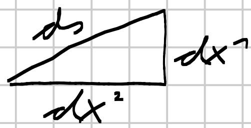
II. Global conformal invariance

II.1 Definition of conformal transformations

Consider a space M (think of $\mathbb{R}^d$ or $\mathbb{R}^{1,d-1}$) with line element

$$ds^2 = \underbrace{g_{\mu\nu}}_{\text{metric}} \, dx^\mu dx^\nu = (dx)^2$$

coordinates



__Prerequisites:__ Scalar product: $v \cdot w = g_{\mu\nu} v^\mu w^\nu$

Length: $|v| = \sqrt{g_{\mu\nu} v^\mu v^\nu}$

Angle: $\cos\theta = v \cdot w / |v| \cdot |w|$

__Def:__ A __global__ conformal transformation is an __invertible__ map $M \to M$ (with $x \to x'$) such that

$$\underbrace{g'_{\mu\nu}(x')}_{\text{transformed metric}} \stackrel{!}{=} \underbrace{\Omega(x)\, g_{\mu\nu}(x)}_{\text{rescaled old metric}}$$

__Remarks:__
- Such transformations preserve angles
- They include the standard QFT-symmetries (which require length preservation, i.e. $g'_{\mu\nu} = g_{\mu\nu}$)
- Angle preserving transformations need to have this form (without proof)

__Need:__ Transformation of the metric ...

<u>Transformation rule:</u> $dx'^{\mu} = d(x'^{\mu}(x^{\alpha})) = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} dx^{\alpha}$

<u>Invariance of line element:</u>

$$ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} \overset{!}{=} g'_{\mu\nu}\, dx'^{\mu} dx'^{\nu}$$

$$= g'_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} dx^{\alpha} dx^{\beta} \qquad (*)$$

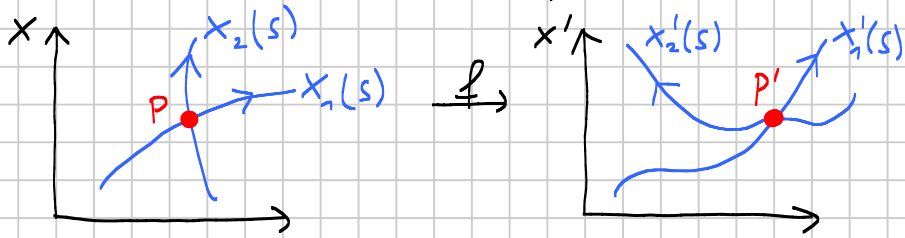
<u>Comparison of coefficients:</u> $g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$

<u>Condition for a conformal transformation:</u>

$$\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x) \overset{!}{=} \Omega(x)\, g_{\mu\nu}(x)$$

Consider two curves γ_1, γ_2 which intersect at a point P

$$[0,1] \xrightarrow{X_1, X_2}$$



<u>Tangent vectors:</u> $v_i^{\mu} = \frac{dX_i^{\mu}}{ds}$

$$v_i'^{\mu} = \frac{dx'^{\mu}(X_i)}{ds} = \frac{\partial x'^{\mu}(X_i)}{\partial x^{\nu}} \frac{dX_i^{\nu}}{ds} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} v_i^{\nu}$$

$$\Rightarrow v_1' \cdot v_2' = v_1'^{\mu} g'_{\mu\nu} v_2'^{\nu} = g'_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\rho}} v_1^{\rho} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} v_2^{\sigma}$$

$$\overset{(*)}{=} \Omega(x)\, g_{\mu\nu} v_1^{\mu} v_2^{\nu} = \Omega(x)\, v_1 \cdot v_2$$

Hence $\frac{v_1 \cdot v_2}{|v_1| \cdot |v_2|}$ is invariant...

II.2 Infinitesimal transformations

Goal: Derive constraints for infinitesimal transformations $x^\mu \to x^\mu + \varepsilon^\mu(x)$ for a flat space with metric $\eta_{\mu\nu}$

The line element changes according to

$$ds^2 \to ds^2 + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) dx^\mu dx^\nu \overset{!}{=} \overbrace{(1 + f(x))}^{\Omega(x)} ds^2$$

This requires: $$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) \cdot \eta_{\mu\nu} \qquad (1)$$

Contraction with $\eta^{\mu\nu}$: $2\,\partial\cdot\varepsilon = f(x)\, tr(1) = d\, f(x)$
(this determines $f(x)$ in terms of $\varepsilon(x)$)

Goal: Derive constraints for $f(x) = \frac{2}{d}\,\partial\cdot\varepsilon$ (2)
and (afterwards) for derivatives of ε...

Act with ∂_ρ on (1):

$$\partial_\rho \partial_\mu \varepsilon_\nu + \partial_\rho \partial_\nu \varepsilon_\mu = \partial_\rho f\, \eta_{\mu\nu} \qquad (3.1)$$

$(\nu \leftrightarrow \rho)$: $$\partial_\rho \partial_\nu \varepsilon_\mu + \partial_\nu \partial_\mu \varepsilon_\rho = \partial_\nu f\, \eta_{\mu\rho} \qquad (3.2)$$

$(\mu \leftrightarrow \nu)$: $$\partial_\rho \partial_\mu \varepsilon_\nu + \partial_\nu \partial_\mu \varepsilon_\rho = \partial_\mu f\, \eta_{\nu\rho} \qquad (3.3)$$

Add up $-(3.1) + (3.2) + (3.3)$:

$$2\,\partial_\mu \partial_\nu \varepsilon_\rho = \partial_\mu f\, \eta_{\nu\rho} + \partial_\nu f\, \eta_{\mu\rho} - \partial_\rho f\, \eta_{\mu\nu} \qquad (4)$$

Contract μ,ν: $\quad 2\Box\varepsilon_\rho = (2-d)\partial_\rho f$

Act with ∂_ν: $\quad 2\Box\partial_\nu\varepsilon_\rho = (2-d)\partial_\nu\partial_\rho f \quad$ (symmetric in $\nu\rho$!)

Act with $\Box$ on (1): $\quad \Box(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = \Box f\,\eta_{\mu\nu}$

One finds: $$(2-d)\partial_\nu\partial_\rho f = \Box f\,\eta_{\nu\rho} \qquad (5)$$

Contract ν,ρ: $$(d-1)\Box f = 0 \qquad (6)$$

<u>Different cases:</u>

a) <u>$d=1$:</u> No constraint (actually: no angles...)

b) <u>$d=2$:</u> Separate discussion

c) <u>$d\geq 3$:</u> Act with ∂_λ on (4) and use (5) and (6)

$$\partial_\lambda\partial_\mu\partial_\nu\varepsilon_\rho = 0 \implies \varepsilon \text{ is a polynomial in } x$$
of degree at most two!

<u>General ansatz:</u> $\quad \varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$

(1) is linear in ε and homogeneous in derivatives

$\Rightarrow$ can discuss each term separately

i) All a_μ satisfy (1) $\longrightarrow$ translations

$$x'^\mu = x^\mu + a^\mu = (1 + i a^\rho P_\rho)x^\mu$$
with momentum operator $P_\rho = -i\partial_\rho$

ii) $\varepsilon_\mu = b_{\mu\nu} x^\nu \Rightarrow \partial_\nu \varepsilon_\mu = b_{\mu\nu}$ and $f = \frac{2}{d} \partial \cdot \varepsilon = \frac{2}{d} b^\rho{}_\rho$

Insert into (1): $b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\rho{}_\rho \eta_{\mu\nu}$

$\Rightarrow$ No constraint on antisymmetric part of $b_{\mu\nu}$ but symmetric part diagonal ($\sim \eta_{\mu\nu}$)

$$b_{\mu\nu} = \underbrace{\alpha \eta_{\mu\nu}}_{\text{dilatation}} + \underbrace{\omega_{\mu\nu}}_{\text{rotation/Lorentz transformation}} \qquad (\text{with } \omega_{\mu\nu} = -\omega_{\nu\mu})$$

$$x'^\mu = x^\mu + \alpha x^\mu + \omega^\mu{}_\nu x^\nu = \left(1 + i\alpha D + \tfrac{i}{2}\omega_{\rho\sigma} L^{\rho\sigma}\right) x^\mu$$

with dilatation operator $D = -i x^\rho \partial_\rho = x \cdot P$

angular momentum operator $L_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho)$

iii) Without proof: The general solution is parametrized by a vector b and reads $\varepsilon_\rho = b_\rho x^2 - 2 x_\rho (b \cdot x)$

$$x'^\mu = x^\mu + b^\mu x^2 - 2x^\mu (b \cdot x) = (1 - i b^\rho K_\rho) x^\mu$$

with generator $K_\rho = -i(2x_\rho x^\sigma \partial_\sigma - x^2 \partial_\rho)$

<u>Check:</u> $\partial_\mu \partial_\nu \varepsilon_\rho = 2 b_\rho \eta_{\mu\nu} - 2\eta_{\mu\rho} b_\nu - 2\eta_{\nu\rho} b_\mu$

$$\partial_\rho f = \frac{2}{d} \partial_\rho \partial^\sigma \left(b_\sigma x^2 - 2x_\sigma (b \cdot x)\right)$$
$$= \frac{2}{d}(2b_\rho - 4d\, b_\rho) = \frac{4}{d}(1 - 2d) b_\rho$$

One finds: $\partial_\mu \varepsilon_\nu = 2 b_\nu x_\mu - 2\eta_{\mu\nu}(b \cdot x) - 2x_\nu b_\mu$

$$f = \frac{2}{d} \partial \cdot \varepsilon = -4\, b \cdot x$$

$$\Rightarrow \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = -4\eta_{\mu\nu}(b \cdot x) = \eta_{\mu\nu} f$$

<u>General proof</u>: Decompose the tensor into its irreducible components...

<u>Remark</u>: With $x'_\mu = x_\mu + b_\mu x^2 - 2x_\mu(b\cdot x)$ one finds

$$\frac{x'_\mu}{x'^2} = \frac{x_\mu}{x^2} + b_\mu \quad \text{(modulo terms of higher order in } b\text{)}$$

<u>Indeed</u>: $x^2 x'_\mu = x^2 x_\mu + b_\mu (x^2)^2 - 2x_\mu (b\cdot x) x^2$

$$\begin{aligned}
x'^2(x_\mu + x^2 b_\mu) &= \Big[x^2 + b^2 (x^2)^2 + \cancel{4x^2(b\cdot x)^2} + \cancel{2(b\cdot x)x^2} \\
&\quad - \cancel{4}^{\,2}(b\cdot x)x^2 - \cancel{4(b\cdot x)^2 x^2}\Big]\Big[x_\mu + x^2 b_\mu\Big] \\
&= x^2 x_\mu + \cancel{x_\mu b^2 (x^2)^2} - 2x_\mu(b\cdot x)x^2 \quad \rightarrow \text{higher order in } b \\
&\quad + b_\mu (x^2)^2 + \cancel{b_\mu b^2 (x^2)^3} - \cancel{2 b_\mu (b\cdot x)(x^2)^2} \\
&= x^2 x'_\mu
\end{aligned}$$

<u>Remark</u>: These results are true for any (Euclidean, Minkowski, ...) signature of the metric...