Classifying representations for conformal field theory

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- The classification

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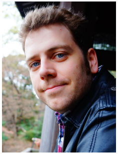


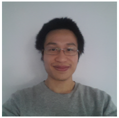

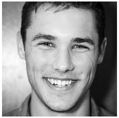

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CFTs are fundamental to statistical mechanics and string theory and (increasingly) to pure mathematics.

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$$\begin{array}{ccccc} \text{states} & \text{state-field correspondence} & \text{fields} & \phi(z)=\sum_n \phi_n z^{-n-h} & \text{modes} \\ |\phi\rangle \in \mathcal{V} & \longleftrightarrow & \phi(z) & \longleftrightarrow & \phi_n \in \mathrm{End}(\mathcal{V}) \end{array}$$

$$\begin{array}{ccccc} \text{mode action} & & \text{operator product} & & \text{commutator} \\ \phi_n|\psi\rangle & \longleftrightarrow & \phi(z)\psi(w) & \longleftrightarrow & [\phi_m, \psi_n] \\ (\textit{eg. } \mathfrak{vir}\text{-module}) & & \text{expansion} & & (\textit{eg. } \mathfrak{vir}) \end{array}$$

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Fields may be differentiated and admit a normally ordered product:

$$:\phi(w)\psi(w): \; = \oint_w \frac{\phi(z)\psi(w)}{z-w} \frac{\mathrm{d}z}{2\pi \mathfrak{i}}.$$

Examples

- The Heisenberg algebra $\widehat{\mathfrak{h}}$:

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- The weight 0 Verma module of the Virasoro algebra $\mathfrak{vir}_c$ has a weight 1 Verma submodule. The quotient is a VOA with OPE

$$L(z)L(w) \sim \frac{\frac{c}{2}\,\mathbf{1}}{(z-w)^4} + \frac{2\,L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}.$$

- The affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)_k$:

$$\left[J^a_m, J^b_n\right] = \left[J^a, J^b\right]_{m+n} + m\kappa\left(J^a, J^b\right)\delta_{m+n=0}k\,\mathbf{1}.$$

Its weight 0 Verma module has a weight -2 Verma submodule. For $k \neq -2$, the quotient is a VOA with OPE

$$J^a(z)J^b(w) \sim \frac{\kappa\left(J^a, J^b\right)k\,\mathbf{1}}{(z-w)^2} + \frac{\left[J^a, J^b\right](w)}{z-w}.$$

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- The $\mathfrak{vir}_c$ and $\widehat{\mathfrak{sl}}(2)_k$ VOAs are not simple iff

$$c = c(p, p') = 1 - \frac{6(p'-p)^2}{pp'}, \quad p, p' \in \mathbb{Z}_{\geq 2},\ \gcd\left\{p, p'\right\} = 1,$$
$$k = k(u, v) = -2 + \frac{u}{v}, \quad u \in \mathbb{Z}_{\geq 2},\ v \in \mathbb{Z}_{\geq 1},\ \gcd\left\{u, v\right\} = 1,$$

respectively. The simple quotients are the minimal models.

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How to do this calculation for general minimal models?

Symmetric polynomials

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Partitions $\lambda = [\lambda_1, \ldots, \lambda_m]$ give more:

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And **monomial**, **Schur**, **Jack**, ... also indexed by partitions.

Schur polynomials

The n-variable Schur polynomials are defined by

$$\mathsf{s}_\lambda(z) = \frac{\det\left(z_i^{\lambda_j+n-j}\right)}{\det\left(z_i^{n-j}\right)}.$$

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Schur polynomials have a remarkable orthonormality property:

$$\langle f, g\rangle_n = \int_{[\Delta_n]} \prod_{1\le i\ne j\le n}\left(1-\frac{z_i}{z_j}\right) f(z_1^{-1},\dots,z_n^{-1})g(z_1,\dots,z_n)\,\frac{\mathrm{d}z_1\cdots\mathrm{d}z_n}{z_1\cdots z_n}$$

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$\int_{[\Delta_n]}$ is normalised so that $\langle \mathsf{s}_0, \mathsf{s}_0\rangle_n = \langle 1, 1\rangle_n = 1$.

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Special case (but crucial later):

$$\mathsf{J}^{\kappa}_{[m^n]}(z_1, \dots, z_n) = \prod_{i=1}^{n} z_i^{m} \quad \text{(for all } \kappa\text{)}.$$

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- For the Heisenberg VOA $\widehat{\mathfrak{h}}$, Zhu's algebra is $\mathbb{C}[a_0]$.
- For the Virasoro VOA $\mathfrak{vir}_c$, Zhu's algebra is $\mathbb{C}[L_0]$.
- For the VOA $\widehat{\mathfrak{sl}}(2)_k$, Zhu's algebra is $\mathcal{U}\mathfrak{sl}(2)$.

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Irreducible modules are under control for these VOAs.

But, what about their minimal models? That's much harder!

Ancient history

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- Few other examples! Projection technology doesn't generalise?

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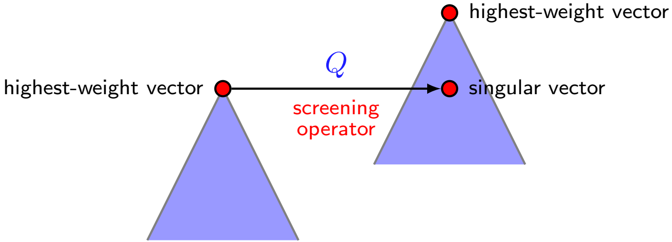
Recent resurgence of interest due to AGT conjecture [Alday-Gaiotto-Tachikawa '09] and "generalised" Jack polynomials [Morozov-Smirnov '13].

Why Jack polynomials?

Free field constructions of singular vectors use screening operators.



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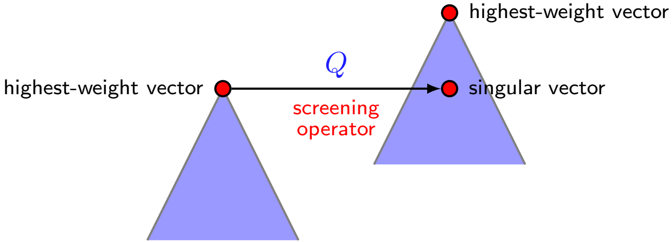

Screening operators intertwine the action of the Virasoro VOA on the free field modules.

Q is the zero mode of a product of $\widehat{\mathfrak{h}}$ vertex operators:

$$V_\alpha(z) = \mathsf{e}^{\alpha q} z^{\alpha a_0} \prod_{m=1}^{\infty} \mathsf{e}^{\alpha a_{-m} z^m / m} \mathsf{e}^{-\alpha a_m z^{-m} / m}.$$

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To get the vacuum singular vector $\lvert\chi\rangle$ of $\mathfrak{vir}_{c(p,p')}$, we take

$$Q = \int_{[\Delta_{p-1}]} V_\alpha(z_1) \cdots V_\alpha(z_{p-1}) \, \mathrm{d}z_1 \cdots \mathrm{d}z_{p-1}, \quad \alpha = \sqrt{\frac{2p'}{p}},$$

$$\lvert\chi\rangle = Q\lvert -(p-1)\alpha\rangle.$$

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Other singular vectors obtained similarly!

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The product may be decomposed into Jacks using specialisation:

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Homogeneity and orthogonality finish the job!

Theorem [Wang, DR-Wood]: The Virasoro minimal model VOA with $c = c(p, p')$ is rational and the irreducibles are precisely those of highest weight

$$h_{r,s} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}, \quad 1 \leq r \leq p-1, \ 1 \leq s \leq p'-1.$$

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Superconformal challenge is “generalised” Jacks.

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The idea is to exploit rigorous free field methods as “brute force” calculations become infeasible at higher rank.

Thank you!