Representations of affine vertex algebras

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August 19, 2019

[arXiv:0810.3532, 1001.3960, 1012.2905, 1205.6513, 1303.0847, 1306.4388, 1501.07318, 1705.04006, 1803.01989, 1806.09146, 1906.02935]

Outline

1. Motivation

2. Affine things, module categories

3. Relaxed things

4. Classifying relaxed modules
   - Simple relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules
   - Some non-simple indecomposable relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules

5. Outlook

Motivation

"... it is a highly nontrivial problem to construct essentially any example of a vertex operator algebra."

"A significant feature of the theory is that the construction of modules for a vertex operator algebra is more subtle than the construction of the algebra itself."

— Jim and Haisheng.

Affine VOAs arguably form the most important class of all.

The rational ones (WZW models) are widely regarded as fundamental (and beautiful) building blocks on which much of our understanding rests.

What about non-rational affine VOAs? They should also be crucial in understanding non-rational (logarithmic) cases. And they should be beautiful. It's a pity we don't understand them at all...

Affine things

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be its affinisation.

Let $\mathbb{C}_k$ denote the 1-dimensional module of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts as 0 and K acts as $\mathsf{k} \cdot \mathbf{1}$ for some $\mathsf{k} \in \mathbb{C}$.

Then, $\mathsf{V}_\mathsf{k}(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_\mathsf{k}$ carries the structure of a vertex algebra of level k, in fact a vertex operator algebra if $\mathsf{k} \neq -\mathsf{h}^\vee$.

Any quotient of a $\mathsf{V}_\mathsf{k}(\mathfrak{g})$ is called an affine vertex algebra.

The level-k simple quotient will be denoted by $\mathsf{L}_\mathsf{k}(\mathfrak{g})$.

Module categories

Conformal field theory seems to need characters because the partition function must be invariant under the standard action of $\mathsf{SL}_2(\mathbb{Z})$.

Consider, for some affine vertex algebra, the category $\mathscr{W}_{\mathsf{k}}$ of smooth weight modules (with finite-dimensional weight spaces).[1]

This is hard to analyse, but we can try to find consistent full subcategories instead.

For $\mathsf{k} \in \mathbb{N}$, the category $\mathscr{KL}_{\mathsf{k}}$ of ordinary $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules is consistent:

- It is closed under twisting by the automorphisms $\sigma = \mathrm{Aut}(\mathfrak{g}) \ltimes \mathsf{P}^\vee$ of $\widehat{\mathfrak{g}}$ that preserve $\mathfrak{h}$.
- It is closed under the VOA tensor product (fusion).
- Its characters form a vector-valued modular form, so the usual sesquilinear combination (partition function) is modular-invariant.

[1] [To save space, let “weight” mean “weight with finite-dimensional weight spaces”.]

Take $\mathsf{k} \notin \mathbb{N}$. Then, $\mathscr{KL}_{\mathsf{k}}$ is not closed under σ-twists and modularity also fails. It is, however, closed under fusion.

Try the BGG category $\mathscr{O}_{\mathsf{k}}$ whose simples are highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules. Not closed under σ-twists or fusion; modularity fails.

Try $\mathscr{O}_{\mathsf{k}}^{\sigma}$ whose simples are σ-twists of highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules. Not closed under fusion; modularity still fails.

Better try $\mathscr{R}_{\mathsf{k}}^{\sigma}$ whose simples are σ-twists of relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules.

Conjecture: $\mathscr{R}_{\mathsf{k}}^{\sigma}$ is closed under fusion and the usual modular transformations preserve the vector space spanned by the characters.

Question: Is $\mathscr{R}_{\mathsf{k}}^{\sigma} = \mathscr{W}_{\mathsf{k}}$?

[I think the answer is yes, by a result of Futorny and Tsylke, but I need to check the details.]

How to relax

Relaxation is good for the soul:

- The relaxed triangular decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = t\mathfrak{g}[t] \oplus (\mathfrak{g} \oplus \mathbb{C}K) \oplus t^{-1}\mathfrak{g}[t^{-1}].$$

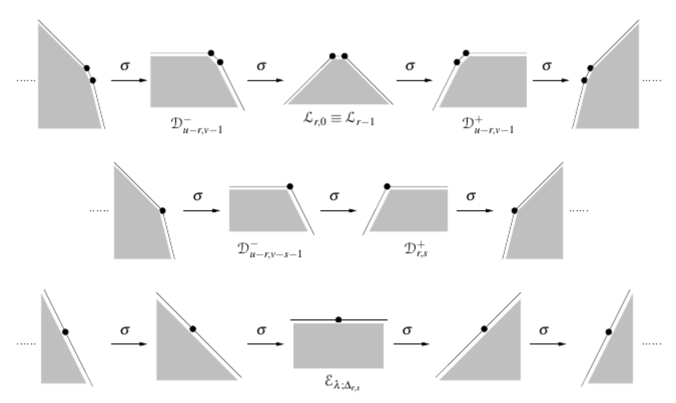
- A relaxed highest-weight vector is
  - an eigenvector of $\mathfrak{h}$;
  - a generalised eigenvector of L_0 (via Sugawara);
  - annihilated by $t\mathfrak{g}[t]$.
- A relaxed highest-weight module[2] is a module that is generated by a single relaxed highest-weight vector.

Relaxed modules differ from parabolic highest-weight modules in that their top spaces (Zhu images) need not be simple nor finite-dimensional.

[2] [We say "relaxed module" for short.]

Example: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ with $\mathsf{k}+2=\frac{u}{v}$, $u,v\in\mathbb{Z}_{\geqslant 2}$ and $(u,v)=1$.

- $u-1$ ordinary modules;
- $(u-1)v$ highest-weight modules;
- $\frac{1}{2}(u-1)(v-1)$ 1-parameter families of (almost-always-simple) relaxed modules;
- σ-twists of the above, parametrised by $\mathsf{P}^\vee\cong\mathbb{Z}$.



Adding rigour — where to start?

We understand the relaxed modules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ very well — they have been classified, their characters are known (rigorously), their modular properties have been calculated. Fusion rules have also been conjectured.

We aim to build and prove a (hopefully) widely applicable formalism. However, we first have to figure out what to prove: need more examples.

In particular, we need a much better understanding of the possible types of relaxed modules. Unfortunately, there is not much literature:

- The simple relaxed modules have only been classified for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [AM,RW], $\mathsf{L}_{\mathsf{k}}(\mathfrak{osp}(1|2))$ [W,CKLR] and $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$ [AFR].
- The characters of these simples have only been computed for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ and $\mathsf{L}_{\mathsf{k}}(\mathfrak{osp}(1|2))$ [A,KR].
- Only the fusion rules of the ordinary modules have been proven [CHY,C].
- Projective covers in $\mathscr{R}_{\mathsf{k}}^{\sigma}$ have also been conjectured for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [CKLR].
- Proving that $\mathscr{R}_{\mathsf{k}}^{\sigma}$ is a rigid braided tensor category is still a dream.

Classifications

We aim to classify the simple relaxed $\mathsf{L_k}(\mathfrak{g})$-modules. (But the method works for any affine vertex algebra!)

As usual, the tool to do so is Zhu's algebra, *eg.*

$$\mathsf{Zhu}[\mathsf{V_k}(\mathfrak{g})] \cong \mathscr{U}(\mathfrak{g}), \quad \mathsf{Zhu}[\mathsf{L_k}(\mathfrak{g})] \cong \frac{\mathscr{U}(\mathfrak{g})}{I_{\mathsf{k}}}.$$

For $\mathsf{V_k}(\mathfrak{g})$, the simple Zhu-modules are then the simple weight $\mathfrak{g}$-modules.

For $\mathsf{L_k}(\mathfrak{g})$, our job is then to determine which of these simple weight $\mathfrak{g}$-modules are annihilated by I_{k}.

But this presupposes that we have a classification of simple weight $\mathfrak{g}$-modules. Luckily, Mathieu completed this classification relatively recently, building on work of Fernando (and others).

Recall that a weight $\mathfrak{g}$-module is dense if its support is $\lambda + \mathrm{Q}$, for some $\lambda \in \mathfrak{h}^*$. For simple weight modules, dense = cuspidal = torsion-free.

Recall also that every parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ contains a Borel, hence a set of simple root vectors e^{α_i}.

The Levi factor of $\mathfrak{p}$ is the subalgebra $\mathfrak{l}$ generated by $\mathfrak{h}$ and the $e^{\pm\alpha_i}$ for which $e^{-\alpha_i} \in \mathfrak{p}$. Levi factors are always reductive.

Recall that parabolic induction means extending an $\mathfrak{l}$-module to a $\mathfrak{p}$-module, by letting all root vectors in $\mathfrak{p} \setminus \mathfrak{l}$ act as 0, then inducing to a $\mathfrak{g}$-module.

Theorem (Fernando '90).

- Every simple weight $\mathfrak{g}$-module is the simple quotient of a parabolic induction of a simple dense $\mathfrak{l}$-module.
- Simple dense $\mathfrak{l}$-modules exist only if all simple ideals of $\mathfrak{l}$ have AC-type.

We say that a $\mathfrak{g}$-module is bounded if it is infinite-dimensional and the dimensions of its weight spaces are uniformly bounded.

Theorem (Mathieu '00).

- Simple dense $\mathfrak{g}$-modules exist iff $\mathfrak{g}$ has AC-type.
- Every simple dense $\mathfrak{g}$-module is the direct summand of a unique irreducible semisimple coherent family of $\mathfrak{g}$-modules.
- A semisimple coherent family of $\mathfrak{g}$-modules always contains a bounded highest-weight submodule.
- Any such bounded highest-weight submodule completely characterises the coherent family (up to isomorphism).

Mathieu also explicitly determined the conditions for a highest-weight $\mathfrak{g}$-module to be bounded.

This completes the classification of simple weight $\mathfrak{g}$-modules. The classification of simple relaxed $\mathsf{V}_k(\mathfrak{g})$-modules now follows.

However, we want to classify simple relaxed modules for $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ (or some other quotient). Define:

- a parabolic family of $\mathfrak{g}$-modules to be the "almost-simple" quotient of the parabolic induction of some coherent family of $\mathfrak{l}$-modules;
- an $\mathfrak{l}$-bounded $\mathfrak{g}$-module to be the "almost-simple" quotient of the parabolic induction of some bounded $\mathfrak{l}$-module.

Theorem (Kawasetsu-DR '19).

The classification of simple relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules may be obtained algorithmically from the classification of simple *highest-weight* $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

More precisely, a simple relaxed $\widehat{\mathfrak{g}}$-module, that is *not* highest-weight wrt any Borel subalgebra, is an $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module iff its Zhu-image is a submodule of an irreducible **semisimple** parabolic family of $\mathfrak{g}$-modules that has a simple $\mathfrak{l}$-bounded highest-weight submodule whose Zhu-induction is an $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module.

Algorithm (Kawasetsu-DR '19).

Assume that we know the simple highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

- For each (non-empty) subset of the simple roots, check if the corresponding parabolic subalgebra $\mathfrak{p}$ has AC-type.
- If so, project the highest weight λ of the Zhu-image of each simple highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module H along each simple ideal $\mathfrak{s}_i$ of $\mathfrak{l}$. Check if *all* projections correspond to bounded $\mathfrak{s}_i$-modules.
- If so, one has an irreducible semisimple coherent family of $\mathfrak{s}_i$-modules containing the bounded $\mathfrak{s}_i$-module, for all i. Tensoring them together, along with an appropriate $\mathfrak{z}(\mathfrak{l})$-module, gives an irreducible semisimple standard parabolic family whose Zhu-induction contains H.
- The direct summands of this induction are all $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

Along with the simple highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules, the direct summands found with this algorithm form a complete set, up to isomorphism, of simple relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules.

Example: Relaxed $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

$\mathsf{L}_{-2}(\mathfrak{so}_8)$ has one simple ordinary module L_0 and four non-ordinary highest-weight modules L_i, $i = 1, 2, 3, 4$. L_0 is invariant under W-twists while L_2 gives 24 twists and the others give 8 each.

The algorithm now gives

simple root subset	$\mathfrak{l}$	# parabolic families
$\{1\}, \{3\}, \{4\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	24 each
$\{2\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	72
$\{1,2\}, \{2,3\}, \{2,4\}$	$\mathfrak{sl}_3 \oplus \mathfrak{gl}_1^{\oplus 2}$	32 each
$\{1,3\}, \{1,4\}, \{3,4\}$	$\mathfrak{sl}_2^{\oplus 2} \oplus \mathfrak{gl}_1^{\oplus 2}$	no families
$\{1,3,4\}$	$\mathfrak{sl}_2^{\oplus 3} \oplus \mathfrak{gl}_1$	no families
$\{1,2,3\}, \{1,2,4\}, \{2,3,4\}$	$\mathfrak{sl}_4 \oplus \mathfrak{gl}_1$	8 each

and each of these families gives either a 1-, 2- or 3-parameter family of (almost-always-simple) relaxed $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

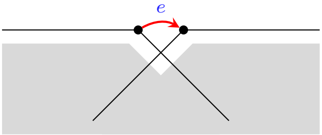
Motivated by ideas for constructing projective covers, we also want to study non-simple indecomposable relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules. Define:

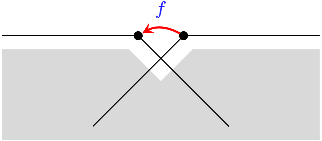
- A weight $\mathfrak{l}$-module to be α-bijective, for some given root α, if the root vector $e^\alpha \in \mathfrak{l}$ acts bijectively.

Theorem (Kawasetsu-DR '19).

Let P be a parabolic family induced from an irreducible α-**bijective** coherent family of $\mathfrak{l}$-modules and suppose that P has an $\mathfrak{l}$-bounded highest-weight submodule whose Zhu-induction is an $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module. Then, the Zhu induction of every subquotient of P is also an $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module.

eg., the indecomposable relaxed $\widehat{\mathfrak{sl}}_2$-modules used in [CR] to prove the modularity of $\mathscr{R}_\mathsf{k}^\sigma$ for k admissible are therefore $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules.



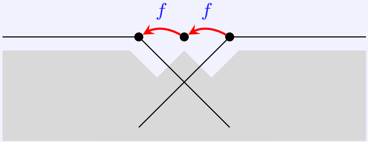


Example: Non-semisimple highest-weight $\mathsf{L}_{-2}(\mathfrak{so}_8)$-modules.

Recall that $\mathsf{L}_{-2}(\mathfrak{so}_8)$ has one simple ordinary module L_0 and four non-ordinary highest-weight modules L_i, $i = 1, 2, 3, 4$.

For the simple root subset $\{1\}$, $\mathfrak{l} \cong \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$ and the relevant coherent family of $\mathfrak{l}$-modules has direct summands with three composition factors.

The corresponding parabolic family of $\mathfrak{so}_8$-modules then does too:



The maximal proper submodule then Zhu-induces to a non-split extension of L_2 by L_1:

$$0 \longrightarrow \mathsf{L}_1 \longrightarrow \mathsf{M} \longrightarrow \mathsf{L}_2 \longrightarrow 0.$$

The previous theorem guarantees that M is an $\mathsf{L}_{-2}(\mathfrak{so}_8)$-module.

Outlook

- These results allow us to explore relaxed module theory in general. In particular, all admissible levels are now accessible (in principle) because of Arakawa's highest-weight classification.
- For example, we intend to isolate the role of W-algebras (and nilpotent orbits) in the characters, modularity and fusion rules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ (*cf.* Kazuya's talk).
- This will require the concurrent investigation of the relaxed modules of these W-algebras.
- Non-semisimple relaxed modules (and their σ-twists) are expected to be crucial in studying projective covers.
- An ultimate goal would be to establish a vertex tensor category structure on $\mathscr{R}_{\mathsf{k}}^{\sigma}$.
- The future of affine vertex algebras is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— M C Escher

The Mathematics of Conformal Field Theory II
July 6–10, 2020
Australian National University, Canberra