A new approach to W-algebras

W-algebras are a type of vertex algebra. So what's a vertex algebra?

STATES	FIELDS	MODES
elements A of some graded vector space $V = \bigoplus_{h \in \mathbb{C}} V_h$	generating functions $A(z)$ in the modes $A(z) = \sum_n A_n z^{-n-\Delta}$	endomorphisms A_n of V that respect the grading: $A_n : V_h \to V_{h-n}$
eg. vacuum $\lvert 0 \rangle$	identity field $\mathbb{1}$	identity $\mathbb{1}_n = \begin{cases} \mathrm{id}_V & n=0 \\ 0 & n \neq 0 \end{cases}$
conformal vector T	energy-momentum tensor $T(z)$	Virasoro modes L_n, $n \in \mathbb{Z}$, $[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3-m}{12} \delta_{m+n,0}\, c\, \mathrm{id}_V$

conformal vertex algebra
= vertex operator algebra (VOA)

central charge

Key advantage of the conformal structure is that the L_0-eigenvalue Δ gives the grading. Natural application: conformal field theory.

Examples:

- Virasoro VOA: Generated by $T(z)$ with conformal weight $\Delta = 2$.
- Affine VOA: Generated by fields $J^a(z)$ of conf. wt. $\Delta = 1$. Modes J^a_n satisfy commutation relations of an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ ($\mathfrak{g}$ a simple Lie algebra):
$$[J^a_m, J^b_n] = f^{ab}{}_c J^c_{m+n} + m \kappa^{ab} \delta_{m+n,0}\, k \cdot \mathrm{id}_V .$$
level
- Superconformal VOAs: Generated by $T(z)$ $(\Delta=2)$, some $J^a(z)$ $(\Delta=1)$ and N fermionic fields $G^a(z)$ with $\Delta = \frac{3}{2}$.
- Zamolodchikov's W-algebra: Generated by $T(z)$ $(\Delta=2)$ and $W(z)$ $(\Delta=3)$. Inspired by the 3-state Potts model.

Originally, "W-algebra" meant basically any VOA but especially those with a generating field of conformal weight > 2.

Now, it usually refers to a much more restricted class of VOAs: those obtained from an affine VOA by quantum hamiltonian reduction.

$$\text{affine VOA } V \xrightarrow{\text{QHR}} \text{W-algebra } W \qquad V\text{-modules} \xrightarrow{\text{QHR}} W\text{-modules}$$

So what's quantum hamiltonian reduction?

simple Lie algebra

level

It's a BRST-cohomology construction applied to an affine VOA $V^k(\mathfrak{g})$.

- Tensor $V^k(\mathfrak{g})$ with an appropriate "ghost" VOA G (cf Faddeev-Popov).
- Grade $V^k(\mathfrak{g}) \otimes G$ by "total ghost number".
- Introduce a differential $d : V^k(\mathfrak{g}) \otimes G \circlearrowleft$ of degree 1.
- Take the (zeroth) cohomology wrt this differential: this is the W-algebra.
- For a module M, replace $V^k(\mathfrak{g})$ by M above.

There are many choices for G and d, parametrised by a choice of nilpotent element $f \in \mathfrak{g}$. The W-algebra is therefore denoted by

$$W^k(\mathfrak{g}; f).$$

Examples:

- $W^k(\mathfrak{g}; 0) = V^k(\mathfrak{g})$.
- $W^k(sl_2; f^\theta)$ is the Virasoro VOA.
- $W^k(osp(1|2); f^\theta)$ is the $N=1$ superconformal VOA.
- $W^k(sl_3; f^{\alpha_1} + f^{\alpha_2})$ is Zamolodchikov's original W-algebra.
- $W^k(sl_3; f^\theta)$ is the Bershadsky-Polyakov algebra.
- $W^k(sl(2|1); f^\theta)$ is the $N=2$ superconformal VOA.

Actually, for given $\mathfrak{g}$, the possible W-algebras are classified by the nilpotent orbits of $\mathfrak{g}$.

For $\mathfrak{g} = sl_n$, each nilpotent orbit $\mathbb{O}$ is specified by a partition λ of n.

adjoint action (conjugation)

$$\begin{matrix} \lambda = [\lambda_1, \lambda_2, \dots] \\ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0 \\ \sum_i \lambda_i = n \end{matrix} \longleftrightarrow f(\lambda) = \begin{array}{c|c} \begin{array}{cccc} \lambda_1 & \lambda_2 & \lambda_3 & \cdots \end{array} & \\ \hline \left[\begin{array}{c|c|c|c} \begin{smallmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{smallmatrix} & & & \\ \hline & \begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{smallmatrix} & & \\ \hline & & \begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix} & \\ \hline & & & \begin{smallmatrix} 0 & \\ & \ddots \end{smallmatrix} \end{array}\right] & \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \end{matrix} \end{array} \longleftrightarrow \mathbb{O}_\lambda = SL_n \cdot f(\lambda).$$

So the sl_n W-algebras are classified by partitions of n (and the level k).

Nilpotent orbits are also ordered by inclusion of their (Zariski) closures. For sl_n, this is the dominance ordering:

$$\lambda \geqslant \mu \quad \text{iff} \quad \lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i \ \forall i.$$

Examples:

- $\mathfrak{g} = \mathfrak{sl}_2$

  [2] regular (principal) " minimal — Virasoro

  [1,1] zero = subregular — $V^k(\mathfrak{sl}_2)$

- $\mathfrak{g} = \mathfrak{sl}_3$

  [3] regular — Zamolodchikov

  [2,1] minimal = subregular — Bershadsky – Polyakov

  [1,1,1] zero — $V^k(\mathfrak{sl}_3)$

- $\mathfrak{g} = \mathfrak{sl}_4$

  [4] regular — Casimir alg $W_4 = W(2,3,4)$

  [3,1] subregular

  [2,2] rectangular

  [2,1,1] minimal

  [1,1,1,1] zero — $V^k(\mathfrak{sl}_4)$

- $\mathfrak{g} = \mathfrak{sl}_6$

  [6] reg. — Casimir alg. $W(2,3,4,5,6) = W_6$

  [5,1] subreg.

  [4,2]

  [4,1,1] [3,3] rect.

  [3,2,1]

  [3,1,1,1] [2,2,2] rect.

  [2,2,1,1]

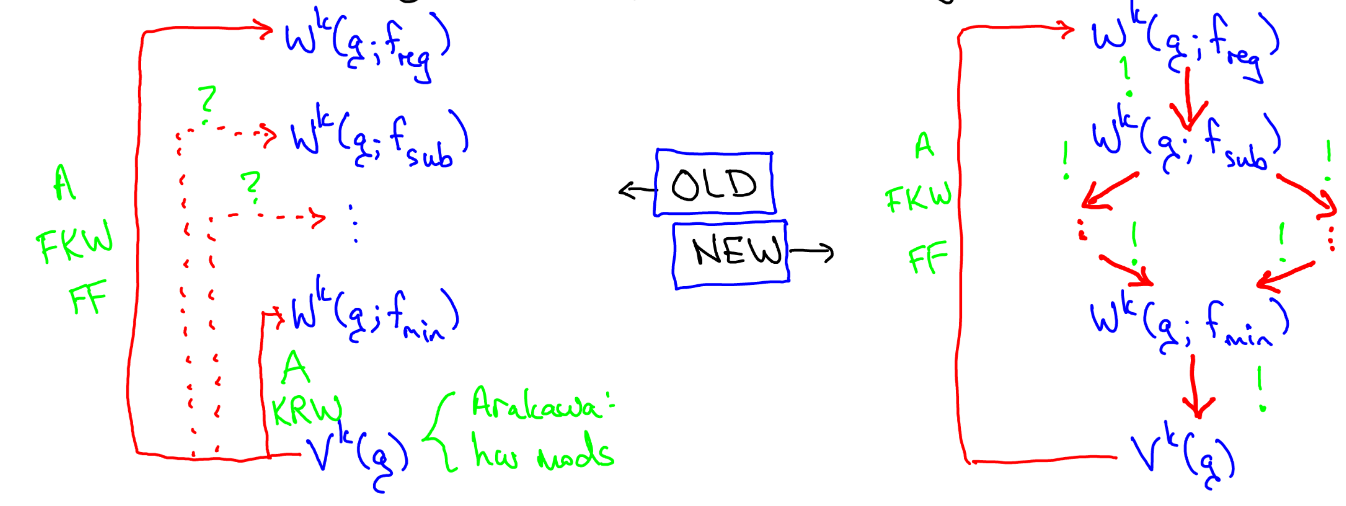
  [2,1,1,1,1] min.

  [1,1,1,1,1,1] zero — $V^k(\mathfrak{sl}_6)$

So what do we know about W-algebras?

KNOWLEDGE	algebraic structure	representation theory
regular	BASICALLY IMPOSSIBLE TO BE EXPLICIT	hw reps known, finite & semisimple (for k admissible).
subregular		occasionally finite & semisimple; usually 1-parameter families of relaxed reps.
⟨others⟩		VERY LITTLE KNOWN
minimal	gen. by fields with $\Delta \leq 2$: OK!	hw reps known, but also ~ (rank $\mathfrak{g}$ − 1)-parameter relaxed families.
zero	gen. by fields with $\Delta = 1$: easy!	hw reps known, but (rank $\mathfrak{g}$)-parameter relaxed families.

Most of what is known was deduced by analysing cohomological details and relations with geometric representation theory.



So, this new approach to W-algebras "builds up" the representation theory, starting from that of the regular W-algebra (which is "nice" - often finite and semisimple) and proceeding down the poset of nilpotent orbits until we arrive back at $V^k(\mathfrak{g})$.

How can we do this?

- In '97, Semikhatov outlined how one could "invert" quantum hamiltonian reduction to construct $V^k(\mathfrak{sl}_2)$ from Virasoro $= W^k(\mathfrak{sl}_2, f_{reg})$.
- In '17, Adamović extracted the mathematical content of this inversion and showed that it induced very simple constructions of certain $V^k(\mathfrak{sl}_2)$-modules from Virasoro modules!

The "certain" modules here are not the familiar highest-weight ones, but are actually the relaxed highest-weight modules.

These form continuous families and are almost always irreducible.

When reducible, the relaxed modules decompose into two highest-weight modules (one twisted by an automorphism). In this way, constructing relaxed modules leads to the highest-weight modules too.

How to do it!

To pass from Virasoro $(= W^k(\mathfrak{sl}_2; f_{reg}))$ to $V^k(\mathfrak{sl}_2)$, we need an isotropic lattice VOA Π:

- Start with the Heisenberg VOA $V^1(\mathfrak{gl}_1 \oplus \mathfrak{gl}_1)$ where $\mathfrak{gl}_1 \oplus \mathfrak{gl}_1 = \mathrm{span}\{c, d\}$ and the "Killing form" is $\langle c, c\rangle = 0 = \langle d, d\rangle$, $\langle c, d\rangle = 2$.
- Equip it with the energy-momentum tensor
$$T(z) = \tfrac{1}{2} :c(z)\, d(z): + \tfrac{k}{4}\partial c(z) - \tfrac{1}{2}\partial d(z).$$
- Extend it by the lattice $\mathbb{Z}c$: the result is Π.
- $c(z)$ and $d(z)$ have conformal weight 1 but that of the lattice field $e^{nc}(z)$ is n $(\forall n \in \mathbb{Z})$.

Π is a bit strange, as VOAs go, because its conformal weights are unbounded below, but this is perfect for constructing relaxed modules!

Thm [Semikhatov, Adamović]: $V^k(\mathfrak{sl}_2) \hookrightarrow W^k(\mathfrak{sl}_2; f_{reg}) \otimes \Pi$.

$\therefore$ tensoring a Virasoro module with a Π-module results in a $V^k(\mathfrak{sl}_2)$-module.

Thm [Adamović]: If M is an irreducible hw Virasoro module, then $R_M(\lambda) = M \otimes \Pi \cdot e^{-a+\lambda c}$, $a = \frac{1}{2}(c+d)$, is a relaxed hw $V^k(\mathfrak{sl}_2)$-module. It is irreducible for almost all $\lambda \in \mathbb{C}$.

Thm [Adamović]: If $k \notin \mathbb{Z}_{\geq 0}$, then the previous theorems still hold if $V^k(\mathfrak{sl}_2)$ and $W^k(\mathfrak{sl}_2; f_{reg})$ are replaced by their simple quotients $L_k(\mathfrak{sl}_2)$ and $W_k(\mathfrak{sl}_2; f_{reg})$.

We therefore have the following program for understanding relaxed hw $L_k(\mathfrak{sl}_2)$-modules:

$$\boxed{\begin{array}{c}\text{determine the hw}\\ L_k(\mathfrak{sl}_2)\text{-mods}\end{array}} \xrightarrow{\text{QHR}} \boxed{\begin{array}{c}\text{compute the hw}\\ W_k(\mathfrak{sl}_2; f_{reg})\text{-mods}\end{array}} \xrightarrow[\text{QHR}]{\text{inverse}} \boxed{\begin{array}{c}\text{construct the relaxed}\\ \text{hw } L_k(\mathfrak{sl}_2)\text{-mods}\end{array}}$$

But this is a bit dull — we actually already have a general result for passing from affine hw modules to relaxed hw mods [Kawasetsu-DR].

Let's kick it up a notch!

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{determine hw}\\ L_k(\mathfrak{sl}_3)\text{-mods}\end{array}} \arrow[r, "\text{QHR}"] & \boxed{\begin{array}{c}\text{compute hw}\\ W_k(\mathfrak{sl}_3; f_{reg})\text{-mods}\end{array}} \arrow[r, "\text{inverse}", "\text{QHR}"'] & \boxed{\begin{array}{c}\text{construct relaxed}\\ \text{hw } W_k(\mathfrak{sl}_3; f_{min})\text{-mods}\end{array}} \arrow[dl, "\text{degeneration}"] \\
 & \boxed{\begin{array}{c}\text{compute hw}\\ W_k(\mathfrak{sl}_3; f_{min})\text{-mods}\end{array}} \arrow[r, "\text{inverse}", "\text{QHR}"'] & \boxed{\begin{array}{c}\text{construct relaxed}\\ \text{hw } L_k(\mathfrak{sl}_3)\text{-mods}\end{array}} \arrow[dl, "\text{degeneration}"] \\
 & \boxed{\begin{array}{c}\text{compute ``semi-relaxed''}\\ \text{hw } L_k(\mathfrak{sl}_3)\text{-mods}\end{array}} \arrow[uul, dashed, bend left, "\begin{array}{c}\text{degeneration}\\ \text{(if you like)}\end{array}"] &
\end{tikzcd}$$

Now we have the chance to explore completely new representation theories!

- hw $W_k(\mathfrak{sl}_3, f_{reg})$-mods are known but the hw $W_k(\mathfrak{sl}_3, f_{min})$-mods are only known for very special k.
- relaxed $L_k(\mathfrak{sl}_3)$- and $W_k(\mathfrak{sl}_3, f_{min})$-mods unknown for all k.

Results thus far

Thm [Adamović – Kawasetsu – DR]: • $W^k(\mathfrak{sl}_3; f_{min}) \hookrightarrow W^k(\mathfrak{sl}_3; f_{reg}) \otimes \Pi$, where Π is equipped with EMT $\frac{1}{2}:cd: + \frac{2k+3}{3}\partial c - \frac{1}{2}\partial d$.

- If $2k+3 \notin \mathbb{Z}_{\geq 0}$, then $W_k(\mathfrak{sl}_3; f_{min}) \hookrightarrow W_k(\mathfrak{sl}_3; f_{reg}) \otimes \Pi$.
- If M is an irreducible hw $W^k(\mathfrak{sl}_3, f_{reg})$-module, then
$$M \otimes \Pi \cdot e^{-j+\lambda c}, \qquad j = \frac{2k+3}{6}c + \frac{1}{2}d,$$
is an indecomposable relaxed hw $W^k(\mathfrak{sl}_3, f_{min})$-module that is irreducible for almost all $\lambda \in \mathbb{C}$.
- If $2k+3 \notin \mathbb{Z}_{\geq 0}$, then the previous statement holds with "W^k" replaced everywhere by "W_k".

This covers (most of) the first "inverse QHR" for $\mathfrak{g} = \mathfrak{sl}_3$. It remains to use these results to construct $V^k(\mathfrak{sl}_3)$- and $L_k(\mathfrak{sl}_3)$-modules.

Comments:

- Our $\mathfrak{g} = \mathfrak{sl}_3$ results develop general tools for inverse QHR that, eg., strengthen the $\mathfrak{sl}_2$ results whilst simplifying the proofs.
- But, all results to date rely on knowing the algebras' defining relations explicitly: need to develop a more general approach (free fields?).
- What are we actually inverting? For $\mathfrak{g} = \mathfrak{sl}_2$, we invert the regular QHR but for $\mathfrak{g} = \mathfrak{sl}_3$, our results invert some (as yet unknown) affine QHR "by stages".
- Opportunity for someone to develop this properly. There should also be an inverse QHR at the level of "finite W-algebras" (Zhu algebras), but this doesn't seem to have been considered yet!
- One big advantage of the inverse construction is that it gives beautiful character formulae for the relaxed modules.

  $\Rightarrow$ modularity & (Grothendieck) fusion coefficients via the "standard module formalism" of logarithmic CFT ...