Resistor networks with distributed breakdown voltages

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As a primitive model for structural breakdown in elastic media, we analyze the failure of random resistor-fuse networks with various distributions of properties. We show that variations in breakdown voltage have a more significant effect than variations in resistance values. This is analogous to the fluid-displacement problem [D.Y.C. Chan, B. D. Hughes, L. Paterson, and C. Sirakoff, Phys. Rev. A **38**, 4106 (1988)], in which variations in fluid capacity have a greater effect on displacement efficiencies than variations in permeability. An exponential distribution of breakdown voltages creates much more disorder than any uniform distribution, but power-law distributions that emphasize weak bonds can create even greater disorder, up to the percolation limit, in which bonds are broken independently at random.

I. INTRODUCTION

Classical theories of failure and degradation of solids by mechanical stress emphasize regular geometries, the prime example being tensile failure tests in notched beams, where the experimental setup is carefully contrived to produce regular fracture patterns. In contrast, failure in engineering or geomechanical contexts is usually associated with erratic fracture patterns. These errative patterns are associated with the fluctuations in the material parameters of the failing continuum. Recently, computer simulation studies of properties of randomly structured materials have become fashionable, and there has been a proliferation of models. In the present paper we report simulations of a simple class of models for failure or fracture of materials which we feel capture many of the key aspects of the problem. If a network of Hookean springs with natural length zero is stretched onto a frame, then there is an exact mapping onto the random resistor network problem.[1] Our model is phrased in terms of the failure of a network of electrical resistors which act as fuses, burning out when an appropriately defined failure criterion is met. However, such networks may also serve as primitive models for fracture in elastic media. The advantage of studying resistor networks is that, because only scalar quantities are involved, the network models are less expensive to simulate. Furthermore, the use of such networks helps to unveil the essential physics without the need to introduce a large number of variables and parameters that a full tensor formalism would require.

Previous studies of resistor networks have concentrated on several types of models in which resistors are burnt out (converted to insulators) one at a time during the breakdown process. The models differ in the criterion for burning of a resistor.

Type 1. One begins with a regular network of *identical* resistors, say a two-dimensional (2D) square lattice confined between two parallel has bus bars.[2] Before the simulation commences, each resistor is inspected and declared to be burnt out already, (i.e., to have infinite resistance) with probability $1-p$. A potential difference is now applied across the bus bars. The resistors are now burnt out one at a time, the resistor selected being that with the greatest voltage across it. The process continues until sufficiently many fuses burn out to disconnect the system and preclude further current flow.

Type 2. The previous model may be generalized[3] by assigning a general probability density function $f(g)$ to the bond conductances g. Model 1 corresponds to the special choice

$$f(g)=(1-p)\delta_{+}(g)+p\delta(g-g_0) . \tag{1}$$

The resistors are now burnt out one at a time, with one of the following three failure criteria being selected: (i) the resistor that carries the greatest current is broken at each step; (ii) the resistor with the greatest voltage drop across it is broken at each step; (iii) the resistor with the greatest power dissipation is broken at each step. A converse of this model, in which the fuses turn into near short circuits when they blow, has also been proposed.[4]

Type 3. In the third model[5] the resistors in a regular network are all assigned the *same* resistance and the stochastic element is introduced by assigning *random breakdown voltages* to the components. At each stage in the simulation, the applied voltage is raised until one resistor suffers a voltage drop that exceeds its breakdown voltage and this resistor is immediately burnt out. As the response of the network is linear, this is equivalent to burning out the resistor which has the largest value for the ratio of the voltage drop to the breakdown voltage.

In all three of these models, there is usually probability zero that the need to break two fuses simultaneously will ever arise. We have chosen to concentrate on the type-3 model to complement previous studies on types 1 and 2, because we wish to examine the relative importance of fluctuations in resistance, which control the current flow patterns at each stage, and fluctuations in the local intrinsic strength of the system. In another context, we have previously shown that when two radically different forms of fluctuation are present, the effects of one may dominate over the other. Specifically, in the study of fluid displacement in porous media, fluctuations in the local fluid capacity (analogous to the breakdown voltage) dominate over fluctuations in the local permeability (resistance).[6,7]

An earlier study by Kahng *et al.*[5] of the breakdown of resistor networks, based on a type-3 model, suggests that the pattern of burnt-out resistors as well as the fraction of resistors burnt at the limit where the entire network becomes disconnected, depends on the form of the distribution function of breakdown voltages as well as the network size. They considered N resistors in a 2D square lattice network in which the breakdown voltage φ of each element is given by the normalized distribution function

$$f(\varphi)=1/w, \quad 1-w/2<\varphi<1+w/2, \quad 0<w\le 2 \ . \tag{2}$$

For small values of w, they observed that when the network breaks down, the pattern of burnt resistors forms a linear chain with the mean number of burnt-out resistors $\langle N_b \rangle$ being proportional to $N^{1/2}$. They called this the "brittle" regime. As w becomes larger, say >1.5, the mean number of burnt-out resistors $\langle N_B \rangle$ increases faster than $N^{1/2}$. The pattern of burnt resistors at breakdown also changes in a qualitative sense. In addition to the main critical cluster of burnt resistors that caused the network to become disconnected, there are also many clusters of burnt resistors which are not attached to the main critical cluster— this was termed the "ductile" regime. Kahng *et al.* conjectured that in the limit $w=2$, the mean number of burnt-out resistors $\langle N_b \rangle$ will increase with the lattice size like N.

In this paper, we consider in more detail the breakdown characteristics of a resistor network as a function of the distribution of breakdown voltages of the elements.

II. PAIRS OF RESISTORS

In a simple random bond percolation model, bonds in a network are selected at random and then broken. The network breaks down when one particular cluster of broken bonds separates the network into two disconnected portions. In a type-3 resistor network prescribed above, the criterion for breaking a particular bond or resistor is controlled by the ratio of the voltage drop across the resistor to the breakdown voltage of the resistor. The magnitude of the breakdown voltage φ depends on the choice of the normalized distribution function $f(\varphi)$, while the voltage drop may be found using the Kirchoff equations for the current configuration of unburnt resistors. To gain insight into the role of the breakdown voltage distribution function, let us first analyze the simple case of two resistors with breakdown voltages φ_1 and φ_2 chosen according to some normalized distribution $f(\varphi)$. Let the voltage drop across the two resistors be V_1 and V_2, respectively. In a type-3 model, we burn out the resistor with the largest value of the ratio (V_i/φ_i). We assume, without loss of generality, that resistor 1 has the larger voltage drop across it, and define the ratio of the voltage drops across the two resistors as $\rho=V_2/V_1<1$. Therefore the probability that the resistor with the *larger* voltage across it will be burnt is given by

$$P_L=\int_0^\infty d\varphi_1 f(\varphi_1)\int_{\rho\varphi_1}^\infty d\varphi_2 f(\varphi_2) \ . \tag{3}$$

Explicit forms for this probability can be obtained for a number of breakdown voltage distribution functions $f(\varphi)$.

Uniform:

$$f(\varphi)=1, \quad 0<\varphi<1; \quad P_L=1-\frac{\rho}{2} \ . \tag{4}$$

Exponential:

$$f(\varphi)=\alpha\exp(-\alpha\varphi), \quad 0<\varphi<\infty; \quad P_L=\frac{1}{1+\rho} \ . \tag{5}$$

Power:

$$f(\varphi)=(1-q)\varphi^{-q}, \quad 0<\varphi<1, \quad q<1; \quad P_L=1-\frac{\rho^{1-q}}{2} \ . \tag{6}$$

In Fig. 1, we show the probability P_L for the uniform, exponential, and power distribution for various values of the index

$$q=(n-1)/n \ . \tag{7}$$

The probability P_L calculated according to the distribution (2) is

$$P_L=\begin{cases}1, & 0<\rho<\rho_1\\ \dfrac{3}{4}+\dfrac{1}{8w^2}[8-\rho(w+2)^2-(w-2)^2/\rho], & \rho_1<\rho<1\end{cases} \tag{8}$$

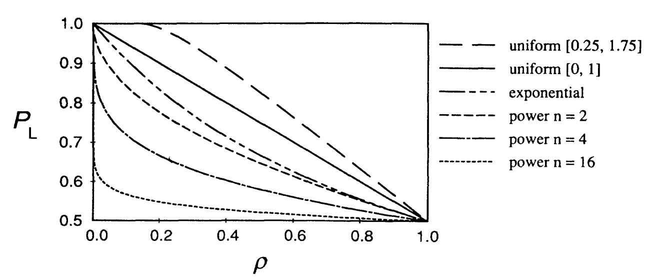

FIG. 1. The probability P_L that, in a pair of resistors, the resistor with larger voltage drop will be burnt, as a function of the ratio ρ of the voltage drops across the two resistors, $0<\rho<1$, for the uniform [0.25,1.75], the uniform [0,1], exponential, and three-power-law distributions with $n=2$, 4, and 16.

where $\rho_1=(1-w/2)/(1+w/2)$. This probability is also shown in Fig. 1 for $w=\frac{3}{2}$ as used by Kahng *et al.*[5]

Since $f(\varphi)$ is assumed to be normalized over some interval, the probability $P_L \rightarrow 0.5$ in the limit $\rho \rightarrow 1$ for all models. A value of $P_L=0.5$ is equivalent to the simple random percolation model in which one of the two resistors is chosen at random to be burnt irrespective of their breakdown voltages. Thus at a given ρ value, the model that has the lowest P_L will yield the most "random" burnt resistor clusters. From Fig. 1 we see that the power distribution (6) is expected to yield the most random distribution of burnt resistor clusters when the network breaks down. We shall present simulation results to support this conclusion. The power distribution has other interesting properties. To quote Stephens and Sahimi:[8] "Halperin, Feng and Sen have shown that $f(\varphi)=(1-q)\varphi^{-q}$, $0<q<1$, describes the distribution of the conductance of the channel random-void ('Swiss cheese') model of continuous media. In this model, spherical holes are randomly placed in a medium having otherwise uniform transport properties. It was proposed many years ago [for example, by Kogut and Straley[9]] that such distributions would give rise to nonuniversal behavior of the conductivity of percolating systems near [the percolation threshold] p_c."

We have included the exponential distribution (5) because of all distributions supported on the interval $0<\varphi<\infty$, with a given mean $\langle\varphi\rangle=1/\alpha$, the exponential distribution is the most random in the sense that it maximizes the Shannon entropy[10] $-f(\varphi)\ln f(\varphi)$. For this distribution, the probability P_L for the exponential distribution is independent of the parameter α. However, we expect from Fig. 1 that the power distribution, with $q>0$, will give a more random distribution of the burnt-out clusters.

III. NUMERICAL SIMULATIONS

We have performed a number of simulations of type-2 and -3 resistor networks. Three examples of networks at the point of failure are shown in Figs. 2–4. In Fig. 2, we show a type-3 network with an exponential distribution of breakdown voltages. In Fig. 3, we show another type-3 network, with breakdown voltages governed by Eqs. (6) and (7), with $n=128$. In this case, the effects of weak bonds were strongly emphasized and a larger number of resistors were burnt out when breakdown occurred. In Fig. 4 we show a type-2(i) network with an exponential distribution of resistance values, displaying a more localized breakdown path than for the distributed breakdown voltages.

In Fig. 5 we show a summary of many of our simulation results of type-3 resistor network for different distributions of the breakdown voltage. The set of linear equations for the voltage drops across each resistor was solved by a successive over-relaxation method. Each data point represents an average over 20 realizations of the same distribution function. The average number of burnt resistors is fitted to the equation

$$\langle N_b \rangle = \mu N^{\nu} \tag{9}$$

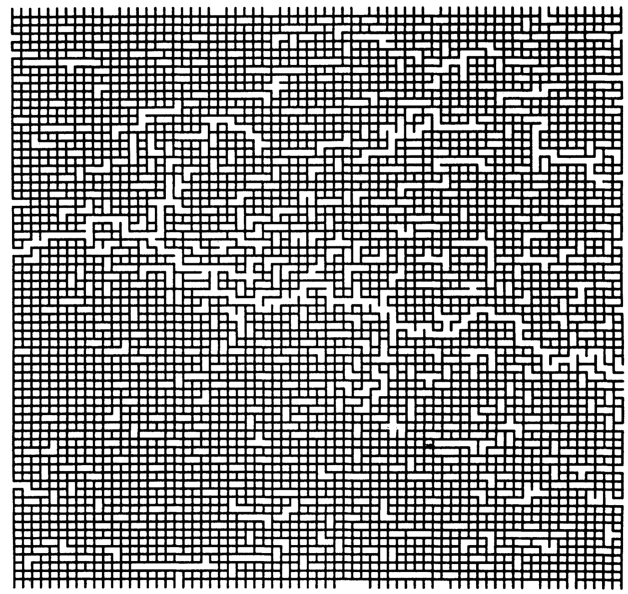

FIG. 2. An example of a simulation on a type-3 network with an exponential distribution of breakdown voltages.

and the results for the parameters μ and ν are summarized in Table I. We also show in Fig. 5 the results for a type-2(i) model, distributing the resistances and at each step breaking the resistor with maximum current. This demonstrates that varying the breakdown voltages has a much greater effect than varying the resistances.

The following question naturally arises: Does the set of burnt-out resistors at failure have nonzero density in

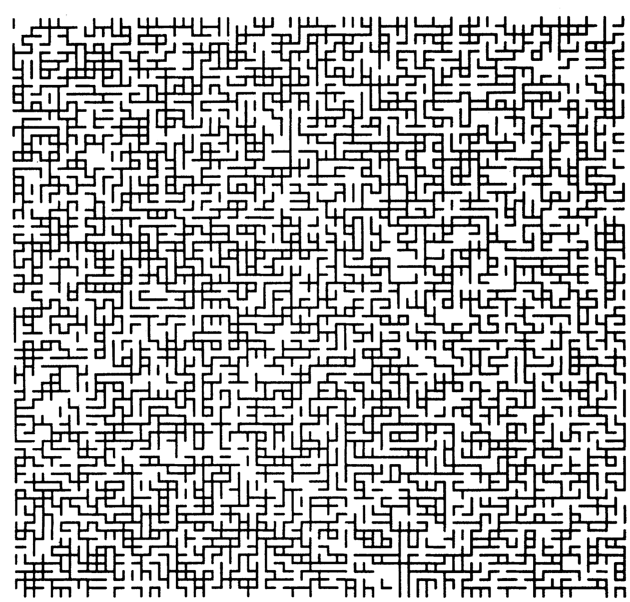

FIG. 3. An example of a simulation on a type-3 network, with breakdown voltages governed by Eqs. (6) and (7), with $n=128$.

FIG. 4. An example of a simulation on a type-2(i) network with an exponential distribution of resistance values.

the limit of an infinite lattice? If the answer is yes, then the mean number $\langle N_b \rangle$ of burnt-out resistors at network breakdown increases linearly with the total number of resistors N in the network. Kahng *et al.*[5] claimed that this seems to happen for the distribution (2) in the limit $w=2$, which corresponds to the power distribution at $q=0$ ($n=1$). From the results in Table I, the dependence is $\langle N_b \rangle \propto N^{0.88}$ for $w=2$, i.e., $\nu=0.88$, which is less than 1. However, for the power-law distributions, the exponent ν is an increasing function of n. For $n \geq 4$ (that is, $q \geq \frac{3}{4}$), $0.97 \leq \nu \leq 1$ and in the case $n=128$, the exponent ν is equal to 1 within the accuracy of our simulation.

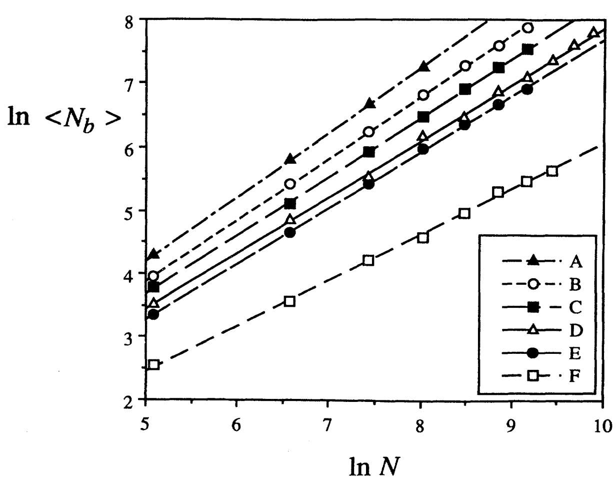

FIG. 5. A summary of simulation results for type-3 resistor networks with various distributions of the breakdown voltages: A power $n=128$; B, power $n=4$; C, power $n=2$; D, exponential distribution; E, power $n=1$ (equal to uniform distribution). Also shown are simulation results for type-2(i) resistor networks with (F) an exponential distribution of resistance values.

TABLE I. Fitted parameters for $\langle N_b \rangle$ defined in Eq. (8) for various distribution functions of breakdown voltages or resistances. The exponent q of the power distribution (6) is given by $q=(n-1)/n$.

Type	Distribution	μ	ν
Breakdown voltage	Exponential	0.37	0.88
Breakdown voltage	Power $n=1$	0.32	0.88
Breakdown voltage	Power $n=2$	0.38	0.93
Breakdown voltage	Power $n=4$	0.38	0.97
Breakdown voltage	Power $n=128$	0.41	1.0
Resistance	Exponential	0.31	0.72

In a classical bond percolation model on the square lattice at the percolation threshold, Eq. (9) is replaced by

$$\langle N_b \rangle \sim \tfrac{1}{2} N \quad \text{as } N \rightarrow \infty \; . \tag{10}$$

In the case $n=128$, where the exponent ν is very close to 1, the coefficient μ (which would be the fraction of burnt resistors if ν were exactly 1) is only 0.41, which is still some way short of the random percolation limit, where $\mu=0.5$ for an infinite system.

In Fig. 6, we show for an 11×9 network consisting of 162 resistors the dependence of the number of burnt resistors $\langle N_b \rangle$ for a power distribution of breakdown voltages on the index $q=(n-1)/n$. We observe that as $q \rightarrow 1-$, that is, when $n \rightarrow \infty$, $\langle N_b \rangle$ satisfies the empirical relation

$$\langle N_b \rangle = 77 - 48 n^{-0.48} \tag{11}$$

so we can see how the coefficient μ in Eq. (9) approaches 0.5 as n becomes large.

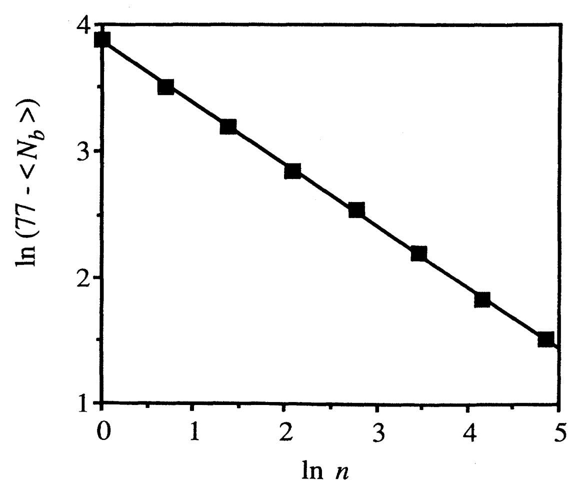

FIG. 6. The dependence of the number of burnt resistors $\langle N_b \rangle$ for a power distribution of breakdown voltages on n for an 11×9 network consisting of 162 resistors.

IV. COMPARISON WITH FLUID DISPLACEMENT IN RANDOM MEDIA

The resistor simulations have much in common with stochastic simulations of fluid displacement in porous media at infinite viscosity ratio. Both involve solving Laplace's equation for the potential function and then changing the state of a component of the system as some function of the gradient of the potential. In flow in porous media, the component which changes state is at the node of the lattice, corresponding to a pore, which may be filled with either the displaced or driving fluid. In the resistor network problem, the changing component is the bond of the lattice corresponding to a resistor which may be either intact or burnt out. In the fluid-displacement problem, an exponential distribution of fluid capacities, and in the network problem, an exponential distribution of breakdown voltages, has a very special property: the component changes state with a probability equal to the magnitude of the potential gradient at that component. The major qualitative difference between the two problems is that, for simulations of flow in porous media, components can only change state if they are adjacent to a component that has already changed state. As a result of this restriction, random walkers can be used to simulate the special case of an exponential distribution of fluid capacity. This permits the use of the simple diffusion-limit aggregation model to simulate fluid displacement in random media. In the resistor network problem, components can change state anywhere in the network so a new random walker model will be required.

As we have shown in Fig. 5, varying the breakdown voltage has a much larger effect on the fraction of burnt-out resistors than varying the resistances in each component. In a general resistor network, both the individual resistances and the breakdown voltages may vary according to their own distributions (which may be related), however, variations in the breakdown will have the dominant effect in dictating the course of the network breakdown. This is directly analogous to the porous medium problem, where varying local fluid capacity has a greater effect on unstable two-fluid displacement processes than varying local permeabilities.[6,7]

All the models (types 1–3) we have outlined here incorporate, to varying degrees, the heterogeneous microstructure in an otherwise statistically homogeneous medium. The stochastic property is necessary to explain the irregular and serrated surfaces which are created in the fracture of natural materials. Knowledge of the geometry of these fracture surfaces is relevant to such processes as sliding (earthquakes) and fragmentation (blasting).

V. CONCLUSIONS

Our simulation results suggest that the breakdown of a resistor network (or equivalently a network of Hookean springs with natural length zero) is more sensitive to the distribution of breakdown voltages (tensile strengths) than to variations in the resistance (the elastic moduli). This observation is mirrored in the study of fluid displacement in random media in which the distribution of fluid capacities is mainly responsible in controlling the displacement efficiency. The exponential distribution, which we found to give the least efficient displacement and to have a one-to-one correspondence to simulation of fluid displacement by the diffusion-limited aggregation algorithm, does not seem to hold the same significance in the resistor breakdown problem. The power distribution of breakdown voltages (6) gives a more random breakdown pattern than the exponential distribution and in the limit $q \to 1-$, appears to produce the expected classical version of percolation theory.

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