Bosonic ghosts: Modularity for a non-rational CFT.

Based on 0810.3532, 1205.6513 & 1306.4388 (w/ T. Creutzig) and 1408.4185 (w/ S. Wood)

Recall: (Suitably nice) rational CFTs are modular:

- The characters of the irreducibles span a rep of $SL(2;\mathbb{Z})$.
- The T-matrix is unitary and diagonal.
- The S-matrix is unitary and symmetric; its square is a permutation matrix (representing conjugation).
- The S-matrix vacuum entries are non-zero: $S_{0\ell} \neq 0 \ \forall \ell$.
- Verlinde's formula relates the modular S-matrix to the fusion coeffs:

$$N_{ij}^{\ k} = \sum_{\ell} \frac{S_{i\ell} S_{j\ell} S_{k\ell}^*}{S_{0\ell}} \quad (\in \mathbb{Z}_{\geq 0}).$$

This modularity facilitates a standard consistency check of CFT: the bulk partition function should be $SL(2;\mathbb{Z})$-invariant.

But this consistency check should also apply more generally...

$\boxed{\text{Q: What type of modularity can we expect for non-rational CFTs??}}$

Explorations with non-rational, but lisse (C_2-cofinite), examples (eg the triplet models) suggest that things get complicated:

- The characters do not span an $SL(2;\mathbb{Z})$-rep.
- Generalising characters to 1-pt torus functions restores $SL(2;\mathbb{Z})$-invariance.
- However, we don't have a canonical basis of 1-pt functions. Generally, the S- and T-matrices do not have nice properties:
  - T needs not be diagonal or unitary.
  - S needs not be unitary or symmetric and its square needs not be a permutation.
  - The vacuum entries of the S-matrix may be 0, hence the Verlinde formula cannot hold in the usual form.
- How to relate 1-pt functions to modular invariant partition functions?

All these issues seem to arise because the CFT is logarithmic — it has reducible but indecomposable modules.

ie there are irreducibles that are not projective and injective!

On the other hand, the free boson is non-rational and its modular properties are exemplary (once sums are replaced by integrals):

- The T-matrix is unitary and diagonal: $T_{\lambda\mu} = e^{i\pi(\lambda^2 - 1/12)} \delta(\lambda - \mu)$.
- The S-matrix is unitary and symmetric, $S_{\lambda\mu} = e^{-2\pi i \lambda\mu}$, and its square is conjugation: $(S^2)_{\lambda\mu} = \int_{\mathbb{R}} S_{\lambda\nu} S_{\nu\mu} \, d\nu = \delta(\lambda + \mu)$.
- The vacuum S-matrix entries are non-zero and Verlinde works:

$$N_{\lambda\mu}{}^{\nu} = \int_{-\infty}^{\infty} \frac{S_{\lambda\rho} S_{\mu\rho} S_{\nu\rho}^*}{S_{0\rho}} \, d\rho = \delta(\lambda + \mu - \nu) \quad \Rightarrow \quad \mathcal{F}_\lambda \times \mathcal{F}_\mu = \int_{-\infty}^{\infty} N_{\lambda\mu}{}^{\nu} \mathcal{F}_\nu \, d\nu = \mathcal{F}_{\lambda+\mu}.$$

Of course, the free boson is not logarithmic, so all the irreducibles are projective and injective. So we should expect good modularity!

To explore modularity for logarithmic CFTs, first investigate examples of "almost-everywhere non-logarithmic CFTs" meaning that almost all of the irreducibles are projective/injective.

The easiest example of such a CFT must be the bosonic ghost system, a.k.a. a $\beta\gamma$-system, symplectic bosons, free superfermions, ...

It arises naturally in Wakimoto-type free field realisations and when gauging (eg. superstring BRST or quantum hamiltonian reduction).

Recall OPEs:

$$\beta(z)\beta(w) \sim 0, \qquad \beta(z)\gamma(w) \sim \frac{-1}{z-w}, \qquad \gamma(z)\gamma(w) \sim 0.$$

$J(z) = :\beta(z)\gamma(z):$ is a current under which β (γ) has charge $+1$ (-1).

We choose $T(z) = - :\beta(z)\partial\gamma(z):$ so $\Delta_\beta = 1$, $\Delta_\gamma = 0$ and $c = 2$.

There is a single highest-weight module - the vacuum module - which is an irreducible Verma module. Its character is

$$\mathrm{ch}[V] = \mathrm{tr}_V \, z^{J_0} q^{L_0 - 1/12} = z^{1/2} \frac{\eta(q)}{i\vartheta_1(z;q)}.$$

Since $S\{\mathrm{ch}[V]\} = e^{i\pi/2} \mathrm{ch}[V]$ and $T\{\mathrm{ch}[V]\} = e^{-i\pi/6} \mathrm{ch}[V]$, the modular properties also seem exemplary! Verlinde works ...

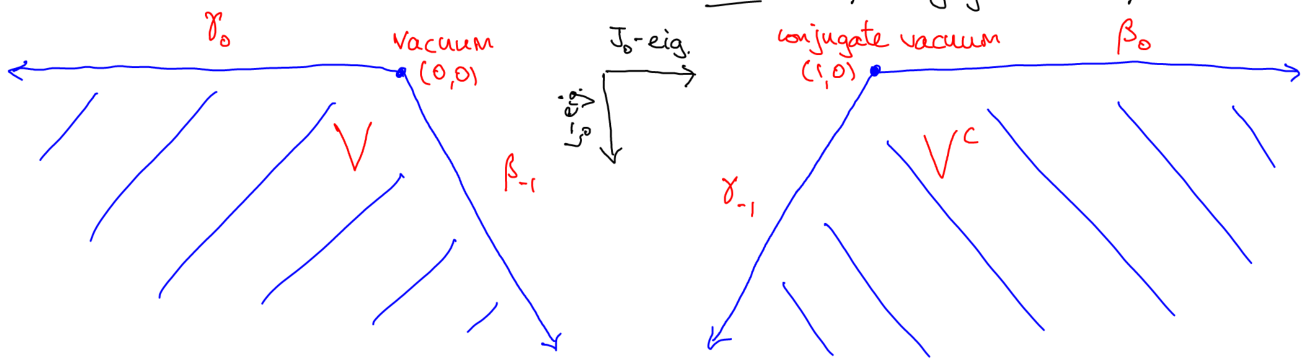
But, there is one weird thing: $S^2 = -1$.

What does it mean for conjugation to negate the vacuum character?

Look at the conjugation automorphism:

$$\beta_n \mapsto \gamma_n, \qquad \gamma_n \mapsto -\beta_n, \qquad J_n \mapsto -J_n + \delta_{n,0}\mathbb{1}, \qquad L_n \mapsto L_n - nJ_n.$$

This makes it clear that the vacuum module is <u>not</u> self-conjugate: $V \not\cong V^c$.



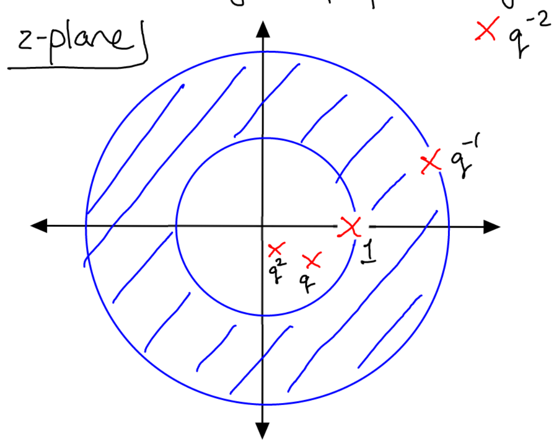
In particular, V^c is <u>not</u> a highest-weight module. Its character is

$$\mathrm{ch}[V^c] = z^{1/2}\frac{\eta(q)}{i\vartheta_1(z^{-1};q)} = -z^{1/2}\frac{\eta(q)}{i\vartheta_1(z;q)} = -\mathrm{ch}[V],$$

explaining why $S^2=-1$.

But, it doesn't explain what it means for a character to be negative!

The answer is a bit subtle and is found in the analytic properties of the vacuum character:



$$\mathrm{ch}[V] = z^{1/2}\frac{\eta(q)}{i\vartheta_1(z;q)} = \frac{q^{-1/12}}{\prod_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^{n-1})}$$

has poles at $z=q^n$, for all $n\in\mathbb{Z}$.

To get the correct power series, we have to expand $\mathrm{ch}[V]$ in the annulus $1<|z|<|q|^{-1}$.

Similarly, to get the character of V^c as a power series, expand $-\mathrm{ch}[V]$ in the annulus $|q|<|z|<1$.

<u>Moral</u>: Characters are power series. If we want to interpret them as meromorphic functions, we have to stipulate the expansion domain!

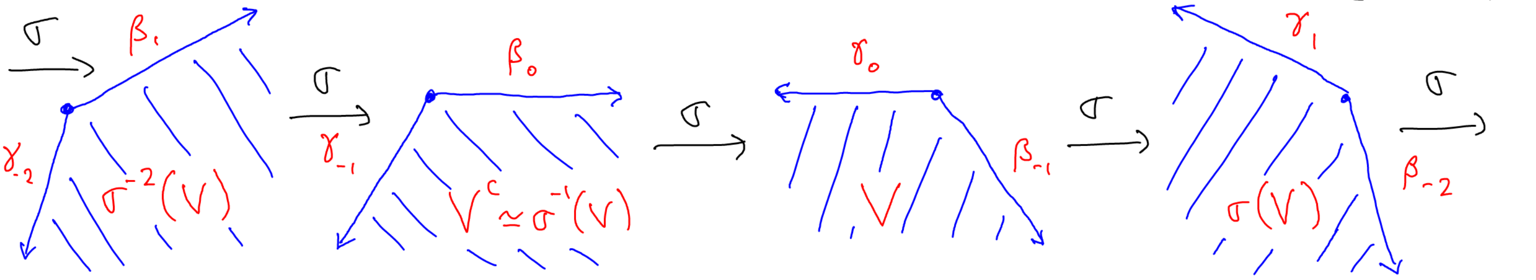
<u>Note</u>: While the T-transform preserves these annuli, the S-transform does not. $\therefore$ any modularity claim based on characters as meromorphic functions is highly suspicious (ie probably wrong)!

What about the other annuli $|q|^{\ell} < |z| < |q|^{\ell-1}$??

Expanding $\pm \mathrm{ch}[V]$ in these annuli gives the characters of the spectral flows of the vacuum module!

The spectral flow automorphisms are given by $(\ell \in \mathbb{Z})$

$$\sigma^{\ell}(\beta_n) = \beta_{n-\ell}, \quad \sigma^{\ell}(\gamma_n) = \gamma_{n+\ell}, \quad \sigma^{\ell}(J_n) = J_n + \ell\,\delta_{n,0}\mathbb{1}, \quad \sigma^{\ell}(L_n) = L_n - \ell J_n - \tfrac{1}{2}\ell(\ell-1)\delta_{n,0}\mathbb{1}.$$

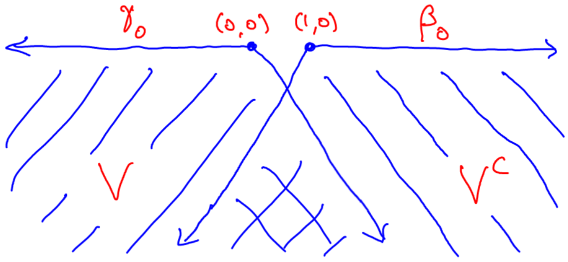


We still don't know how to compute modular transforms of the characters of the $\sigma^{\ell}(V)$, $\ell \in \mathbb{Z}$, but we have an infinite number of irreducibles.

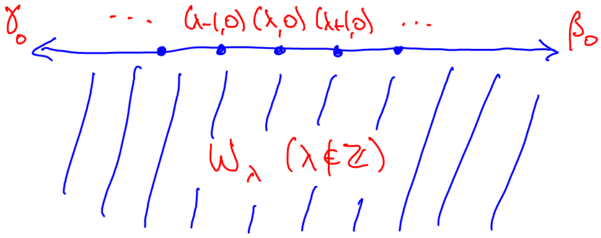
$$\boxed{\therefore \text{ The bosonic ghost CFT is not rational.}}$$

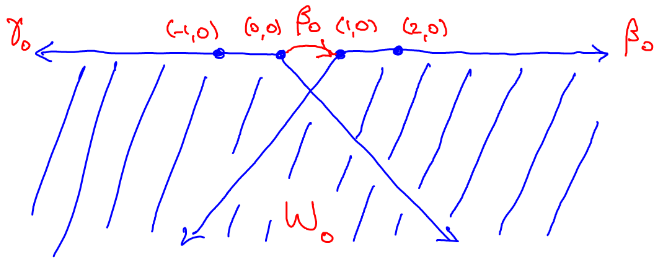
Is it logarithmic? Yes: and this even helps with modularity!!

Consider $V \oplus V^c$: Can we modify this rep so that β_0 maps the vacuum to the conjugate vacuum (or γ_0 maps vice versa)??



Yes: studying the rep theory of the zero mode subalgebra (or employing Mathieu's twisted localisation) gives an uncountable number of new ghost reps W_λ, $\lambda \in \mathbb{R}$, and their spectral flows!





The W_λ, $\lambda \notin \mathbb{Z}$, are irreducible! $\qquad 0 \to V^c \to W_0 \to V \to 0$

The bosonic ghost CFT is logarithmic because W_0 is reducible but indecomposable!

Note: 1) We also have the conjugate W_0^c s.t. $0 \to V \to W_0^c \to V^c \to 0$.

2) $W_\lambda \simeq W_\mu$ iff $\lambda - \mu \in \mathbb{Z}$.

Q: how does this help with modularity?

As meromorphic functions (wrong!!), we have

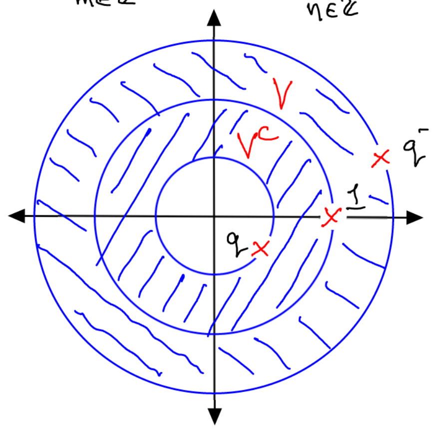
$$\mathrm{ch}[W_0] = \mathrm{ch}[V^c] + \mathrm{ch}[V] = 0.$$

But, being careful (using PBW bases) gives instead

$$\mathrm{ch}[W_0] = \sum_{m\in\mathbb{Z}} \frac{z^m q^{-1/12}}{\prod_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^n)} = \sum_{m\in\mathbb{Z}} \frac{z^m q^{-1/12}}{\prod_{n=1}^{\infty}(1-q^n)^2} = \frac{1}{\eta(q)^2}\sum_{m\in\mathbb{Z}} z^m .$$

We can't treat this as a meromorphic function in z! The best we can do is treat it as a distribution: $\sum_{m\in\mathbb{Z}} z^m = \sum_{m\in\mathbb{Z}} e^{2\pi i \zeta m} = \sum_{n\in\mathbb{Z}} \delta(\zeta - n)$.

Note that $\zeta = n \in \mathbb{Z} \Rightarrow z = e^{2\pi i n} = 1$: the support of $\mathrm{ch}[W_0]$ is the pole separating the expansion regions of $\mathrm{ch}[V]$ and $\mathrm{ch}[V^c]$!



Main result:

The $\mathrm{ch}[\sigma^\ell(W_\lambda)] = \dfrac{e^{i\pi\ell(\ell-1)\tau}}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}} e^{2\pi i m\lambda}\,\delta(\zeta+\ell\tau-m)$,

with $\ell\in\mathbb{Z}$ and $\lambda\in\mathbb{R}$, span a rep of $SL(2;\mathbb{Z})$ with

$S_{(\ell,\lambda)(m,\mu)} = (-1)^{\ell+m} e^{-2\pi i(\ell\mu+m\lambda)}$ and $T_{(\ell,\lambda)(m,\mu)} = e^{i\pi[\ell(\ell-1+2\lambda)-\frac{1}{6}]}\,\delta_{\ell m}\,\delta(\lambda-\mu \bmod \mathbb{Z})$.

Here, $S\{\mathrm{ch}[\sigma^\ell(W_\lambda)]\} = \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}} S_{(\ell,\lambda)(m,\mu)}\,\mathrm{ch}[\sigma^m(W_\mu)]\,d\mu$, etc...

- T is diagonal and unitary!
- S is unitary and symmetric; $S^2_{(\ell,\lambda)(m,\mu)} = \delta_{\ell,-m}\,\delta(\lambda+\mu \bmod \mathbb{Z})$ is conjugation!

However, $(\ell,\lambda) = (0,0)$ is W_0, not the vacuum V.

To get the correct modular transforms for V, note that we have a resolution of V in terms of the $\sigma^\ell(W_0)$:

$$\cdots \longrightarrow \sigma^{-3}(W_0) \longrightarrow \sigma^{-2}(W_0) \longrightarrow \sigma^{-1}(W_0) \longrightarrow W_0 \longrightarrow V \longrightarrow 0.$$

$$\therefore \quad \mathrm{ch}[V] = \sum_{\ell=0}^{\infty} (-1)^{\ell}\, \mathrm{ch}[\sigma^{-\ell}(W_0)] \qquad (\in \text{some completion of the span!})$$

$$\Rightarrow \quad S_{V(m,\mu)} = \sum_{\ell=0}^{\infty} (-1)^{\ell}\, S_{(-\ell,0)(m,\mu)} = (-1)^{m-1} \frac{e^{-\pi i \mu}}{e^{\pi i \mu} - e^{-\pi i \mu}}.$$

- We always expand the S-transform of a character in the basis given by the $\mathrm{ch}[\sigma^{\ell}(W_{\lambda})]$. We call the $\sigma^{\ell}(W_{\lambda})$ the standard modules. They are irreducible, projective and injective for almost all λ.
- The vacuum S-matrix entries $S_{V(m,\mu)}$ are non-zero, but there is a pole at $\mu = 0$ $(\in \mathbb{R}/\mathbb{Z})$, ie the poles form a measure zero set!

(Grothendieck) fusion is now determined by the standard Verlinde formula:

$$A \times B = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} N_{AB}^{(n,\nu)}\, \sigma^{n}(W_{\nu})\, d\nu, \qquad N_{AB}^{(n,\nu)} = \sum_{r \in \mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{S_{A(r,\rho)} S_{B(r,\rho)} S^{*}_{(n,\nu)(r,\rho)}}{S_{V(r,\rho)}}\, d\rho.$$

Results: The fusion coefficients are non-negative integers!

$$\boxed{\begin{gathered} \sigma^{\ell}(V) \times \sigma^{m}(V) = \sigma^{\ell+m}(V), \qquad \sigma^{\ell}(V) \times \sigma^{m}(W_{\mu}) = \sigma^{\ell+m}(W_{\mu}), \\ \sigma^{\ell}(W_{\lambda}) \times \sigma^{m}(W_{\mu}) = \sigma^{\ell+m}(W_{\lambda+\mu}) + \sigma^{\ell+m-1}(W_{\lambda+\mu}). \end{gathered}}$$

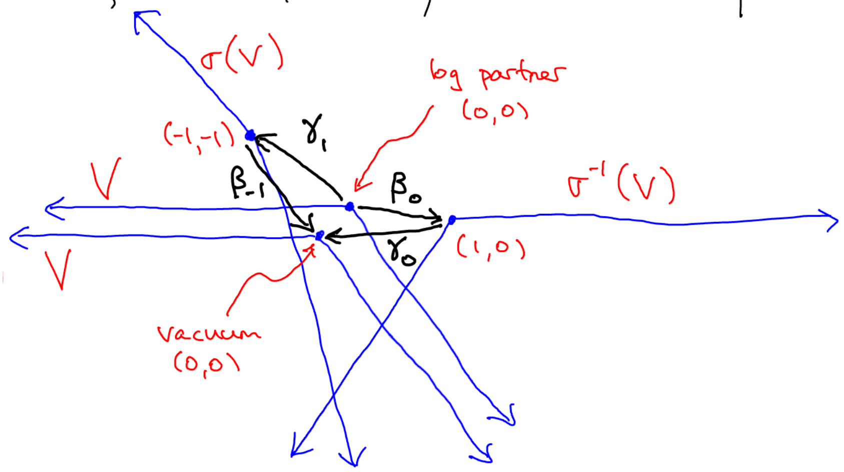
- This last sum is only direct if $\lambda + \mu \neq 0 \bmod \mathbb{Z}$.
- These results were checked explicitly using the Nahm-Gaberdiel-Kausch algorithm. It shows that when $\lambda + \mu = 0 \bmod \mathbb{Z}$, the fusion product is indecomposable:

eg. $W_{\lambda} \times W_{-\lambda} = \sigma^{-1}(P)$, where

$$0 \to \sigma(W_0) \to P \to W_0 \to 0.$$

P:

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ \sigma^{-1}(V) \arrow[dr] & & \sigma(V) \arrow[dl] \\ & V & \end{tikzcd}$$



- L_0 acts non-diagonalisably on P.
- P is the projective cover/injective hull of the vacuum V. [Allen–Wood 2001.05986]
- The bulk state space is $\bigoplus_{\ell \in \mathbb{Z}} \oint_{0<\lambda<1} W_{\lambda} \otimes W_{\lambda}\, d\lambda \oplus \mathbb{P}$, where $\mathbb{P}$ is an indecomposable constructed from P and its spectral flows.

In summary:

- The bosonic ghost CFT is an example of a logarithmic CFT with excellent modular properties.
- Because the "logarithmic bits" are confined to a set of measure zero, the formalism is not too dissimilar to that of rational CFTs.
- Other "almost-everywhere non-logarithmic" CFTs include:
  - fractional-level Wess-Zumino-Witten models.
  - W-algebras at non-rational levels, eg. Bershadsky-Polyakov, $N=2$, ...
  - Singlet models.
- The triplet models don't fall into this class as their logarithmic bits form a set of positive measure. However, the modularity of the triplet models can be deduced from that of the singlet models ...