Chemical distance for the half-orthant model

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Abstract

The half-orthant model is a partially oriented model of a random medium involving a parameter $p \in [0, 1]$, for which there is a critical value $p_c(d)$ (depending on the dimension d) below which every point is reachable from the origin. We prove a limit theorem for the graph-distance (or "chemical distance") for this model when $p < p_c(2)$, and also when $1-p$ is larger than the critical parameter for site percolation in $\mathbb{Z}^d$. We extend this to a shape theorem and describe some features of the shape.

1 Introduction and main results

The half-orthant model is an i.i.d. site-based model of a random environment defined as follows: Independently at each site $x \in \mathbb{Z}^d$, with probability $p \in [0, 1]$ insert arrows from x pointing to each of the neighbours $x+e_i$, $i \in [d] := \{1, 2, \ldots, d\}$, and otherwise (with probability $1 - p$) insert arrows from x to all neighbours $x \pm e_i$ of x. Here $(e_i)_{i=1}^d$ are the canonical basis vectors.

Let ω_x denote the set of arrows from x. Then $\boldsymbol{\omega} = (\omega_x)_{x\in\mathbb{Z}^d}$ is a random configuration of arrows. These types of site-based i.i.d. environments (that have been called *degenerate random environments* in recent years) arise naturally in the study of random walks in i.i.d. *non-elliptic* (i.e. some steps are not available from some sites) random environments, see e.g. [20, 22].

For fixed $d \geq 2$, let $\boldsymbol{\omega} = (\omega_x)_{x\in\mathbb{Z}^d}$ be such a configuration (or "environment") and let $\gamma = (\gamma_i)_{i=0}^{\ell}$ be a nearest-neighbour path of length $\ell < \infty$ in

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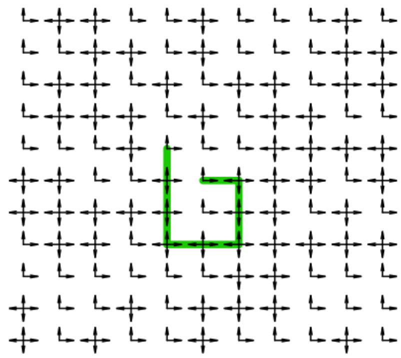


Figure 1: An example of a finite piece of the environment for the half-orthant model, with the shortest path from the origin (centre) to $(-1, 1)$ highlighted.

$\mathbb{Z}^d$. For $j \in [\ell]$, the j-th step of the path $\gamma_j - \gamma_{j-1}$ is *consistent with* $\boldsymbol{\omega}$ if $\omega_{\gamma_{j-1}}$ contains the arrow pointing in direction $\gamma_j - \gamma_{j-1}$. The path γ is *consistent with* $\boldsymbol{\omega}$ if the j-th step is consistent with $\boldsymbol{\omega}$ for every $j \in [\ell]$. Figure 1 shows an example of a finite piece of an environment for this model, together with a path of length 8 (starting from the centre of the figure) that is consistent with this environment.

Consider the set of points $\mathcal{C}_o$ reachable from the origin $o = (0, 0, \ldots, 0)$ by following arrows. This is exactly the set of points x for which there exists a finite path from o to x that is consistent with the environment. When $p = 1$ this set is the positive orthant, i.e. $\mathcal{C}_o(1) = \mathbb{O}_+ := \{x \in \mathbb{Z}^d : x^{\{i\}} \geq 0 \text{ for } i \in [d]\}$, where $x^{\{i\}}$ denotes the i-th coordinate of x. When $p = 0$ this set is all of $\mathbb{Z}^d$, i.e. $\mathcal{C}_o(0) = \mathbb{Z}^d$. There is a natural coupling of environments for all values of p, such that the cluster $\mathcal{C}_o(p)$ is decreasing in p. Moreover there exists a critical value $p_c(d) \in (0.57, 1)$ such that almost surely $\mathcal{C}_o(p) = \mathbb{Z}^d$ if $p < p_c(d)$ and almost surely $\mathcal{C}_o(p) \neq \mathbb{Z}^d$ if $p > p_c(d)$. Here, the cluster $\mathcal{C}_o$ is always infinite. Various other degenerate random environments (with $\mathcal{C}_o$ possibly finite) have been studied in recent years, including so-called *lattice k-neighbour percolation* [8, 26] and the corrupted compass model [5, 18].

In this paper we consider the lengths of shortest paths (or *chemical distances*) between vertices in the half-orthant model. The chemical distance is a fundamental notion in the first passage percolation literature (see e.g. [2]). A substantial difference is that in the half-orthant model edges are directed. First passage percolation on directed edges has been considered recently in

[15] but our model is a site-based i.i.d. environment (in [15] the environment is i.i.d. over edges), so edges are locally correlated in our setting.

Understanding the behaviour of the chemical distances to distant points in *different directions* leads to what is sometimes called a *shape theorem*. A different kind of shape theorem for the forward cluster $\mathcal{C}_o$ (viewed as a set of points) has been proved for this model when p is large [24]. That result is similar to the kind of shape theorems available for oriented percolation and the contact process (see e.g. [11, 12]) - morally it says that the cluster looks like a cone from far away. The fact that the half-orthant model is only partly oriented presents different challenges compared to the oriented percolation setting. For example, the result in [24] currently does not extend to all $p > p_c(d)$ when $d \geq 3$ in part because of a lack of a "sharpness" result for this phase transition[1].

For $u, v \in \mathbb{Z}^d$ and an environment $\boldsymbol{\omega}$ let $T_{u,v} = T_{u,v}(\boldsymbol{\omega})$ denote the length of any shortest path from u to v that is consistent with the environment. If no such path exists then set $T_{u,v} = \infty$. We will also refer to $T_{u,v}$ as the *passage time* (from u to v). For $v \in \mathbb{Z}^d$, let $T_v = T_{o,v}$. In Figure 1, $T_{(-1,1)} = T_{o,(-1,1)} = 8$. When $p < p_c(d)$ we have that $\mathcal{C}_o = \mathbb{Z}^d$ almost surely, so T_v is a.s. finite. By translation invariance, $T_{u,v}$ is almost surely finite for every $u, v \in \mathbb{Z}^d$. Clearly, for any $v \in \mathbb{Z}^d$ we have $T_v \geq \|v\|_1 := \sum_{i=1}^d |v^{\{i\}}|$. In order to visualise our main result, it is useful to define $A_n = \{x \in \mathbb{Z}^d : T_x = n\}$, which is the random subset of sites x for which the shortest path from the origin to x has length n. When $p = 0$ this set is the ℓ_1 sphere $B_n := \{x \in \mathbb{Z}^d : \|x\|_1 = n\}$, and when $p = 1$ this set is $B_n^+ := \{x \in \mathbb{O}_+ : \|x\|_1 = n\}$. Indeed, it is elementary that for every p, all points $v \in \mathbb{Z}^d$ in the positive orthant $\mathbb{O}_+$ can be reached in $\|v\|_1$ steps. The other directions are the interesting ones for this model. Figure 2 shows a simulation of the set A_{2000} for each of $p = 0.25, 0.5, 0.75$ respectively. The "shape" of A_n remains consistent for large values of n. This suggests the existence of a *limiting* shape. Note also several features of these shapes. In each case there are clear "flat" regions. For large p these flat regions are restricted to directions in a neighbourhood of the positive orthant. For small p there is an additional flat region in directions in a neighbourhood of $(-1, -1)$.

The main results of this paper are the law of large numbers for the chem-

[1] Beekenkamp [4], has shown that the relevant exponential decay of certain connection probabilities holds for all p larger than some $p_c'(d)$ which is believed (but not proved) to be equal to $p_c(d)$.

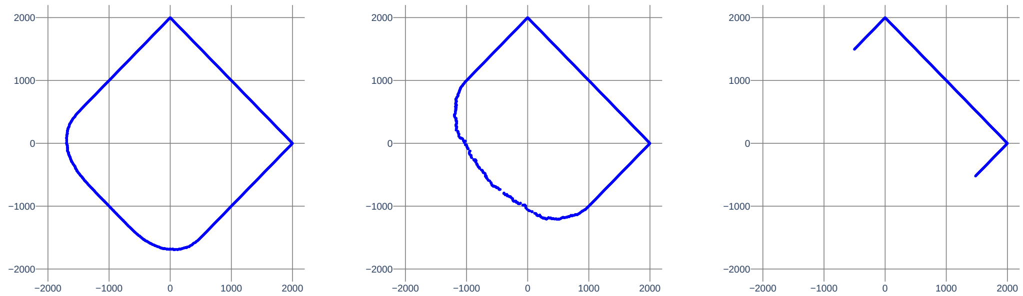


Figure 2: Simulation of the set of points whose distance from the origin within the random environment is exactly $n = 2000$, for $p = 0.25, p = 0.5, p = 0.75$ (left to right) respectively. Note the apparent flat and curved regions in these shapes.

ical distance (Theorem 1.1 below), the corresponding shape theorem (Theorem 1.3 below), and features of the shape (Theorem 1.2 below). We will utilise several facts about site percolation and oriented site percolation on $\mathbb{Z}^d$. Let $p_*(d)$ denote the critical parameter for site percolation on $\mathbb{Z}^d$, and $p_\dagger(d)$ denote the critical parameter for *oriented* site percolation on $\mathbb{Z}^d$.

Our first main result is the following theorem.

Theorem 1.1 (Chemical distance)**.** *Fix $d \geq 2$. For p such that $1 - p > \min\{1 - p_c(2), p_*(d)\}$ there exists a function $\zeta_p : \mathbb{Z}^d \to \mathbb{R}_+$ such that for each $v \in \mathbb{Z}^d$*

$$\lim_{n\to\infty} n^{-1} T_{nv}(p) = \zeta_p(v), \quad \textit{almost surely and in } L^1.$$

The claim with $v = o$ is trivial with $\zeta_p(o) = 0$. For other v this result follows from an application of the subadditive ergodic theorem. The non-obvious condition to check is that the expected passage times are finite. For $1 - p > 1 - p_c(2)$ we verify this via a familiar duality argument (based on duality with an oriented percolation model in 2 dimensions). Call the sites that get all outward arrows *full sites* and the others *half sites*. For $1 - p > p_*(d)$ we (roughly speaking) verify finiteness of expected passage times by showing that to reach one vertex from another one can "mostly" follow a path in a supercritical percolation cluster of full sites.

For $u \in \mathbb{Q}^d$, let $m_u = \inf\{m \in \mathbb{N} : mu \in \mathbb{Z}^d\}$ (so $m_u = 1$ if $u \in \mathbb{Z}^d$). We extend the domain of ζ_p to $\mathbb{Q}^d$ by defining

$$\zeta_p(u) = \zeta_p(m_u u)/m_u. \tag{1.1}$$

It is then immediate from Theorem 1.1 that for all $u \in \mathbb{Q}^d$,

$$\lim_{n\to\infty} (nm_u)^{-1} T_{nm_u u}(p) = \zeta_p(u), \quad \text{almost surely and in } L^1.$$

For fixed dimension $d \geq 2$ and any orthogonal set $O \subset \mathcal{E} := \{\pm e_i : i \in [d]\}$ with $|O| = d$ (i.e. the elements of O form a basis for $\mathbb{Z}^d$), we can consider *oriented site percolation on* $\mathbb{Z}^d$ with orientation given by O. In other words there are (directed) connections from occupied sites x to neighbouring occupied sites of the form $x + e$ for each $e \in O$. These oriented percolation models are identical in law, except for the change of orientation. It is known (see e.g. [11, 12, 17]) for supercritical oriented site percolation (i.e. with parameter $q > p_\dagger(d)$) with orientation O, that conditional on the percolation cluster of the origin being infinite, the cluster viewed from far away is a deterministic cone with axis $\sum_{e\in O} e$. We call this cone O_q. As $q \uparrow 1$ this cone increases to the whole orthant $\{\sum_{e\in O} \lambda_e e : \lambda_e \geq 0, \forall e \in O\}$. Let

$$\mathcal{O} = \{O \subset \mathcal{E} : O \text{ is a basis for } \mathbb{Z}^d\}.$$

Let $\mathbb{D} = \mathbb{Q}^d \setminus \{o\}$, and let

$$\begin{aligned} \mathcal{S}_p^{\text{good}} = \mathbb{D} \cap \Bigg\{ \sum_{i=1}^d (\alpha_i - \beta_i) e_i : \quad & a \in [p,1], \quad 0 \leq \alpha_i, \beta_i \ \forall i \in [d], \text{ and} \\ & \sum_{i=1}^d \alpha_i = a = 1 - \sum_{i=1}^d \beta_i, \text{ and} \\ & \alpha_i \beta_i = 0 \text{ for each } i \in [d] \Bigg\}. \end{aligned} \tag{1.2}$$

Then $\mathcal{S}_p^{\text{good}}$ is a subset of the boundary of the ℓ_1 ball and $\mathcal{S}_p^{\text{good}}$ contains the intersection of this boundary and $\mathbb{D}_+ = \{x \in \mathbb{D} : x^{\{i\}} \geq 0 \text{ for every } i\}$for every p (take $a = 1$). See Figure 3 for an example in 3 dimensions.

Let $\mathcal{S}_p = \{u \in \mathbb{D} : \zeta_p(u) = \|u\|_1\}$. Our second main result reveals features of the shape including specifying certain directions in $\mathcal{S}_p$ and its complement.

Theorem 1.2 (Features of the shape)**.** *Fix $d \geq 2$, and $1-p > \min\{1-p_c(2), p_*(d)\}$. Then the function $\zeta_p : \mathbb{Q}^d \to \mathbb{R}$ in Theorem 1.1 and (1.1) satisfies the following:*

(a) (i) *for $q \in \mathbb{Q}_+$, and $u \in \mathbb{Q}^d$, $\zeta_p(qu) = q\zeta_p(u)$,*

(ii) *ζ_p is subadditive, i.e. $\zeta_p(u+v) \leq \zeta_p(u) + \zeta_p(v)$,*

(iii) *ζ_p is uniformly continuous.*

(b) *For each fixed $v \in \mathbb{Z}^d$, $\zeta_p(v)$ is non-decreasing in p.*

(c) *For $u \in \mathbb{D}$ we have $u \in \mathcal{S}_p$ in the following cases:*

(i) *$u \in \mathcal{S}_p^{good}$,*

(ii) *$1-p > p_\dagger(d)$ and $u \in \bigcup_{O \in \mathcal{O}} O_{1-p}$.*

(d) *For $u \in \mathbb{D}$ with $\|u\|_1 = 1$ we have $u \notin \mathcal{S}_p$ (i.e. $\zeta_p(u) > 1$) in the following cases:*

(i) *for any $i = 1, \dots, d$, $\|u+e_i\|_1 < \varepsilon$ for ε sufficiently small depending on d, p,*

(ii) *$d = 2$ and $u = (-(1-s), s)$ for $s \in [0, p) \cap \mathbb{Q}$.*

Some parts of this theorem are standard (or at least well understood (see e.g. [2])). The most novel component of this theorem (or more precisely, its proof) is most likely (d)(ii). Part (c)(i) is perhaps novel in the sense that this kind of feature is not present in classical models where chemical distance is examined. To prove (c)(ii) we will use a supercritical oriented percolation fact, Lemma 5 below, which is "well known", but we have not found a statement of this result when $d > 2$.

Remark 1. *Part (a)(iii) of the Theorem allows one to extend the domain of ζ_p to $\mathbb{R}^d$ by taking limits through rationals. It then follows from part (a) that the resulting function ζ_p is convex.*

Let $B_p(n) = \{x \in \mathbb{Z}^d : T_x \leq n\}$, and let $\mathfrak{B}_p = \{\frac{\delta u}{\zeta(u)} : \delta \in [0,1], \|u\|_1 = 1\}$. Then $\mathfrak{B}_p$ is a deterministic compact set. Our final main result is the following.

Theorem 1.3 (Shape theorem)**.** *Fix $d \geq 2$. For p as in Theorem 1.1, for every $\varepsilon \in (0, 1/2)$,*

$$\mathbb{P}\Big((1-\varepsilon)n\mathfrak{B}_p \cap \mathbb{Z}^d \subset B_p(n) \subset (1+\varepsilon)n\mathfrak{B}_p \cap \mathbb{Z}^d \text{ for all } n \text{ sufficiently large}\Big) = 1.$$

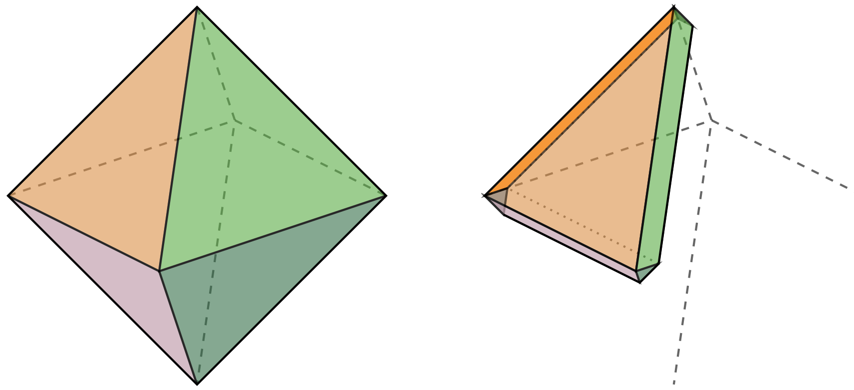
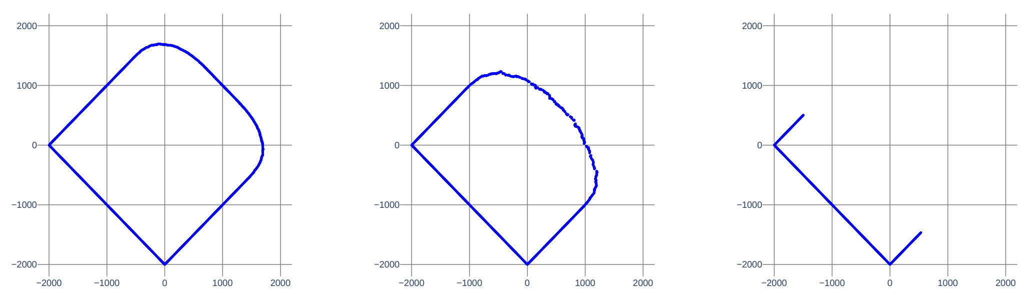


Figure 3: An illustration of the region $\mathcal{S}_p^{\text{good}}$ in 3 dimensions. The ℓ_1 ball in 3 dimensions appears on the left, with an example of $\mathcal{S}_p^{\text{good}}$ depicted on the right.

Remark 2 (Backward connections)**.** One can also ask about *backward* clusters and connections *to* the origin. Let $\hat{T}_v = T_{v,o}$. The subadditive ergodic theorem also applies to $\hat{T}_v$, and since $\hat{T}_v$ has the same distribution as T_{-v} one can immediately infer a limit theorem (a.s. and in L^1) for $\hat{T}_v$. However, e.g. $\mathbb{P}(T_{-e_1} = 1, T_{-e_2} = 1) = 1-p \neq (1-p)^2 = \mathbb{P}(\hat{T}_{e_1} = 1, \hat{T}_{e_2} = 1)$ so the joint distributions are different. We leave it as an exercise for the interested reader to modify the proof of Theorem 1.3 to prove the corresponding shape theorem (with "shape" given by $-\zeta_p$ as depicted below with $p = 0.25, 0.5, 0.75$):



The remainder of this paper is organised as follows. In Section 2 we present several open problems. In Section 3 we prove Theorem 1.1, by invoking the subadditive ergodic theorem and using a familiar duality argument in 2 dimensions. The proofs of Theorems 1.2 and 1.3 appear in Sections

4 and 5 respectively. Some standard arguments, as well as some proofs of "well-known" or technical results, are included in the Appendix.

2 Open problems

Several open problems for the half-orthant model and other similar models are discussed e.g. in [23, 24], and we do not repeat those here. Instead we present some open problems regarding chemical distance and related properties. We start with the half-orthant model and then discuss other degenerate random environments.

2.1 Half-orthant model

For dimensions $d > 2$, Theorem 1.1 is not believed to be sharp.

Open Problem 1. Prove that the conclusion of Theorem 1.1 holds when $p < p_c(d)$ (or $p < p_c'(d)$).

Theorem 1.2 asserts directions u in which vertices of the form $nm_u u$ can be reached from the origin in time of order $n\|m_u u\|_1$ (part (c) of the theorem), as well as directions in which this does not hold (part (d) of the theorem). The two sets of directions do not exhaust all directions in $\mathbb{D}$. This observation motivates the following question.

Open Problem 2. Does the subset of $\mathcal{S}_p$ presented in Theorem 1.2(c) together with the positive orthant in fact contain all directions in $\mathcal{S}_p$? See e.g. [29] in the case of first passage percolation in 2 dimensions.

For $d \geq 2$ and $p > p_c(d)$, not all vertices are reachable from the origin. Indeed in this setting, for "most" directions only finitely many points in that direction are reachable from the origin, so there is no interesting shape theorem in these directions. On the other hand, as long as p is strictly smaller than 1 there exists a neighbourhood of the positive orthant, such that for directions u in this neighbourhood, $\limsup_{n\to\infty}(m_u n)^{-1}T_{nm_u u} < \infty$. For directions in $\mathcal{S}_p^{\text{good}}$ one can show that $\limsup_{n\to\infty}(m_u n)^{-1}T_{nm_u u} = 1$. However, for all $u \in \mathbb{D} \setminus \mathbb{D}_+$ it is the case that with positive probability $nm_u u$ is not reachable from the origin. This means that the subadditive ergodic theorem (as stated herein) is not applicable. Nevertheless it should be true that limit theorems for the chemical distance hold in directions for which (a.s.) $T_{nm_u u}$ is finite for all n sufficiently large.

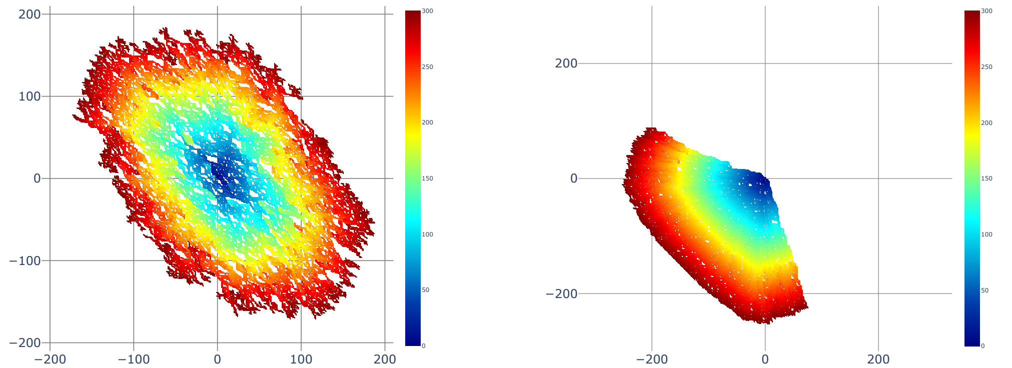
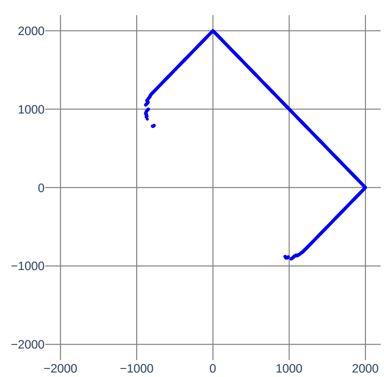


Figure 4: Simulation of the set of points reachable in n steps for the orthant model with $p = 1/2$ (left) and $p = 1/4$ (right) and $n \leq 300$. White patches surrounded by colour are not reachable from the origin.

Open Problem 3. Prove that $n^{-1}T_{nv}(p) \to \zeta_p(v) > 0$ a.s. for the half-orthant model in a non-trivial subset of directions when $p > p_c(2)$. See the short curved region in the following ($p = 0.6$).

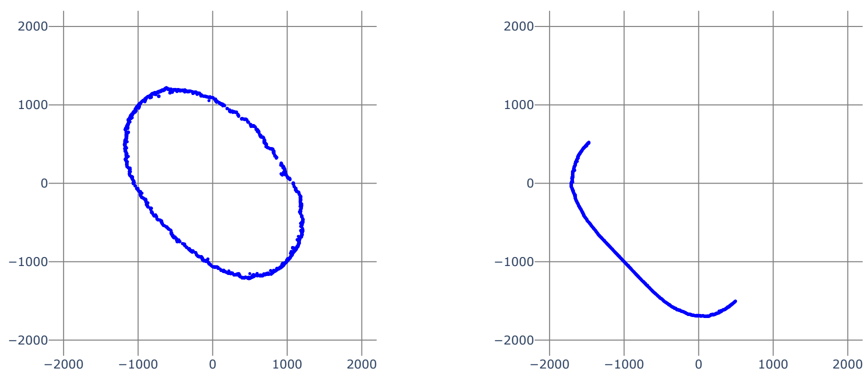


2.2 Other models

For the orthant model (where full sites are replaced with half-sites of opposite orientation), in any direction $u \in \mathbb{Q}^2 \setminus \{o\}$ there are a.s. infinitely many points that are not reachable from the origin. Thus the limit as presented in

Theorem 1.1 fails in this model in every direction. Nevertheless it is clear from e.g. Figure 4 that there is still a "shape" to the set $A_n = \{x : T_x = n\}$ when n is large. This setting is similar to the setting of first-passage percolation when passage times across edges can be infinite with positive probability (see e.g. [7]).

Open Problem 4. Prove a version of a shape theorem for the orthant model in dimensions $d \geq 2$. It seems natural to start with $p \in (1 - p_c(2), p_c(2))$, which is depicted on the left of both Figure 4 and below:



There are many more examples of degenerate random environments where shape theorems are relevant and interesting. An example that has a similar flavour to what has been presented in this paper is the case where we have just the arrow e_1 with probability p and full sites otherwise.

Open Problem 5. Investigate shape theorems in more general degenerate random environments.

Figure 5 shows the "shape" for k-neighbour percolation on $\mathbb{Z}^2$ (see e.g. [8, 26], and note that there is no parameter p in this model) with $k = 2, 3$.

3 Proof of Theorem 1.1

The proof is an application of the celebrated subadditive ergodic theorem, which we now state.

Theorem 3.1 (Kingman, Liggett). *Let* $(X_{m,n})_{0 \leq m < n}$ *be a family of random variables satisfying the following:*

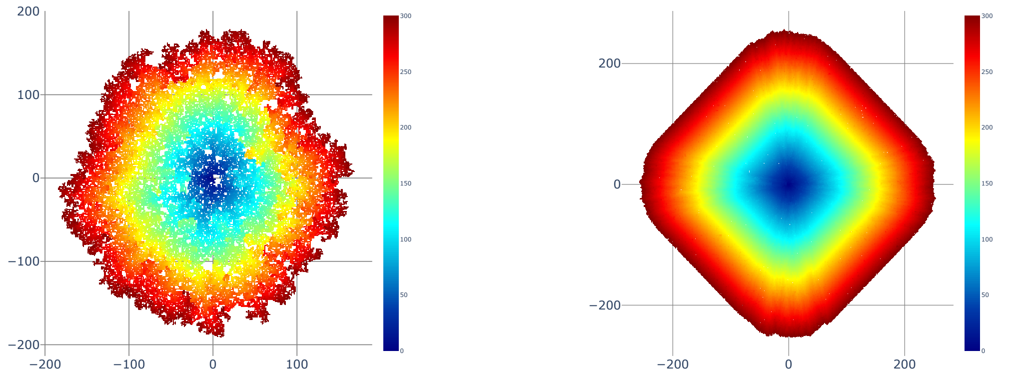


Figure 5: The shapes for 2-neighbour and 3-neighbour percolation on $\mathbb{Z}^2$.

(i) $X_{0,n} \leq X_{0,k} + X_{k,n}$ *for all* $0 < k < n$*;*

(ii) the distribution of the sequence $(X_{m,m+k})_{k\geq 1}$ *does not depend on* m*;*

(iii) for each n*,* $\mathbb{E}[|X_{0,n}|] < \infty$ *and* $\mathbb{E}[X_{0,n}] \geq cn$ *for some constant* $c > -\infty$*;*

(iv) for each k*,* $(X_{nk,(n+1)k})_{n\geq 1}$ *is a stationary and ergodic sequence.*

Then $X = \lim_{n\to\infty} X_{0,n}/n$ *exists a.s. and in* L^1*, and* X *is deterministic.*

We apply this theorem to the random variables $X_{m,n} = T_{mx,nx}$ (the minimum number of steps to reach nx, starting from mx) for arbitrary but fixed $x \in \mathbb{Z}^d \setminus \{o\}$. Conditions (ii) and (iv) in the theorem follow from the fact that the environment is i.i.d., while condition (i) is also straightforward since if γ' is a path from o to kx that is consistent with the environment and γ'' is a path from kx to nx that is consistent with the environment then the concatenation of the two paths yields a path γ from o to nx that is consistent with the environment. Trivially $\mathbb{E}[T_{a,b}] \geq 0$ since $T_{a,b} \geq 0$. Therefore, in order to apply the theorem, it remains to prove that $\mathbb{E}[T_{o,nx}] < \infty$ for every n. Since we also want this for every x, our goal is to show that $\mathbb{E}[T_{o,x}] < \infty$ for every x. Since every x is a sum of $\|x\|_1$-many vectors $e \in \mathcal{E}$, it is in fact sufficient to show that $\mathbb{E}[T_{o,e}] < \infty$ for every $e \in \mathcal{E}$.

For the half-orthant model, $T_{o,e} = 1$ if $e \in \mathcal{E}_+ = \{e_i : i \in [d]\}$ and by symmetry $\mathbb{E}[T_{o,-e_j}] = \mathbb{E}[T_{o,-e_1}]$ so we need only show that $\mathbb{E}[T_{o,-e_1}] < \infty$. We will verify this in the following two sections (for fixed $d \geq 2$), in the cases $p < p_c(2)$ and $1 - p > p_*(d)$ respectively.

3.1 Finite expectation for $p < p_c(2)$

It is sufficient to prove the result in the case $d = 2$, since in higher dimensions the time $T_{o,-e_1}$ is less than or equal to the time it takes to reach $-e_1$ from o using only moves that stay in $\mathbb{Z}^2 \times \{0\}^{d-2}$. So for the remainder of this section we restrict our attention to $d = 2$.

Let Ω_+ be the set of half sites. Define a random set $\mathfrak{C} \subset \mathbb{D}_{-,+} := \{x \in \mathbb{Z}^2 : x^{\{1\}} \leq 0, x^{\{2\}} \geq 0\}$ as follows. Let $\mathfrak{C}_0 = \{o\} = \{(0,0)\}$.

- If $o \notin \Omega_+$ then let $\mathfrak{C} = \mathfrak{C}_0 = \{o\}$.
- Otherwise $o \in \Omega_+$. In this case let $\mathfrak{C}_1$ denote the set of $x \in \mathbb{D}_{-,+} \cap \Omega_+$ with $\|x\|_1 = 1$. For $n \geq 2$, we define $\mathfrak{C}_n$ to be the set of $x \in \mathbb{D}_{-,+} \cap \Omega_+$ with $\|x\|_1 = n$ such that either $\{x+e_1, x-e_2\} \cap \mathfrak{C}_{n-1} \neq \varnothing$ or $x+e_1-e_2 \in \mathfrak{C}_{n-2}$. Let $\mathfrak{C} = \cup_{n \geq 0} \mathfrak{C}_n$.

It follows that $\mathfrak{C}$ is the cluster of the origin for an oriented site percolation model on the triangular lattice with parameter p; the occupied sites are Ω_+ and the connections are in the three directions $-e_1, +e_2, -e_1 + e_2$ of a triangular lattice. We note that this construction appears to be quite similar to a duality transformation detailed in [10][2]; however the particular result we require is not obtained in [10] and so we will proceed here with $\mathfrak{C}$ as defined above.

If $p < p_c(2) \approx 0.59$ then since $p_c(2)$ is also the critical value for oriented site percolation on the triangular lattice [19], the cluster $\mathfrak{C}$ is finite. Thus $K := \sup\{k : \mathfrak{C}_k \neq \varnothing\} < \infty$ almost surely. Moreover (see e.g. [11, Section 7 (6)]) there exists $c(p) > 0$ such that for all n,

$$\mathbb{P}(|\mathfrak{C}(p)| > n) \leq e^{-c(p)n}. \tag{3.1}$$

Define the set of *awesome* points $\mathbb{A}$ by

$$\mathbb{A} := \{x \in \mathbb{D}_{-,+} \cap \mathfrak{C} : x \in \Omega_+ \text{ but none of } x - e_1, x + e_2, x - e_1 + e_2 \text{ is in } \Omega_+\}.$$

Note that if $K \geq 1$ then $\mathfrak{C}_K \subset \mathbb{A}$ since $\mathfrak{C}_K \subset \Omega_+$ and $\mathfrak{C}_{K+1}$ and $\mathfrak{C}_{K+2}$ are empty.

Lemma 1. *There is a path from o to $-e_1$, consistent with the environment, of length at most $4|\mathfrak{C}|$.*

[2] We thank an anonymous reviewer of a previous version of this paper for making this observation.

Proof. We prove by induction on $n = |\mathfrak{C}|$ that there is a path of length at most $4n$ from o to $-e_1$.

If $|\mathfrak{C}| = 1$ then either $o \notin \Omega_+$ or o is awesome. In the former case there is a path of length 1 consistent with the environment $\boldsymbol{\omega}$, from o to $-e_1$, consisting of the step $-e_1$. In the latter case both e_2 and $e_2 - e_1$ are full sites, so there is a path of length 3 consistent with the environment $\boldsymbol{\omega}$ from o to $-e_1$ consisting of the steps $e_2, -e_1, -e_2$ in that order.

Next let $n \geq 1$ and assume the result is true for every possible cluster of size n. Now suppose that we are given a cluster $\mathfrak{C}$ of size $|\mathfrak{C}| = n + 1$. Since this cluster is finite but non-empty, it necessarily contains an awesome point $x \neq o$ (in particular, since $n + 1 > 1$ we have that $K \geq 1$ and every point in $\mathfrak{C}_K$ is awesome).

Since x is awesome, by changing the status of x from half-site to full-site, we obtain a new environment (call it $\boldsymbol{\omega}^x$), and a new cluster $\mathfrak{C}^x$ of size $|\mathfrak{C}^x| = n$. It follows from the induction hypothesis that there exists a path γ consistent with the environment $\boldsymbol{\omega}^x$ from o to $-e_1$ of length at most $4n$. By deleting loops if necessary we may assume that γ is simple, so it passes through each point at most once. We will find a path γ' in the original environment $\boldsymbol{\omega}$ from o to $-e_1$, by modifying γ.

Suppose that γ does not pass through (i.e. enter and exit) x. Then the path γ is also consistent with $\boldsymbol{\omega}$ so we take $\gamma' = \gamma$. Otherwise γ enters x from some direction and exits from some other direction. If γ exits in the direction e_1 or e_2 then (since these arrows are still available from x in the environment $\boldsymbol{\omega}$) γ is still consistent with $\boldsymbol{\omega}$, so we take $\gamma' = \gamma$ again.

If γ exits x in direction $-e_1$, then $\gamma_k = x$, and $\gamma_{k+1} = x - e_1$ for some $k < 4n$. In this case we define γ' by $\gamma'_j = \gamma_j$ for $j \leq k$, $\gamma'_{k+1} = x + e_2$, $\gamma'_{k+2} = x + e_2 - e_1$, $\gamma'_{k+3} = x + e_2 - e_1 - e_2$, and then γ' follows the remainder of the path γ from the point $x - e_1 = x + e_2 - e_1 - e_2$. All of the added steps are consistent with $\boldsymbol{\omega}$ since x was awesome in this environment. The resulting path γ' (which may or may not be simple) has two extra steps than γ.

Similarly, if γ exits x in direction $-e_2$, then we set $\gamma' = \gamma$ until the hitting time of x, then we make γ' take the steps $e_2, -e_1, -e_2, -e_2, e_1$ (all these steps are consistent with $\boldsymbol{\omega}$ since x was awesome in $\boldsymbol{\omega}$) and then γ' follows γ from the point $x + e_2 - e_1 - e_2 - e_2 + e_1 = x - e_2$. In this case γ' (which may or may not be simple) contains 4 more steps than γ.

It follows that we have a path γ' consistent with the environment $\boldsymbol{\omega}$, from o to $-e_1$, whose length satisfies $|\gamma'| \leq 4 + |\gamma| \leq 4 + 4n = 4(n + 1)$. This completes the proof. ■

We have verified that for every $x \in \mathbb{Z}^d$, $\mathbb{E}[T_x] < \infty$. Applying the subadditive ergodic theorem then yields the following.

Lemma 2. *The claim of Theorem 1.1 holds for* $p < p_c(2)$*.*

3.2 Finite expectation for $1 - p > p_*(d)$

Lemma 2 and the following lemma together imply that Theorem 1.1 holds.

Lemma 3. *The claim of Theorem 1.1 holds for* $1 - p > p_*(d)$*.*

Let $\mathbb{P}_q^*$ denote the law of site percolation on $\mathbb{Z}^d$, with parameter q. For supercritical site percolation (i.e. with $q > p_*(d)$) let $\mathcal{P}$ denote the (unique) infinite cluster, and let $N^+ = \inf\{n \geq 0 : ne_1 \in \mathcal{P}\}$. We will use the following result, which is well-understood (but we have not seen an explicit statement in the literature). The proof is in the Appendix.

Lemma 4. *Fix* $d \geq 2$*. For* $q > p_*(d)$ *there exist* $c, C > 0$ *such that for all* $n > 0$*,*

$$\mathbb{P}_q^*(N^+ > n) \leq Ce^{-cn}.$$

Proof of Lemma 3. Fix $d \geq 2$, and $1 - p > p_*(d)$. Then the full sites $(\Omega_+)^c$ percolate in the sense of site percolation on $\mathbb{Z}^d$. Let $\mathcal{P}$ denote the infinite cluster (in the sense of site percolation) of the set of full sites. Let $N^+ = \inf\{n \geq 0 : ne_1 \in \mathcal{P}\}$, and let $N^- = \inf\{n > 0 : -ne_1 \in \mathcal{P}\}$. Let Γ_N denote any shortest path in $\mathcal{P}$ from N^+e_1 to $-N^-e_1$, and let $M = |\Gamma_N|$.

Now consider the path Γ from the origin to $-e_1$ as follows. Starting from o, the path takes N^+ steps in the direction e_1 to the point N^+e_1. It then follows the path Γ_N from N^+e_1 to $-N^-e_1$. It then takes $N_- - 1$ steps in direction e_1 to reach the point $-e_1$. This entire path is consistent with the environment and the total length is at most $N^+ + M + N^-$. Thus, $T_{o,-e_1} \leq N^+ + N^- + M$ and it suffices to show that the three random variables on the right all have finite expectation. Finiteness of the first two expectations follows from Lemma 4 (or indeed from the ergodic theorem as $\mathbb{E}[N^+]$ and $\mathbb{E}[N^-]$ are both at most $1/\mathbb{P}(o \in \mathcal{P})$). It remains to show that $\mathbb{E}[M] < \infty$.

For $n \in \mathbb{N}$,

$$\mathbb{P}(M > n) \leq \mathbb{P}\Big(M > n, N^+ + N^- \leq \sqrt{n}\Big) + \mathbb{P}\Big(N^+ > \frac{\sqrt{n}}{2}\Big) + \mathbb{P}\Big(N^- > \frac{\sqrt{n}}{2}\Big). \tag{3.2}$$

The last two terms are summable in n by Lemma 4. It remains to show that the first term on the right of (3.2) is summable in n. This term is at most

$$
\begin{aligned}
&\sum_{n_+=0}^{\sqrt{n}} \sum_{n_-=1}^{\sqrt{n}} \mathbb{P}^*_{1-p}\Big(n_+e_1 \leftrightarrow -n_-e_1, T^*_{n_+e_1,-n_-e_1} > n\Big) \\
&= \sum_{n_+=0}^{\sqrt{n}} \sum_{n_-=1}^{\sqrt{n}} \mathbb{P}^*_{1-p}\Big(o \leftrightarrow (n_+ + n_-)e_1, T^*_{o,(n_++n_-)e_1} > n\Big) \qquad (3.3)
\end{aligned}
$$

where now we are only considering connections using full sites. We now use the argument of [1, Theorems 1.1 and 1.2].[3] Specifically we use the inequality [1, (4.49)] with the box radius N therein chosen so that [1, (4.47)] holds, and setting y in [1, (4.49)] equal to me_1. Note that in [1, (4.49)] their l is our n, while their n is equal to $|\boldsymbol{a}(y)| \le |y| = m$. So [1, (4.49)] together with Markov's inequality applied to $e^{h(|\mathbf{C}^*_{\mathbf{0}}|+1)}$ therein gives the existence of constants $C^*, c^* > 0$ such that for all $m, n \in \mathbb{N}$,

$$\mathbb{P}^*_{1-p}(o \leftrightarrow me_1, T^*_{o,me_1} > n) \le C^* m e^{-c^* n/m}.$$

Setting $m = n_+ + n_- \le 2\sqrt{n}$ it follows that there exist $a, b > 0$ such that for all $n \in \mathbb{N}$, (3.3) is at most

$$an\sqrt{n}e^{-b\sqrt{n}}.$$

This proves that $\mathbb{P}(M > n)$ is summable in n, so $\mathbb{E}[M] < \infty$, which completes the proof. ■

4 Features of the shape for $p < p_c(2)$

In this section we prove the non-standard parts of Theorem 1.2, namely (c) and (d). Part (a) is standard and the proof is relegated to the Appendix, while the proof of (b) is via a coupling argument that is both elementary and standard, and is therefore omitted.

[3] This paper proves the result for bond percolation, but also points out that the same arguments also prove the result for site percolation.

4.1 Proof of Theorem 1.2(c)

Proof of Theorem 1.2(c)(i). Fix $d \geq 2$ and p as in the theorem. Note first that by continuity of ζ it is sufficient to prove the result for v with rational coordinates, and with $a \in (p, 1)$.

Fix $v \in \mathcal{S}_p^{\text{good}}$ (with rational coordinates). Then we can write v as

$$v = \sum_{i=1}^{d} (a\alpha_i' - (1-a)\beta_i')e_i,$$

where $\alpha_i' = \alpha_i/a$ and $\beta_i' = \beta_i/(1-a)$. Since v has rational coordinates it follows that $m_v := \inf\{k \in \mathbb{N} : kv \in \mathbb{Z}^d\} < \infty$, and the points $nm_v v$ are in $\mathbb{Z}^d$ for $n \in \mathbb{Z}_+$. It therefore suffices to show that there is a constant $C > 0$ such that almost surely for each $\varepsilon > 0$, $T_{m_v nv} \leq nm_v + C\varepsilon n$ infinitely often. We will prove this by constructing a path Γ consistent with the environment along which every site x is reachable in time $\|x\|_1$, and such that the points $m_v nv$ are reachable by following Γ for a long time and then taking $+e_i$ steps for a relatively short time.

We start by enlarging the probability space to include the random environment, as well as some i.i.d. standard uniform random variables $(U_n)_{n\geq 0}$ that are also independent of the environment. Proving the result in this enlarged space establishes the desired result (since the desired result is simply an a.s. statement about the environment).

Let $\varepsilon < a - p$ and $b \in (a - \varepsilon, a)$. Set $\Gamma_0 = o$ and define Γ_n for $n \geq 1$ recursively as follows:

- if $\Gamma_{n-1} \in \Omega_+$ then we choose our next step to be e_i with probability α_i' independent of the past (here we use the random variable U_n),
- if $\Gamma_{n-1} \notin \Omega_+$ then we choose our next step to be e_i with probability $(b-p)\alpha_i'/(1-p)$, and $-e_i$ with probability $(1-b)\beta_i'/(1-p)$, independent of the past, (using U_n).

By (1.2) if this path ever takes a step e then it a.s. never takes a step $-e$. It follows that $\|\Gamma_n\|_1 = n$ for every n, and that Γ is a self-avoiding path, so the environment seen at every time is a half site with probability p (independent of the past). It follows that Γ is a random walk (i.e. it has i.i.d. increments) with

$$\mathbb{P}(\Gamma_n - \Gamma_{n-1} = e_i) = p\alpha_i' + (1-p)(b-p)\alpha_i'/(1-p) = b\alpha_i', \quad \text{and} \tag{4.1}$$

$$\mathbb{P}(\Gamma_n - \Gamma_{n-1} = -e_i) = (1-p)(1-b)\beta_i'/(1-p) = (1-b)\beta_i'. \tag{4.2}$$

The expected increment is then

$$\mu_b := \sum_{i=1}^{d} (b\alpha_i' - (1-b)\beta_i')e_i.$$

Note that

$$\mu_b - v = \sum_{i=1}^{d} (b-a)(\alpha_i' + \beta_i')e_i.$$

If for some j, $\alpha_j' = \beta_j' = 0$ then for every n, $\Gamma_n \cdot e_j = 0 = v \cdot e_j$. For all other coordinate directions, the law of large numbers for Γ and the fact that $(\mu_b - v)\cdot e_i = (b-a)(\alpha_i' + \beta_i') < 0$ imply that for all n sufficiently large, $n^{-1}\Gamma_n \cdot e_i < v \cdot e_i$. Moreover, since $a - b < \varepsilon$ we have that $n^{-1}\Gamma_n \cdot e_i > v \cdot e_i - 2\varepsilon$ for all n sufficiently large. Thus for all n sufficiently large, $v \cdot e_i - 2\varepsilon < \Gamma_{m_v n} \cdot e_i/(m_v n) \leq v \cdot e_i$ for every i. For such n we can reach $m_v nv$ by following the path Γ to $\Gamma_{m_v n}$ and then taking at most $2d\varepsilon m_v n$ steps in positive coordinate directions (recall that such steps are possible from every site) to $m_v nv$. Thus for all n sufficiently large we have that

$$T_{m_v nv} \leq T_{\Gamma_{m_v n}} + 2d\varepsilon m_v n = m_v n + 2d\varepsilon m_v n,$$

which completes the proof. ■

Let $\mathbb{P}_q^\dagger$ denote the law of oriented (according to $O \in \mathcal{O}$) site percolation on $\mathbb{Z}^d$ with parameter q. Theorem 1.2(c)(ii) is a consequence of the following "well-known" result. The proof is in the Appendix.

Lemma 5. *Fix $d \geq 2$ and let $q > p_\dagger(d)$. For $v \in \mathbb{Z}^d$ in the interior of the deterministic cone O_q,*

$$\mathbb{P}_q^\dagger(o \to nv \textit{ infinitely often}) \geq \inf_{n \in \mathbb{Z}_+} \mathbb{P}_q^\dagger(o \to nv) =: \eta(q) > 0. \tag{4.3}$$

Proof of Theorem 1.2(c)(ii). Fix p as in the theorem and such that $1 - p > p_\dagger(d)$. Let $O \in \mathcal{O}$ and let $v \in \mathbb{Z}^d$ be in the interior of the deterministic cone O_{1-p}. By Lemma 5, with probability at least $\eta(1-p) > 0$, for infinitely many n, one can reach nv from o by following a path of full sites using only steps in O. On this event we therefore have that $T_{nv} = \|nv\|_1$ infinitely often. Since $T_{nv}/n \to \zeta_p(v)$ this proves that $v \in \mathcal{S}_p$. For $u \in \mathbb{Q}^d$ in the interior of O_{1-p} the result follows since $v = m_u u \in \mathbb{Z}^d$. ■

4.2 Proof of Theorem 1.2(d)

The proof of (d)(i) is standard, but we include it in the Appendix for completeness, and to help the reader understand why the more unusual argument that we present to prove (d)(ii) is used.

The proof of Theorem 1.2(d)(i) involves counting "short" paths from the origin to a point u close to $-e_i$. The number of such paths grows exponentially in n, but the exponential growth can be made as close to 1 as we like by taking ε small and u very close to $-e_1$ (i.e. δ small). In part (d)(ii) of the theorem we cannot take u close to $-e_1$, so the number of paths grows at an exponential rate that cannot be made close to 1. For example, the number of self-avoiding paths of length exactly n from o to $(-(1-s)n, sn)$ (where $s \in \mathbb{Q} \in (0,1)$ and $sn \in \mathbb{Z}$) is $\binom{n}{sn}$ which grows exponentially fast with growth rate depending on s. Thus, in order to prove (d)(ii) of the theorem we need a more sophisticated argument than simple enumeration of paths. This is the content of the remainder of the section. The proof as written works for two dimensions. It would be of interest to see whether modifications of these arguments can be used to obtain improvements to (d)(i) in higher dimensions.

Henceforth we fix $d = 2$. Let $b(\gamma)$ denote the number of $\rightarrow, \downarrow$ steps of a path γ. Given an environment $\boldsymbol{\omega}$, a path γ in $\mathbb{Z}^2$ starting from o is said to be $\boldsymbol{\omega}$-*good* if all $\leftarrow$ steps of the path are consistent with the environment $\boldsymbol{\omega}$. Let $G(\boldsymbol{\omega})$ be the set of (finite) $\boldsymbol{\omega}$-good paths.

Given $\boldsymbol{\omega}$ and $\gamma \in G(\boldsymbol{\omega})$ define the *westernisation* of γ to be the path $\tilde{\gamma} \in G(\boldsymbol{\omega})$ of the same length as γ such that the times and types of step in $\{\rightarrow, \downarrow\}$ are identical for the two paths and such that outside of these times, $\tilde{\gamma}$ always takes the $\leftarrow$ step when the environment allows (and $\uparrow$ when it does not). Recall that the coordinates of a point $x \in \mathbb{Z}^2$ are denoted by $x = (x^{\{1\}}, x^{\{2\}})$.

Lemma 6. *Let $\boldsymbol{\omega}$ be an environment and $\gamma \in G(\boldsymbol{\omega})$. Then γ and its westernisation $\tilde{\gamma}$ satisfy*

$$\gamma_m^{\{1\}} - \tilde{\gamma}_m^{\{1\}} = \gamma_m^{\{2\}} - \tilde{\gamma}_m^{\{2\}} \geq 0, \qquad \textit{for all } m \geq 0.$$

Proof. Proof by induction. The claim is trivially true at time 0. Suppose the claim is true up to and including time $m-1$. If at time m one of the paths takes a step in $\{\rightarrow, \downarrow\}$ then the other takes the same step, so the result is also trivially true at time m.

Otherwise, if $\gamma^{\{1\}}_{m-1} - \tilde{\gamma}^{\{1\}}_{m-1} = \gamma^{\{2\}}_{m-1} - \tilde{\gamma}^{\{2\}}_{m-1} = 0$ then both paths are at the same location x_{m-1}. If the local environment $\omega(x_{m-1})$ does not have $\leftarrow$ then both paths take $\uparrow$, and the claim continues to hold (since neither coordinatewise difference changes). Otherwise if the local environment does have $\leftarrow$ it may be that both paths take this arrow (so the claim continues to hold) or γ takes $\uparrow$ and $\tilde{\gamma}$ takes $\leftarrow$ in which case each coordinatewise difference increases by 1, so the claim still holds.

Otherwise if $\gamma^{\{1\}}_{m-1} - \tilde{\gamma}^{\{1\}}_{m-1} = \gamma^{\{2\}}_{m-1} - \tilde{\gamma}^{\{2\}}_{m-1} > 0$, then the only way that either of these increments can change is if one path takes $\leftarrow$ and the other takes $\uparrow$. But in this case either both differences increase by 1, or both differences decrease by 1, so the claim still holds. ■

Let $\tilde{G}(\boldsymbol{\omega}) \subset G(\boldsymbol{\omega})$ denote the set of paths in $G(\boldsymbol{\omega})$ that never take a $\uparrow$ step from a location where $\leftarrow$ is available.

Proof of Theorem 1.2(d)(ii). Let p be as in the statement of the theorem, and $s \in [0,p) \cap \mathbb{Q}$. Let $\varepsilon \in (0,(p-s)/10)$. Let $n \in \mathbb{N}$ be such that ns in an integer. Set $u = (-(1-s), s)$, as in the statement of the theorem.

Any path γ starting from the origin with length $\ell(\gamma) \leq n(1+\varepsilon)$ and taking more than εn $\downarrow, \rightarrow$ steps (i.e. $b(\gamma) \geq \varepsilon n$) cannot reach a point on the boundary of the ℓ_1 ball $B_1(o,n)$ with first coordinate non-positive and second coordinate non-negative (i.e. in the northwest quadrant). Let

$$J_n = \Big\{\exists \gamma \text{ consistent with } \boldsymbol{\omega} \text{ with } \ell = \ell(\gamma) \leq n(1+\varepsilon) \text{ and } b(\gamma) \leq \varepsilon n \text{ such that} \tag{4.4}$$

$$0 \leq \gamma^{\{2\}}_\ell \leq n(s+\varepsilon/4), \gamma^{\{1\}}_\ell \leq -n(1-s-\varepsilon/4), \|\gamma_\ell\|_1 \geq n\Big\}. \tag{4.5}$$

We will show that there exist $C, c > 0$ such that

$$\mathbb{P}(J_n) \leq Ce^{-cn}, \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

To see why this is sufficient to prove the result, note that if $\zeta_p(u) = 1$ then for all but finitely many n, $nm_u u$ is reached within time $nm_u(1+\varepsilon)$. But $0 \leq nm_u u^{\{2\}} \leq nm_u(s+\varepsilon/4)$, $nm_u u^{\{1\}} \leq -nm_u(1-s-\varepsilon/4)$ and $\|nm_u u\|_1 = nm_u$, so this means that J_{nm_u} occurs. By (4.6) and Borel-Cantelli this can only occur for finitely many n, giving a contradiction.

The left hand side of (4.6) is at most

$$\mathbb{P}\Big(\exists\gamma \in G(\boldsymbol{\omega}) \text{ with } \ell(\gamma) \leq n(1+\varepsilon) \text{ and } b(\gamma) \leq \varepsilon n \text{ such that}$$
$$0 \leq \gamma_\ell^{\{2\}} \leq n(s+\varepsilon/4), \gamma_\ell^{\{1\}} \leq -n(1-s-\varepsilon/4), \|\gamma_\ell\|_1 \geq n\Big). \tag{4.7}$$

By Lemma 6 if there exists a γ as in this event then its westernisation $\tilde{\gamma}$ also satisfies the constraints in (4.7) except that we may have $\tilde{\gamma}_\ell^{\{2\}} < 0$. In particular we have that for such a γ,

$$\begin{aligned}
\|\tilde{\gamma}_\ell\|_1 &= -\tilde{\gamma}_\ell^{\{1\}} + |\tilde{\gamma}_\ell^{\{2\}}| \\
&= -\gamma_\ell^{\{1\}} + (\gamma_\ell^{\{1\}} - \tilde{\gamma}_\ell^{\{1\}}) + |\tilde{\gamma}_\ell^{\{2\}}| \\
&= -\gamma_\ell^{\{1\}} + (\gamma_\ell^{\{2\}} - \tilde{\gamma}_\ell^{\{2\}}) + |\tilde{\gamma}_\ell^{\{2\}}| \\
&\geq -\gamma_\ell^{\{1\}} + \gamma_\ell^{\{2\}} = \|\gamma_\ell\|_1 \geq n.
\end{aligned}$$

Thus, (4.7) is at most

$$\mathbb{P}\Big(\exists\gamma \in \tilde{G}(\boldsymbol{\omega}) \text{ with } \ell(\gamma) \leq n(1+\varepsilon) \text{ and } b(\gamma) \leq \varepsilon n \text{ such that}$$
$$\gamma_\ell^{\{2\}} \leq n(s+\varepsilon/4), \gamma_\ell^{\{1\}} \leq -n(1-s-\varepsilon/4), \|\gamma_\ell\|_1 \geq n\Big). \tag{4.8}$$

A path γ such as that in (4.8) need not be self-avoiding. If it is not self-avoiding then by removing loops, it contains a shorter path γ' with $\ell(\gamma') \leq \ell(\gamma)$ and $b(\gamma') \leq b(\gamma)$ that ends at the same point as γ. Let $\tilde{S}(\boldsymbol{\omega}) \subset \tilde{G}(\boldsymbol{\omega})$ denote the set of paths in $\tilde{G}(\boldsymbol{\omega})$ that are self-avoiding.

Then (4.8) is at most

$$\mathbb{P}\Big(\exists\gamma \in \tilde{S}(\boldsymbol{\omega}) \text{ with } \ell(\gamma) \leq n(1+\varepsilon) \text{ and } b(\gamma) \leq \varepsilon n \text{ such that}$$
$$\gamma_\ell^{\{2\}} \leq n(s+\varepsilon/4), \gamma_\ell^{\{1\}} \leq -n(1-s-\varepsilon/4), \|\gamma_\ell\|_1 \geq n\Big)$$
$$\leq \sum_{l=n}^{n(1+\varepsilon)} \sum_{b=0}^{\varepsilon n} \mathbb{P}\Big(\exists\gamma \in \tilde{S}(\boldsymbol{\omega}) \text{ with } \ell(\gamma) = l \text{ and } b(\gamma) = b \text{ such that} \tag{4.9}$$
$$\gamma_l^{\{2\}} \leq n(s+\varepsilon/4), \gamma_l^{\{1\}} \leq -n(1-s-\varepsilon/4), \|\gamma_l\|_1 \geq n\Big).$$

The upper limits of these sums are $\lfloor n(1+\varepsilon)\rfloor$ and $\lfloor n\varepsilon\rfloor$ respectively since l and b are integer-valued. Now we sum over the times $\vec{t}(\gamma) = \{t_1(\gamma), \dots, t_b(\gamma)\}$

at which the path γ takes a $\rightarrow,\downarrow$ step and a sum over the exact sequence $\vec{x}(\gamma) \in \{\rightarrow,\downarrow\}^b$ of such steps to see that (4.9) is at most

$$\sum_{l=n}^{n(1+\varepsilon)} \sum_{b=0}^{\varepsilon n} \sum_{\vec{t}} \sum_{\vec{x}} \mathbb{P}\Big(\exists \gamma \in \tilde{S}(\boldsymbol{\omega}) \text{ with } \ell(\gamma) = l, \vec{t}(\gamma) = \vec{t} \text{ and } \vec{x}(\gamma) = \vec{x} \text{ such that}$$
$$\gamma_l^{\{2\}} \leq n(s + \varepsilon/4), \gamma_l^{\{1\}} \leq -n(1 - s - \varepsilon/4), \|\gamma_l\|_1 \geq n\Big). \tag{4.10}$$

Now consider how to determine whether the above event occurs. Construct a random path $\Gamma := \Gamma(\boldsymbol{\omega}, \vec{t}, \vec{x})$ of length l as follows. Starting from the origin at time 0 (set $\Gamma_0 = o$), if $j = t_i \in \vec{t}$ then let Γ take step x_i (i.e. $\Gamma_{j+1} = \Gamma_j + x_i$). If not, look at the environment ω_{Γ_j}: if this contains $\leftarrow$ then $\Gamma_{j+1} = \Gamma_j - e_1$, and otherwise $\Gamma_{j+1} = \Gamma_j + e_2$. If this path meets itself at some time then it is not self-avoiding, so the event has not occurred. As long as this path never meets itself then (each new environment that is looked at has the usual law, is independent of the past and) we can ask whether $\Gamma_l^{\{2\}} \leq n(s + \varepsilon/4)$ and $\Gamma_l^{\{1\}} \leq -n(1 - s - \varepsilon/4)$, and $\|\Gamma_l\|_1 \geq n$. If all of these are true then the event has occurred. Otherwise the event has not occurred. It follows that the event in (4.10) is equal to

$$\Big\{\Gamma \text{ is self-avoiding, with } \Gamma_l^{\{2\}} \leq n(s + \varepsilon/4), \Gamma_l^{\{1\}} \leq -n(1 - s - \varepsilon/4), \|\Gamma_l\|_1 \geq n\Big\}. \tag{4.11}$$

Now, let $\boldsymbol{X} := (X_i)_{i \geq 0}$ be an i.i.d. sequence with $\mathbb{P}(X_i = \leftarrow) = 1 - p = 1 - \mathbb{P}(X_i = \uparrow)$, and fix b and $\vec{t}$ and $\vec{x}$. Let $\bar{\Gamma} = \bar{\Gamma}(\boldsymbol{X}, \vec{t}, \vec{x})$ be a path of length l in $\mathbb{Z}^2$ defined as follows:

- $\bar{\Gamma}_0 = o$
- if $j = t_i \in \vec{t}$ then $\bar{\Gamma}_{j+1} - \bar{\Gamma}_j$ is equal to x_i
- otherwise $\bar{\Gamma}_{j+1} - \bar{\Gamma}_j$ is equal to X_j.

The probability in (4.10) is equal to

$$\begin{aligned}
&\mathbb{P}\Big(\bar{\Gamma} \text{ is self-avoiding, } \bar{\Gamma}_l^{\{2\}} \leq n(s + \varepsilon/4), \bar{\Gamma}_l^{\{1\}} \leq -n(1 - s - \varepsilon/4), \|\bar{\Gamma}_l\|_1 \geq n\Big) \\
&\leq \mathbb{P}\Big(\bar{\Gamma}_l^{\{2\}} \leq n(s + \varepsilon/4), \bar{\Gamma}_l^{\{1\}} \leq -n(1 - s - \varepsilon/4), \|\bar{\Gamma}_l\|_1 \geq n\Big) \\
&\leq \mathbb{P}\Big(\bar{\Gamma}_l^{\{2\}} \leq n(s + \varepsilon/4)\Big).
\end{aligned} \tag{4.12}$$

Now note that $\bar{\Gamma}$ is a (biased) simple random walk of length $l - b$, with b deterministic $\downarrow, \rightarrow$ steps inserted at specific times in the path. Next note that b is very small compared to l, and we are asking for this walk to deviate far from its mean at time l of order n. We now show that this probability is exponentially small with exponent that can be taken uniform in small ε. To be precise, we will show that there exist $c, C > 0$ depending on p, s (but not ε) such that for all $l \in [n, n(1+\varepsilon)] \cap \mathbb{Z}_+$, $b \le n\varepsilon$, $\vec{t}$, and $\vec{x}$, and all $\varepsilon > 0$ sufficiently small,

$$\mathbb{P}\Big(\bar{\Gamma}_l^{\{2\}} \le n(s + \varepsilon/4)\Big) \le Ce^{-cn}. \tag{4.13}$$

Let us assume that (4.13) holds and explain how to proceed.

Using (4.13) we have that (4.10) is at most

$$\sum_{l=n}^{n(1+\varepsilon)} \sum_{b=0}^{\varepsilon n} \sum_{\vec{t}} \sum_{\vec{x}} Ce^{-cn}. \tag{4.14}$$

The sum over $\vec{x}$ gives a contribution at most $2^b \le 2^{\varepsilon n}$. The sum over $\vec{t}$ contributes at most $\binom{\lfloor n(1+\varepsilon) \rfloor}{\lceil n\varepsilon \rceil}$ when n is large and $\varepsilon < 1/3$. This combinatorial factor is at most

$$\frac{C(n+n\varepsilon)^{n+n\varepsilon+1/2}}{(n\varepsilon)^{n\varepsilon+1/2}(n-1)^{n-1+1/2}} \le \frac{C(n+n\varepsilon)^{n+n\varepsilon}}{(n\varepsilon)^{n\varepsilon}(n-1)^{n-1}}, \tag{4.15}$$

for all n sufficiently large depending on ε. This is at most

$$\frac{C' n^{n+n\varepsilon}(1+\varepsilon)^{n+n\varepsilon}}{n^{n\varepsilon}\varepsilon^{n\varepsilon}n^{n-1}} = \frac{C' n(1+\varepsilon)^{n+n\varepsilon}}{\varepsilon^{n\varepsilon}} = C' n\left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\varepsilon^{\varepsilon}}\right)^n. \tag{4.16}$$

The exponential growth rate of this quantity is

$$a_\varepsilon := (1+\varepsilon)(1+1/\varepsilon)^\varepsilon = (1+\varepsilon)e^{\varepsilon \log(1+1/\varepsilon)}, \tag{4.17}$$

which approaches $1 \times e^0 = 1$ as $\varepsilon \downarrow 0$. Thus for small $\varepsilon > 0$, for all n sufficiently large (4.14) is at most

$$\sum_{l=n}^{n(1+\varepsilon)} \sum_{b=0}^{\varepsilon n} C' n(1+a_\varepsilon)^n 2^{n\varepsilon} Ce^{-cn} \le C'' n^3 e^{-c' n} \le C''' e^{-c'' n}. \tag{4.18}$$

We have shown that (4.6) holds (and hence the theorem is proved), assuming (4.13). It therefore remains to prove that for some $c, C > 0$ (not depending on ε) (4.13) holds for all $l, b, \vec{t}, \vec{x}$. Fix $l, b, \vec{t}, \vec{x}$ and let $N_r = \sum_{i=1}^{r} \mathbb{1}_{\{X_i=\uparrow\}}$ be the number of $\uparrow$ steps among the first r of the X's. Then (4.12) is at most

$$\mathbb{P}\Big(N_{l-b} \leq n(s+\varepsilon/4)+n\varepsilon\Big) = \mathbb{P}\Big(N_{l-b} \leq p(l-b) - (p(l-b) - n(s+5\varepsilon/4))\Big). \tag{4.19}$$

Here, N_{l-b} is a sum of $l-b$ i.i.d. (0-1-valued) random variables each with expectation p, so $\mathbb{E}[N_{l-b}] = p(l-b)$. Since $b \leq \varepsilon n$ and $l \geq n$ we have

$$p(l-b) - n(s+5\varepsilon/4) \geq pn(1-\varepsilon) - n(s+5\varepsilon/4) \tag{4.20}$$

$$= n(p(1-\varepsilon) - s - 5\varepsilon/4). \tag{4.21}$$

Since $\varepsilon < (p-s)/10$ we have

$$p(1-\varepsilon) - s - 5\varepsilon/4 > p(1-(p-s)/10) - s - 5(p-s)/10 \tag{4.22}$$

$$= (p-s)[1-(p+5)/10] \tag{4.23}$$

$$> 4(p-s)/10. \tag{4.24}$$

This means that

$$p(l-b) - n(s+5\varepsilon/4) \geq 4n(p-s)/10 \geq 2(l-b)(p-s)/10,$$

where we have used the fact that $l-b \leq l \leq n(1+\varepsilon) \leq 2n$.

Thus (4.19) is at most

$$\mathbb{P}\Big(N_{l-b} \leq (l-b)[p - 2(p-s)/10]\Big) \leq Ce^{-c(l-b)}, \tag{4.25}$$

for some C, c depending only on p, s (e.g. by Hoeffding's inequality). Since $l-b \geq n(1-\varepsilon) > n/2$ (for all ε sufficiently small), this verifies (4.13) as claimed. ∎

5 The "shape" theorem

Fix $d \geq 2$. For fixed $m \in d\mathbb{N}$, let $\tilde{U}_m = \{x \in \mathbb{R}^d : \|x\|_1 = m\}$ and $U_m = \tilde{U}_m \cap \mathbb{Z}^d$. For $x \in \mathbb{Z}^d$ let $x_+ = \{y \in \mathbb{Z}^d : y^{\{i\}} \geq x^{\{i\}}, i = 1, \dots, d\}$. Every point in x_+ can be reached from x by using only steps in positive directions. Let $\|\cdot\| = \|\cdot\|_1$. For any $x \neq o$ let $x^* = x/\|x\|$. We will prove Theorem 1.3 assuming the following geometrical lemma, whose proof is in the appendix.

Lemma 7. *Let $\alpha \in (0,1/d)$. Then there exists $m = m(\alpha) \in \mathbb{N}$ and $c = c(\alpha) > 1$ such that for all $y \in \mathbb{Z}^d$ with $\|y\| > cm$:*
the hypercube H_y^- with opposite vertices y and $y - \vec{1}\|y\|\alpha$, and
the hypercube H_y^+ with opposite vertices y and $y + \vec{1}\|y\|\alpha$,
each contain at least one point $y_m \in \mathbb{Z}_+ U_m \subset \mathbb{Z}^d$.

Proof of Theorem 1.3. Fix $\varepsilon \in (0,1/2)$, and fix p as in the statement of the theorem. Let $\delta > 0$ be such that

$$\text{if } u,v \in \tilde{U}_1 \text{ and } \|u-v\| \le \delta \text{ then} \qquad \zeta(u)/\zeta(v) \in (1-\varepsilon/10, 1+\varepsilon/10). \tag{5.1}$$

(This is possible by uniform continuity of ζ and the fact that $\zeta(v) \ge 1$ for any $v \in \tilde{U}_1$.)

Let $\alpha < (\varepsilon \wedge \delta)/(100d)$. Let $m = m(\alpha)$ and $c = c(\alpha)$ be as in Lemma 7. Since m is finite and $\lim_{n\to\infty} T_{nv}/n = \zeta(v)$ for every $v \in \mathbb{Z}^d$ (Theorem 1.1), there exists a random $\kappa_0 \ge 2cm/\alpha$ so that whenever $z \in \mathbb{Z}_+ U_m$ and $\|z\| > \kappa_0$ we have $T_z \in \zeta(z^*)\|z\|\Big(1 - \dfrac{\varepsilon}{10}, 1 + \dfrac{\varepsilon}{10}\Big)$. Let $\mathcal{T} = \max\{T_u : u \in \mathbb{Z}^d, \|u\| \le 2\kappa_0\} < \infty$.

Let $n \ge \mathcal{T}$ and $y \in (1-\varepsilon)n\mathfrak{B}_p$. We want to show that $y \in B(n)$ (i.e. that $T_y \le n$). If $\|y\| \le 2\kappa_0$ then $T_y \le \mathcal{T} \le n$ so $y \in B(n)$. Otherwise $\|y\| > 2\kappa_0 > cm$. By Lemma 7, the hypercube H_y^- contains a point $y_m \in \mathbb{Z}_+ U_m$. Since $y_m \in H_y^-$ we have that $\|y - y_m\| \le d\|y\|\alpha$. Thus,

$$\begin{aligned} \|y_m^* - y^*\| = \frac{\big\|\|y_m\|y\| - \|y_m\|y\big\|}{\|y\|\|y_m\|} &\le \frac{\big\|\|y_m\|y\| - \|y_m\|y_m\big\| + \big\|\|y_m\|y_m - \|y_m\|y\big\|}{\|y\|\|y_m\|} \\ &\le \frac{2\|y_m - y\|}{\|y\|} \le 2d\alpha \le \delta. \end{aligned} \tag{5.2}$$

Since $y \in (y_m)_+$ and $+$ steps are available from every site we have

$$T_y \le T_{y_m} + \|y - y_m\| \le T_{y_m} + \|y\|d\alpha. \tag{5.3}$$

It also follows from $|\,\|y\| - \|y_m\|\,| \le \|y - y_m\|$ (and $y_m \in H_y^-$) that

$$\|y_m\| \in \|y\|\Big(1 - d\alpha, 1 + d\alpha\Big). \tag{5.4}$$

Since $y \in (1-\varepsilon)n\mathfrak{B}_p$ we have that

$$\|y\| \le \frac{(1-\varepsilon)n}{\zeta(y^*)}.$$

Therefore from (5.3) (recall also that $\zeta(x^*) \geq 1$ for every $x \neq o$)

$$T_y \leq T_{y_m} + \|y\| d\alpha \leq T_{y_m} + \frac{d\alpha(1-\varepsilon)n}{\zeta(y^*)} \leq T_{y_m} + n\frac{\varepsilon}{10}.$$

Now $\|y_m\| \geq \|y\|(1-d\alpha) \geq 2\kappa_0(1-d\alpha) \geq \kappa_0$ so

$$\begin{aligned} T_{y_m} &\leq \zeta(y_m^*)\|y_m\|(1+\frac{\varepsilon}{10}) \\ &\leq \zeta(y_m^*)\|y\|(1+d\alpha)(1+\frac{\varepsilon}{10}) \\ &\leq \frac{\zeta(y_m^*)}{\zeta(y^*)}(1-\varepsilon)n(1+d\alpha)(1+\frac{\varepsilon}{10}) \\ &\leq (1+\frac{\varepsilon}{10})(1-\varepsilon)n(1+d\alpha)(1+\frac{\varepsilon}{10}) \leq (1-\varepsilon/2)n, \end{aligned}$$

where in going to the last line we have used (5.1) and (5.2). Thus $T_y \leq (1-\frac{\varepsilon}{2}+\frac{\varepsilon}{10})n \leq n$ so $y \in B(n)$ as required.

For the other result we want to show that for all n sufficiently large, if $y \notin (1+\varepsilon)n\mathfrak{B}_p$ then $T_y > n$. So let $n > 4\kappa_0 \max\{\zeta(u) : \|u\| = 1\}$ and suppose that $y \notin (1+\varepsilon)n\mathfrak{B}_p$. Then

$$\|y\| > \frac{(1+\varepsilon)n}{\zeta(y^*)} > 4(1+\varepsilon)\kappa_0.$$

Proceeding as above, by Lemma 7 we can find $w_m \in \mathbb{Z}_+ U_m$ in the hypercube H_y^+, and such that $\|y - w_m\| \leq \|y\| d\alpha$ and $\|w_m^* - y^*\| \leq \delta$. Since $w_m \in y_+$ we have

$$T_y \geq T_{w_m} - \|y - w_m\| \tag{5.5}$$

But $\|w_m\| \geq \|y\|(1-d\alpha) \geq \kappa_0$, whence the right hand side of (5.5) is at least

$$\begin{aligned} &\zeta(w_m^*)\|w_m\|(1-\frac{\varepsilon}{10}) - \|y\| d\alpha \\ &\geq \zeta(w_m^*)\|y\|(1-d\alpha)(1-\frac{\varepsilon}{10}) - \zeta(w_m^*)\|y\| d\alpha \\ &\geq \frac{\zeta(w_m^*)}{\zeta(y^*)}(1+\varepsilon)n\Big[(1-d\alpha)(1-\frac{\varepsilon}{10}) - d\alpha\Big] \\ &\geq (1-\frac{\varepsilon}{10})(1+\varepsilon)n\Big[(1-d\alpha)(1-\frac{\varepsilon}{10}) - d\alpha\Big] \\ &> n, \end{aligned}$$

as required.

■

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Appendix

Proof of Lemma 4. (Supplied by G. Grimmett) First assume $d \geq 3$, and let $q > p_*(d)$. Let $S_k = [1, k] \times \mathbb{Z}^{d-1}$ be the slab of thickness k. By [13] there exists k_0 such that $q > p_*(S_{k_0})$ (the critical probability for percolation in the slab). Let $\mathbb{P}^0_q$ denote the law of site percolation in this slab. Then there exists $a > 0$ such that $\mathbb{P}^0_q(re_1 \leftrightarrow \infty) > a$ (for every $r \in [1, k_0]$). The interval $J_n = [1, nk_0e_1]$ traverses n disjoint copies of S_{k_0}. The contents of these slabs are independent (since they are vertex disjoint), so that $\mathbb{P}^*_q(N^+ > mk_0) \leq \mathbb{P}^*_q(J_m \nleftrightarrow \infty) \leq (1-a)^m$. For $n \in ((m-1)k_0, mk_0]$ we then have

$$\mathbb{P}^*_q(N^+ > n) \leq ((1-a)^{1/k_0})^n(1-a)^{-1},$$

which proves the result.

Now take $d = 2$ and $q > p_*(2)$. If $I_n = [0, ne_1]$ is not joined to ∞, then all points in I_n are in finite open clusters. Let $\mathcal{J}$ be the union of these clusters, and let $\mathcal{K}$ be the exterior site boundary of $\mathcal{J}$ in $\mathbb{Z}^2$. Let L be the matching lattice of $\mathbb{Z}^2$ (i.e. $\mathbb{Z}^2$ with all diagonals added). It is "standard" that $\mathcal{K}$ is connected in L. Since $q > p_*(2)$, and $p_*(2) + p_c(L) = 1$ (see e.g. [16]), the closed sites are subcritical for site percolation on L, and therefore the diameter of the closed cluster containing $(-m, 0)$ has exponential tail. Thus there exist $C, c > 0$ such that for all n, the probability that the diameter of the (closed) cluster of $(-m, 0)$ in L exceeds n is at most Ce^{-cn}. Now note that $\mathcal{K}$ is (part of) a closed cluster in this subcritical site model. Let $(-b, 0)$ be the leftmost vertex of $\mathcal{K}$ on the axis. Then the closed cluster of $(-b, 0)$ in L has diameter at least $b + n$. It follows that $\mathbb{P}^*(I_n \nleftrightarrow \infty) \leq \sum_{b \geq 0} Ce^{-c(n+b)}$ as required. ■

For $y \in \mathbb{R}^d$, let $[y] \in \mathbb{Z}^d$ denote the unique lattice point such that $y \in [y] + [0,1)^d$. We next note the following standard extension of Theorem 1.1.

Lemma 8. *Fix $d \geq 2$ and $1-p > \min\{1-p_c(2), p_*(d)\}$. Then for every $u \in \mathbb{Q}^d$,*

$$\lim_{x\to\infty} \frac{T_{[xu]}}{x} = \zeta_p(u), \quad \textit{almost surely, and in } L^1, \tag{5.6}$$

where we are taking x through $\mathbb{R}$ rather than just $\mathbb{Z}$.

Proof of Lemma 8. The claim is trivial at $u = o$. Otherwise we first restrict to $\mathbb{Z}^d$ and then at the end generalise to $\mathbb{Q}^d$. Suppose that $v \in \mathbb{Z}^d \setminus \{o\}$. By subadditivity,

$$T_{[xv]} \leq T_{\lfloor x \rfloor v} + T_{\lfloor x \rfloor v,[xv]} \tag{5.7}$$

$$T_{\lfloor x \rfloor v} \leq T_{[xv]} + T_{[xv],\lfloor x \rfloor v}. \tag{5.8}$$

Thus

$$T_{\lfloor x \rfloor v} - T_{[xv],\lfloor x \rfloor v} \leq T_{[xv]} \leq T_{\lfloor x \rfloor v} + T_{\lfloor x \rfloor v,[xv]}. \tag{5.9}$$

Now note that by Theorem 1.1

$$\frac{T_{\lfloor x \rfloor v}}{x} = \frac{\lfloor x \rfloor}{x} \cdot \frac{T_{\lfloor x \rfloor v}}{\lfloor x \rfloor} \to \zeta_p(v) \quad \text{as } x \to \infty, \text{ a.s. and in } L^1. \tag{5.10}$$

Thus, to prove the claim on $\mathbb{Z}^d$ it remains to show that almost surely

$$\lim_{x\to\infty} \frac{T_{[xv],\lfloor x \rfloor v}}{x} = \lim_{x\to\infty} \frac{T_{\lfloor x \rfloor v,[xv]}}{x} = 0, \tag{5.11}$$

and that the limits of the corresponding expected values are also 0.

We will explain the argument in detail for $d = 2$ and then explain how to generalise to $d > 2$. Let $s_i = \operatorname{sgn}(v^{\{i\}})$. Define the set of vertices

$$B_x(v) := \left\{ \lfloor x \rfloor v + i s_1 e_1 + j s_2 e_2 : i, j \in \mathbb{Z} \text{ and } 0 \leq i \leq |v^{\{1\}}|, 0 \leq j \leq |v^{\{2\}}| \right\}. \tag{5.12}$$

That is, $B_x(v)$ is a box of width $|v^{\{1\}}|$ and height $|v^{\{2\}}|$, positioned so that $\lfloor x \rfloor v$ is its corner closest to the origin. (Note that if some component of v is 0 this "box" is a line segment.) Since

$$\lfloor x \rfloor |v^{\{i\}}| \leq x|v^{\{i\}}| \leq (\lfloor x \rfloor + 1)|v^{\{i\}}| \tag{5.13}$$

we can conclude that $[xv] \in B_x(v)$. We thus have the bound

$$\frac{T_{\lfloor x \rfloor v, [xv]}}{x} \leq \frac{\max\{T_{\lfloor x \rfloor v, u} : u \in B_x(v)\}}{x} \tag{5.14}$$

$$\leq \sum_{\substack{0 \leq i \leq |v^{\{1\}}| \\ 0 \leq j \leq |v^{\{2\}}|}} \frac{T_{\lfloor x \rfloor v, \lfloor x \rfloor v + i s_1 e_1 + j s_2 e_2}}{x}. \tag{5.15}$$

Similarly

$$\frac{T_{[xv], \lfloor x \rfloor v}}{x} \leq \sum_{\substack{0 \leq i \leq |v^{\{1\}}| \\ 0 \leq j \leq |v^{\{2\}}|}} \frac{T_{\lfloor x \rfloor v + i s_1 e_1 + j s_2 e_2, \lfloor x \rfloor v}}{x}. \tag{5.16}$$

By translation invariance we have $T_{u,u+v} \overset{d}{=} T_v$ for $u, v \in \mathbb{Z}^d$. Thus for any $n \in \mathbb{Z}_+$ and $\varepsilon > 0$ with fixed $0 \leq i \leq |v^{\{1\}}|, 0 \leq j \leq |v^{\{2\}}|$ (excluding $i = j = 0$, which is trivial)

$$\mathbb{P}\Big(\frac{T_{nv, nv + i s_1 e_1 + j s_2 e_2}}{n} > \varepsilon\Big) = \mathbb{P}(T_{i s_1 e_1 + j s_2 e_2} > n\varepsilon) \tag{5.17}$$

$$\leq \mathbb{P}(T_{i s_1 e_1} > n\varepsilon \cdot \tfrac{i}{i+j}) + \mathbb{P}(T_{i s_1 e_1, i s_1 e_1 + j s_2 e_2} > n\varepsilon \cdot \tfrac{j}{i+j}) \tag{5.18}$$

$$= \mathbb{P}(T_{i s_1 e_1} > n\varepsilon \cdot \tfrac{i}{i+j}) + \mathbb{P}(T_{j s_2 e_2} > n\varepsilon \cdot \tfrac{j}{i+j}) \tag{5.19}$$

$$\leq i\mathbb{P}(T_{s_1 e_1} > \tfrac{n\varepsilon}{i+j}) + j\mathbb{P}(T_{s_2 e_2} > \tfrac{n\varepsilon}{i+j}). \tag{5.20}$$

If $s_i = 0$ then of course $T_{s_i e_i} = 0$, while if $s_i = 1$ then $T_{s_i e_i} = 1$. Otherwise we have $s_i = -1$. Note that $\sum_{n \geq 1} \mathbb{P}(T_{-e_1} > n) \leq \mathbb{E}[T_{-e_1}] < \infty$ by Lemmas 2 and 3. Then because $X = T_{-e_1}(i+j)/\varepsilon \geq 0$ we have

$$\sum_{n \geq 1} \mathbb{P}\Big(T_{-e_1} > \frac{n\varepsilon}{i+j}\Big) = \sum_{n \geq 1} \mathbb{P}(X > n) \leq \mathbb{E}[X] = \frac{i+j}{\varepsilon} \cdot \mathbb{E}[T_{-e_1}] < \infty.$$

With E_n denoting the event $\{\frac{T_{nv, nv + i s_1 e_1 + j s_2 e_2}}{n} > \varepsilon\}$, it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty, \tag{5.21}$$

and then by the Borel-Cantelli lemma $\mathbb{P}(E_n \text{ occurs infinitely often}) = 0$. Since this is true for any $\varepsilon > 0$, we must have

$$\lim_{n\to\infty} \frac{T_{nv,nv+is_1e_1+js_2e_2}}{n} = 0, \quad \text{a.s.} \tag{5.22}$$

Switching back to real x, we have that almost surely

$$\lim_{x\to\infty} \frac{T_{\lfloor x\rfloor v,\lfloor x\rfloor v+is_1e_1+js_2e_2}}{x} = \lim_{x\to\infty} \frac{T_{\lfloor x\rfloor v,\lfloor x\rfloor v+is_1e_1+js_2e_2}}{\lfloor x\rfloor} \cdot \frac{\lfloor x\rfloor}{x} = 0. \tag{5.23}$$

Applying this to (5.16) shows that

$$\lim_{x\to\infty} \frac{T_{\lfloor x\rfloor v,[xv]}}{x} = 0, \quad \text{a.s.} \tag{5.24}$$

Similar arguments show the corresponding result for $T_{[xv],\lfloor x\rfloor v}$. Expectations are handled similarly. Take the expectation of both sides of (5.15) (resp. (5.16)) and note that the expectations of the numerator terms on the right hand side of (5.15) are equal to $\mathbb{E}[T_{is_1e_1+js_2e_2}] < \infty$. Since the sums are finite and $x \to \infty$ we see that $\mathbb{E}[\frac{T_{\lfloor x\rfloor v,[xv]}}{x}] \to 0$ as $x \to \infty$ and similarly $\mathbb{E}[\frac{T_{[xv],\lfloor x\rfloor v}}{x}] \to 0$. This completes the proof for $d = 2$.

The generalisation to $d > 2$ is straightforward: the box $B_x(v)$ is just the higher-dimensional analogue, with the point $\lfloor x\rfloor v$ the corner closest to the origin. Then again $[xv] \in B_x(v)$, and to get from $\lfloor x\rfloor v$ to $[xv]$ we again get an upper bound by considering each dimension separately.

To upgrade the result to $u \in \mathbb{Q}^d$ simply note that $v = m_u u \in \mathbb{Z}^d \setminus \{o\}$, and

$$\frac{T_{[xu]}}{x} = \frac{1}{m_u} \frac{T_{[x/m_u \cdot v]}}{x/m_u}.$$

As $x \to \infty$ also $x/m_u \to \infty$, and the limit of the above is therefore $\zeta_p(v)/m_u = \zeta_p(u)$ as required. ∎

Proof of Theorem 1.2(a)(i). This follows from Lemma 8 since

$$\zeta_p(qu) = \lim_{x\to\infty} \frac{T_{[xqu]}}{x} = q \lim_{x\to\infty} \frac{T_{[xqu]}}{xq} = q\zeta_p(u).$$

∎

Proof of Theorem 1.2(a)(ii). By Theorem 1.1 and subadditivity for T, for $x, y \in \mathbb{Z}^d$ we have

$$\begin{aligned}\zeta_p(x+y) &= \lim_{n\to\infty} \frac{\mathbb{E}[T_{n(x+y)}]}{n} \\ &\leq \lim_{n\to\infty} \frac{\mathbb{E}[T_{nx} + T_{nx,n(x+y)}]}{n} \\ &= \lim_{n\to\infty} \frac{\mathbb{E}[T_{nx}] + \mathbb{E}[T_{ny}]}{n} \\ &= \zeta_p(x) + \zeta_p(y).\end{aligned}$$

For $x, y \in \mathbb{Q}^d$ there exists an $m = m(x, y) \in \mathbb{N}$ such that $mx, my \in \mathbb{Z}^d \setminus \{o\}$ and then from (i)

$$\begin{aligned}\zeta_p(x+y) = \zeta_p(m^{-1}(mx+my)) &= m^{-1}\zeta_p(mx+my) \\ &\leq m^{-1}(\zeta_p(mx) + \zeta_p(my)) = \zeta_p(x) + \zeta_p(y).\end{aligned}$$

■

Proof of Theorem 1.2(a)(iii). Let $x, h \in \mathbb{Q}^d$. From (ii) we have

$$\zeta_p(x+h) \leq \zeta_p(x) + \zeta_p(h), \qquad \zeta_p(x) \leq \zeta_p(x+h) + \zeta_p(-h), \tag{5.25}$$

from which it follows that

$$|\zeta_p(x+h) - \zeta_p(x)| \leq \max\{\zeta_p(h), \zeta_p(-h)\}. \tag{5.26}$$

Now we always have $\zeta_p(e_i) = 1$ and $\zeta_p(-e_i) \geq 1$. Thus, using (i) and (ii),

$$\zeta_p(h) = \zeta_p\left(\sum_{i=1}^d h^{\{i\}} e_i\right) \leq \sum_{i=1}^d \zeta_p(h^{\{i\}} e_i) \leq \sum_{i=1}^d |h^{\{i\}}| \zeta_p(-e_i) = \|h\|_1 \zeta_p(-e_1).$$

The same upper bound applies to $\zeta_p(-h)$. Hence

$$|\zeta_p(x+h) - \zeta_p(x)| \leq \|h\|_1 \zeta_p(-e_1). \tag{5.27}$$

This verifies the claim. ■

Proof of Lemma 5. The inequality is a standard consequence of continuity of probability measures. To show that the infimum is positive we will make use of the "shape theorem" and "complete convergence theorem" for oriented site percolation in general dimensions. For this purpose it is convenient to express each point z in the positive orthant as $z = (z_{d-1}, \|z\|_1)$, where the last coordinate represents the time associated to the point and $z_{d-1} \in \mathbb{R}^{d-1}$. Here z_{d-1} should be interpreted as the projection of z onto the hyperplane $x_1 + \cdots + x_d = 0$, i.e. the hyperplane orthogonal to the vector $e_1 + \cdots + e_d$ which contains the origin.

Let v be in the interior of O_q. Let $U \subset \mathbb{R}^{d-1}$ denote the asymptotic shape (as in e.g. [6, Theorem 5] or [17, Theorem 1]) conditional on the cluster of the origin being infinite. Note that $O_q = \cup_{t\geq 0}(tU)$. Then $v = (v_{d-1}, k)$ where $k = \|v\|_1 > 0$ and there exists $\varepsilon > 0$ such that $v_{d-1} \in (1-\varepsilon)kU$. Fix this ε, and let $n \in \mathbb{N}$. Then $nv_{d-1} \in (1-\varepsilon)nkU$. Let $t_n = \inf\{j : o \to (nv_{d-1}, j)\}$. Now observe that

$$\mathbb{P}_q^\dagger(o \to nv) \geq \sum_{\ell \leq nk} \mathbb{P}_q^\dagger\big(t_n = \ell, (nv_{d-1}, \ell) \to (nv_{d-1}, nk)\big) \tag{5.28}$$

$$= \sum_{\ell \leq nk} \mathbb{P}_q^\dagger(t_n = \ell)\mathbb{P}_q^\dagger\big((nv_{d-1}, \ell) \to (nv_{d-1}, nk)\big) \tag{5.29}$$

$$= \sum_{\ell \leq nk} \mathbb{P}_q^\dagger(t_n = \ell)\mathbb{P}_q^\dagger\big(o \to (o, nk-\ell)\big), \tag{5.30}$$

where the sum is over ℓ such that $nk - \ell$ is divisible by d, and the first equality holds by independence. By the complete convergence theorem (e.g. [6, Theorem 4]) we have that $\lim_{n\to\infty} \mathbb{P}_q^\dagger\big(o \to (o, dn)\big) = c > 0$. Since each point in the positive orthant can be reached with positive probability this implies that $\delta := \inf_{n\geq 0} \mathbb{P}_q^\dagger\big(o \to (o, dn)\big) > 0$. Thus

$$\mathbb{P}_q^\dagger(o \to nv) \geq \delta \mathbb{P}_q^\dagger(t_n \leq nk). \tag{5.31}$$

By the shape theorem, a.s. on the event $\{o \to \infty\}$ we have that $t_n \leq nk$ for all n sufficiently large. So for every $\eta > 0$ there exists some n_0 such that for all $n \geq n_0$,

$$\mathbb{P}_q^\dagger(o \to nv) \geq \delta \mathbb{P}_q^\dagger(t_n \leq nk) \geq \delta(1-\eta)\mathbb{P}_q^\dagger(o \to \infty). \tag{5.32}$$

Since $\inf_{n\leq n_0} \mathbb{P}_q^\dagger(o \to nv) > 0$ this completes the proof. ■

Proof of Theorem 1.2(d)(i). Fix p as in the theorem. Without loss of generality we may assume that $i = 1$. Let $\varepsilon > 0$ be sufficiently small so that

$$\eta(\varepsilon, \delta) := \Big(\frac{(2d-1)(1+\varepsilon)e}{\varepsilon + \delta}\Big)^{\varepsilon+\delta}(1-p)^{1-\delta} < 1 \tag{5.33}$$

and $(1-2\delta-\varepsilon)>0$ for any δ with $|\delta|<\varepsilon$. This can be done since $1-p<1$ and $(b/x)^x \to 1$ as $x \to 0$ for any $b>0$.

Let $u \in \mathbb{D}$ with $\|u\|_1 = 1$ and $u^{\{1\}} = -(1-\delta)$, where $|\delta| < \varepsilon$. Suppose that we can reach $un = um_u n'$ in at most $(1+\varepsilon)n$ steps. Then there must be a self-avoiding path γ starting at the origin and ending at un, of length $\ell(\gamma) \in [n,(1+\varepsilon)n]$, that is consistent with the environment. Let D_n be the event that such a path exists. We will show that $\mathbb{P}(D_n)$ is summable in n and hence by Borel-Cantelli $\mathbb{P}(D_n \text{ i.o.}) = 0$. This shows that $\zeta_p(v) \geq (1+\varepsilon)$ and therefore completes the proof.

Let us verify the claim that $\mathbb{P}(D_n)$ is summable. Let $A_{n,k}$ denote the set of self-avoiding paths γ from o to nu, with $\ell(\gamma)=k$, and let $B_n = \cup_{k=n}^{\lfloor(1+\varepsilon)n\rfloor} A_{n,k}$. For a path $\gamma \in A_{n,k}$, let $w(\gamma)$ denote the number of steps in direction $-e_1$ taken by the path. The number of paths of length k with exactly w steps in direction $-e_1$ is at most $\binom{k}{k-w}(2d-1)^{k-w}$ since there are at most $2d-1$ choices for each of the $k-w$ other steps. For any path $\gamma \in A_{n,k}$, $w(\gamma) \in [n(1-\delta),k]$. Now observe that for any $k \in [n, n(1+\varepsilon)]$ and $w \in [n(1-\delta),k]$ we have $0 \leq k-w \leq n(\varepsilon+\delta)$.

It follows that

$$\begin{aligned}
|B_n| &\leq \sum_{k=n}^{\lfloor(1+\varepsilon)n\rfloor} \sum_{w=\lceil n(1-\delta)\rceil}^{k} \binom{k}{k-w}(2d-1)^{k-w} \\
&\leq \sum_{k=n}^{\lfloor(1+\varepsilon)n\rfloor} \sum_{w=\lceil n(1-\delta)\rceil}^{k} \binom{k}{\lceil n(\varepsilon+\delta)\rceil}(2d-1)^{n(\varepsilon+\delta)},
\end{aligned}$$

where we have used the fact that for w,k contributing to the sums, $k-w < k/2$ since $(1-2\delta-\varepsilon)>0$. This is at most

$$\sum_{k=n}^{\lfloor(1+\varepsilon)n\rfloor} \sum_{w=\lceil n(1-\delta)\rceil}^{k} \binom{\lfloor(1+\varepsilon)n\rfloor}{\lceil n(\varepsilon+\delta)\rceil}(2d-1)^{n(\varepsilon+\delta)}.$$

Now use the fact that for $k \in 1,\ldots,n$, $\binom{n}{k} \leq (ne/k)^k$ (e.g. proof by induction on k) to see that

$$|B_n| \le \sum_{k=n}^{\lfloor (1+\varepsilon)n \rfloor} \sum_{w=\lceil n(1-\delta) \rceil}^{k} \Big(\frac{(1+\varepsilon)e}{\varepsilon+\delta}\Big)^{1+n(\varepsilon+\delta)} (2d-1)^{n(\varepsilon+\delta)}$$
$$\le \frac{(1+\varepsilon)e}{\varepsilon+\delta} \Bigg(\Big(\frac{(2d-1)(1+\varepsilon)e}{\varepsilon+\delta} \Big)^{(\varepsilon+\delta)} \Bigg)^n \cdot (1+n\varepsilon) \cdot (1+n(\delta+\varepsilon)). \tag{5.34}$$

Finally, note that any $\gamma \in B_n$ is consistent with the environment with probability at most $(1-p)^{w(\gamma)} \le (1-p)^{n(1-\delta)}$. Therefore

$$\mathbb{P}(D_n) \le (1-p)^{n(1-\delta)} |B_n| \le (\eta(\varepsilon,\delta))^n \frac{(1+\varepsilon)e}{\varepsilon+\delta} (1+n\varepsilon)(1+n(\varepsilon+\delta))$$
$$\le 2n^2 (\eta(\varepsilon,\delta))^n$$

for n sufficiently large (depending on ε and δ). By (5.33), this is exponentially small in n, hence summable, and the claim is proved. ■

Proof of Lemma 7. Any d-dimensional hypercube H with centre u and side lengths $\ell > \beta \|u\|$ contains the d-hypersphere S with centre u and radius $\beta\|u\|/2$. Let S' be the d-hypersphere with centre u and radius $\beta\|u\|/4$. (Here we mean S and S' to be hyperspheres in the ℓ_2 norm.) Note that any line segment from the boundary of S to the boundary of S' has length at least half the radius of S. Suppose that H lies entirely outside $\tilde{U}_1$ (the ℓ_1 ball of radius 1), and let P denote the stereographical projection of S' onto $\tilde{U}_1$. Let u^* be the projection of u. Then P consists of exactly those points w of $\tilde{U}_1$ whose angle differs from u^* by at most $\arctan(\beta/4) > 0$.

Let $\alpha \in (0, 1/d)$, and choose $c = 4/\alpha$. Let $y \in \mathbb{Z}^d$ be such that H_y^- lies entirely outside of $\tilde{U}_1$. Let x be the centre of $H = H_y^-$. So $x = y - \vec{1}\|y\|\alpha/2$, and therefore

$$\|y\|(1-\alpha d/2) \le \|x\| \le \|y\|(1+\alpha d/2).$$

It follows that the side length of H_y^- is

$$\alpha\|y\| \ge \frac{\alpha\|x\|}{1+\alpha d/2} = \alpha'\|x\|,$$

where $\alpha' = \frac{\alpha}{1+\alpha d/2} \in (2\alpha/3, \alpha)$. By the above argument, the projection of H onto $\tilde{U}_1$ contains every point in $\tilde{U}_1$ within angle $\theta := \arctan(\alpha'/4) > 0$ of the projection x^* of the centre x of H.

Choose m large enough so that every point in $\tilde{U}_m$ is within angle $\theta/2$ of some point in U_m. Let $v \in U_m$. Let L_v be the (infinite) ray starting at the origin and passing through v. Then every segment of this line of length $> m$ contains a point in $\mathbb{Z}_+ U_m$ (the ℓ_1 distance between integer points on this line is at most m).

Let $\|y\| > cm$, and let $H = H_y^-$, with centre x. Let S and S' be the corresponding spheres in H as above. Choose $v \in U_m$ such that v is within angle $\theta/2$ of mx^*. The (infinite) ray from o through v hits S'. It therefore has a segment from the boundary of S to the boundary of S', which is necessarily of length at least half the radius of S, i.e. at least $\alpha\|y\|/4 > \alpha cm/4 = m$. Thus this line segment contains a point y_m in $\mathbb{Z}_+ U_m$.

This proves the result for H_y^-. The proof for H_y^+ is almost identical, but with $x = y + \vec{1}\|y\|\alpha/2$ in this case. ■

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