactly one line
- Can represent a line of $U_H(q)$ by the homogenous equation of a line of $\mathrm{PG}(2, q^2)$
- E.g. $x + 0y - z = 0$

- $\mathrm{Aut}(U_H(q)) = \mathrm{P\Gamma U}(3, q)$
- $G = \mathrm{PGU}(3, q) \rtimes \langle \psi^r \rangle$
- $U_H(q)$ is $(G, 2)$-point-transitive, G-block transitive and G-flag transitive
- Flag = incident point-line pair
- $$V(q) = \text{the set of flags of } U_H(q)$$

Definition of a unitary graph

Definition

$q = p^e > 2$; $r \geq 1$ a divisor of $2e$
$\lambda \in \mathbb{F}_{q^2}^*$: λ^q is in the $\langle \psi^r \rangle$-orbit on $\mathbb{F}_{q^2}$ containing λ

Unitary graph $\Gamma_{r,\lambda}(q)$:
Vertex set $V(q)$, $(\langle a_1, b_1, c_1 \rangle, L_1) \sim (\langle a_2, b_2, c_2 \rangle, L_2)$ iff there exist $0 \leq i < 2e/r$ and nonisotropic $(a_0, b_0, c_0) \in V(3, q^2)$ orthogonal to both (a_1, b_1, c_1) and (a_2, b_2, c_2) such that L_1 and L_2 are given by:

$$L_1 : \begin{vmatrix} x & a_1 & a_0 + a_2 \\ y & b_1 & b_0 + b_2 \\ z & c_1 & c_0 + c_2 \end{vmatrix} = 0$$

$$L_2 : \begin{vmatrix} x & a_2 & a_0 + \lambda^{qp^{ir}} a_1 \\ y & b_2 & b_0 + \lambda^{qp^{ir}} b_1 \\ z & c_2 & c_0 + \lambda^{qp^{ir}} c_1 \end{vmatrix} = 0$$

- Requirement on $\lambda \Leftrightarrow \lambda^{p^{tr}} = \lambda^q$ for some $0 \leq t < 2e/r$
- $\Gamma_{r,\lambda}(q)$ is independent of the choice of t
- $\exists j$, $(j+t)r = 2e$, so that $\lambda = \lambda^{qp^{jr}}$
- Rewriting the equations of L_1 and L_2:

$$L_2 : \begin{vmatrix} x & a_2 & \lambda a_0 + \lambda^{qp^{ir}+1} a_1 \\ y & b_2 & \lambda b_0 + \lambda^{qp^{ir}+1} b_1 \\ z & c_2 & \lambda c_0 + \lambda^{qp^{ir}+1} c_1 \end{vmatrix} = 0$$

$$L_1 : \begin{vmatrix} x & \lambda^{qp^{ir}+1} a_1 & \lambda a_0 + \lambda^{qp^{jr}} a_2 \\ y & \lambda^{qp^{ir}+1} b_1 & \lambda b_0 + \lambda^{qp^{jr}} b_2 \\ z & \lambda^{qp^{ir}+1} c_1 & \lambda c_0 + \lambda^{qp^{jr}} c_2 \end{vmatrix} = 0.$$

- So the adjacency relation of $\Gamma_{r,\lambda}(q)$ is symmetric

Example

One can see that

$$\infty = \langle 0, 1, 0\rangle \in X; \qquad 0 = \langle 0, 0, 1\rangle \in X$$

and

$$L : x = z; \qquad N : y = \lambda^q x$$

are lines of $U_H(q)$. Since L passes through ∞ and N passes through 0, we have

$$(\infty, L), (0, N) \in V(q).$$

One can check that (∞, L) and $(0, N)$ are adjacent in $\Gamma_{r,\lambda}(q)$ for any feasible r and λ.

Example

$q = 3 \Rightarrow r = 1, 2$

$r = 1$: $\Gamma_{1,\lambda}(3)$ is well-defined for every $\lambda \in \mathbb{F}_9^*$ ($\lambda^{3^t} = \lambda^3$ for $t = 1$)

$r = 2$: $\Gamma_{2,1}(3)$ and $\Gamma_{2,\omega^4}(3)$ only ($k = 0$, and $\lambda = \lambda^3$ iff $\lambda = 1$ or ω^4, where ω is a primitive element of $\mathbb{F}_9$)

Characterisation of unitary graphs

A graph Γ is G-arc transitive if $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set of arcs (order pairs of adjacent vertices) of Γ

A partition $\mathcal{P}$ of $V(\Gamma)$ is G-invariant if for any element of G maps blocks of $\mathcal{P}$ to blocks of $\mathcal{P}$

$\Gamma_{\mathcal{P}}$: quotient graph

Almost multicover: Γ is an almost multicover of $\Gamma_{\mathcal{P}}$ if, for any two adjacent $P, Q \in \mathcal{P}$, all vertices of P except only one have neighbours in Q

$B(\sigma)$: set of flags of $U_H(q)$ with point-entry $\sigma \in X$

$\mathcal{B} = \{B(\sigma) : \sigma \in X\}$: a partition of $V(q)$ into $q^3 + 1$ blocks each with size q^2

$L(\sigma\tau)$: Unique line of $U_H(q)$ through distinct $\sigma, \tau \in X$

$$\begin{aligned} k_{r,\lambda}(q) &= |\langle \psi^r \rangle| / |\langle \psi^r \rangle_\lambda| \\ &= \text{least } j \geq 1 \text{ such that } \lambda^{p^{jr}} = \lambda \end{aligned}$$

Theorem

(GMPZ 2010) Denote $G = \mathrm{PGU}(3, q) \rtimes \langle \psi^r \rangle$ and $k = k_{r,\lambda}(q)$.

(a) *$\Gamma_{r,\lambda}(q)$ is a G-arc transitive graph of degree $kq(q^2 - 1)$ that admits $\mathcal{B}$ as a G-invariant partition such that*
   - *$\Gamma_{r,\lambda}(q)_{\mathcal{B}}$ is a complete graph, and*
   - *$\Gamma_{r,\lambda}(q)$ is an almost multicover of $\Gamma_{r,\lambda}(q)_{\mathcal{B}}$.*

   Moreover, for distinct σ, τ, $(\sigma, L(\sigma\tau))$ is the only vertex in $B(\sigma)$ that has no neighbour in $B(\tau)$.

(b) *If Γ is a G-arc transitive graph that admits a nontrivial G-invariant partition $\mathcal{P}$ of block size at least 3 such that $\Gamma_{\mathcal{P}}$ is complete, Γ is an almost multicover of $\Gamma_{\mathcal{P}}$, and a certain design $\mathcal{D}(\Gamma, \mathcal{P})$ is a linear space, then Γ is isomorphic to some $\Gamma_{r,\lambda}(q)$.*

Classification of a family of arc-transitive graphs

- Γ: G-arc transitive
- $\mathcal{P}$: nontrivial G-invariant partition of $V(\Gamma)$ such that Γ is an almost multicover of $\Gamma_{\mathcal{P}}$
- $\mathcal{D}(\Gamma, \mathcal{P})$: incidence structure with point-set $\mathcal{P}$ and blocks $(\mathcal{P}(\alpha) \cup \{B\})^g$, $g \in G$
- where $\alpha \in B$ is fixed, and $\mathcal{P}(\alpha)$ is the set of blocks of $\mathcal{P}$ containing no neighbours of α in Γ
- Special case: $\Gamma_{\mathcal{P}}$ is complete $\Rightarrow \mathcal{D}(\Gamma, \mathcal{P})$ is a 2-design and $G \leq \mathrm{Aut}(\mathcal{D}(\Gamma, \mathcal{P}))$ is 2-point-transitive and block-transitive

Problem

(Z, 2000) Classify such graphs Γ *when* $\mathcal{D}(\Gamma, \mathcal{P})$ *is a linear space.*

Done when $\mathcal{D}(\Gamma, \mathcal{P})$ is a trivial linear space (i.e. complete graph)

All nontrivial 2-point-transitive linear spaces are known (Kantor 1985, CFSG used).

The corresponding group G is almost simple (that is, G has a nonabelian simple normal subgroup N such that $N \trianglelefteq G \leq \mathrm{Aut}(N)$) or contains a regular normal subgroup which is elementary abelian.

We give the classification in the almost simple case. (The affine case is open!)

Theorem

(GMPZ 2010) Let $(\Gamma, G, \mathcal{P})$ *be such that* $\Gamma_{\mathcal{P}}$ *is a complete graph,* Γ *is an almost multicover of* $\Gamma_{\mathcal{P}}$ *and* $\mathcal{D}(\Gamma, \mathcal{P})$ *is a nontrivial linear space. Suppose* G *is almost simple. Then one of the following occurs:*

(a) Γ *is isomorphic to one of the two graphs associated with* $\mathrm{PG}(d-1, q)$*, and* $\mathrm{PSL}(d, q) \trianglelefteq G \leq \mathrm{P\Gamma L}(d, q)$*, for some* $d \geq 3$ *and prime power* q*;*

(b) Γ *is isomorphic to a unitary graph and* $\mathrm{PGU}(3, q) \trianglelefteq G \leq \mathrm{P\Gamma U}(3, q)$*, for a prime power* $q > 2$*;*

(c) Γ *is isomorphic to some small graphs from* $\mathcal{D} = \mathrm{PG}(3, 2)$*.*

Tools used

Lemma

(Z, 2000) All $(\Gamma, G, \mathcal{P})$ such that $\Gamma_{\mathcal{P}}$ is a complete graph and Γ is an almost multicover of $\Gamma_{\mathcal{P}}$ can be constructed by using 2-point-transitive and block-transitive designs.

Theorem

(Kantor 1985) All 2-point-transitive linear spaces are known. In the almost simple case, they are:

(a) $\mathcal{D} = \mathrm{PG}(d-1, q)$, $N = \mathrm{PSL}(d, q)$, $d \geq 3$;

(b) $\mathcal{D} = U_H(q)$, $N = \mathrm{PSU}(3, q)$, $q > 2$ a prime power;

(c) $\mathcal{D} =$ Ree unital $U_R(q)$, $N = {}^2G_2(q)$ is the Ree group, $q = 3^{2s+1} \geq 3$;

(d) $\mathcal{D} = \mathrm{PG}(3, 2)$, $N = A_7$.