WEIGHT MODULES FOR $\mathfrak{sl}_3$ MINIMAL MODELS

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5/6/2021 OR 7/5/2021

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Motivation

Vertex operator algebras (VOAs) are interesting for many reasons, one of which is conformal field theory (CFT).

Given a VOA, CFT requires a category of VOA-modules satisfying:

- It's braided tensor (tensor product = fusion product).
- It's closed under twisting by automorphisms of the vertex algebra.
- The partition function (coend character?) is modular-invariant.
- 4-pt correlators satisfy crossing symmetry.

⋮

Want to study affine VOAs $V^k(\mathfrak{g})$ and associated W-algebras.

But $\mathrm{Zhu}[V^k(\mathfrak{g})] \simeq \mathcal{U}(\mathfrak{g})$, so representation theory is unconstrained.

More interesting to study simple quotients $L_k(\mathfrak{g}) \neq V^k(\mathfrak{g})$.

eg. $\mathfrak{g}$ simple and $k \in \mathbb{Z}_{\geq 0}$ $\Rightarrow$ category is finite semisimple.

Other interesting cases include the admissible levels,

eg. $\mathfrak{g} = \mathfrak{sl}_n$ and $k+n = \frac{u}{v}$ with $u \in \mathbb{Z}_{\geq n}$, $v \in \mathbb{Z}_{\geq 1}$, $(u,v)=1$.

I like to call these VOAs the $\mathfrak{sl}_n$ minimal models, denoted by

$$L_k(\mathfrak{sl}_n) = A_{n-1}(u,v).$$

What's known?

- $A_{n-1}(u,1)$ is rational & C_2-cofinite ($SU(2)_k$ WZW model).

  $A_{n-1}(u,1)$-mod is a modular tensor category and its irreducibles are integrable hw $\widehat{sl}_n$-modules (of level $k=u-n$).

- The irreducible $A_{n-1}(u,v)$-modules in $\hat{\mathcal{O}}_{u,v}$ were classified by Arakawa (2012). There are finitely many and $\hat{\mathcal{O}}_{u,v}$ is semisimple, but this category is <u>not</u> tensor or closed under twisting by automorphisms, if $v \neq 1$.

- For $v \neq 1$, $A_1(u,v)$ is not rational or C_2-cofinite.

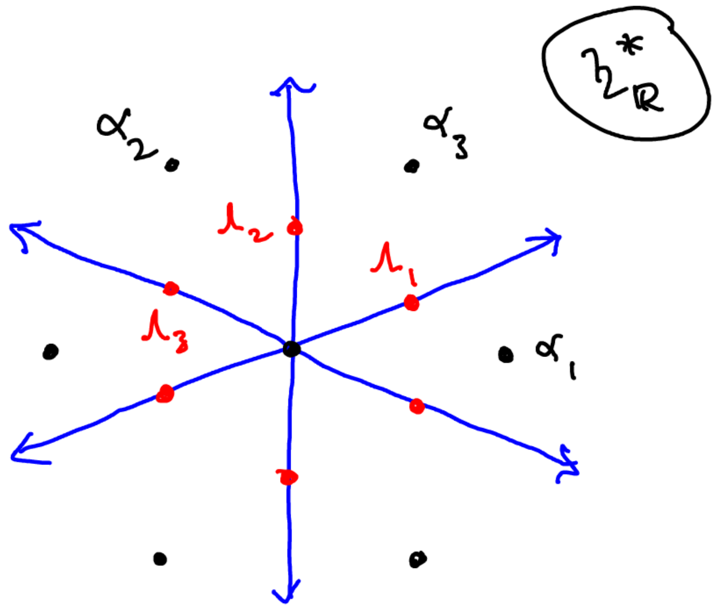
  Moreover, the characters of the irreducibles in $\hat{\mathcal{O}}_{u,v}$ aren't modular. The smallest acceptable category of $A_1(u,v)$-modules seems to be $\hat{\mathcal{W}}_{u,v}$ (weight modules with finite multiplicities).

Aside from this, little is known.

Recently, tools have been developed to understand the representation theory of category $\hat{\mathcal{W}}_{u,v}$.

Time for examples!

sl_3 minimal models

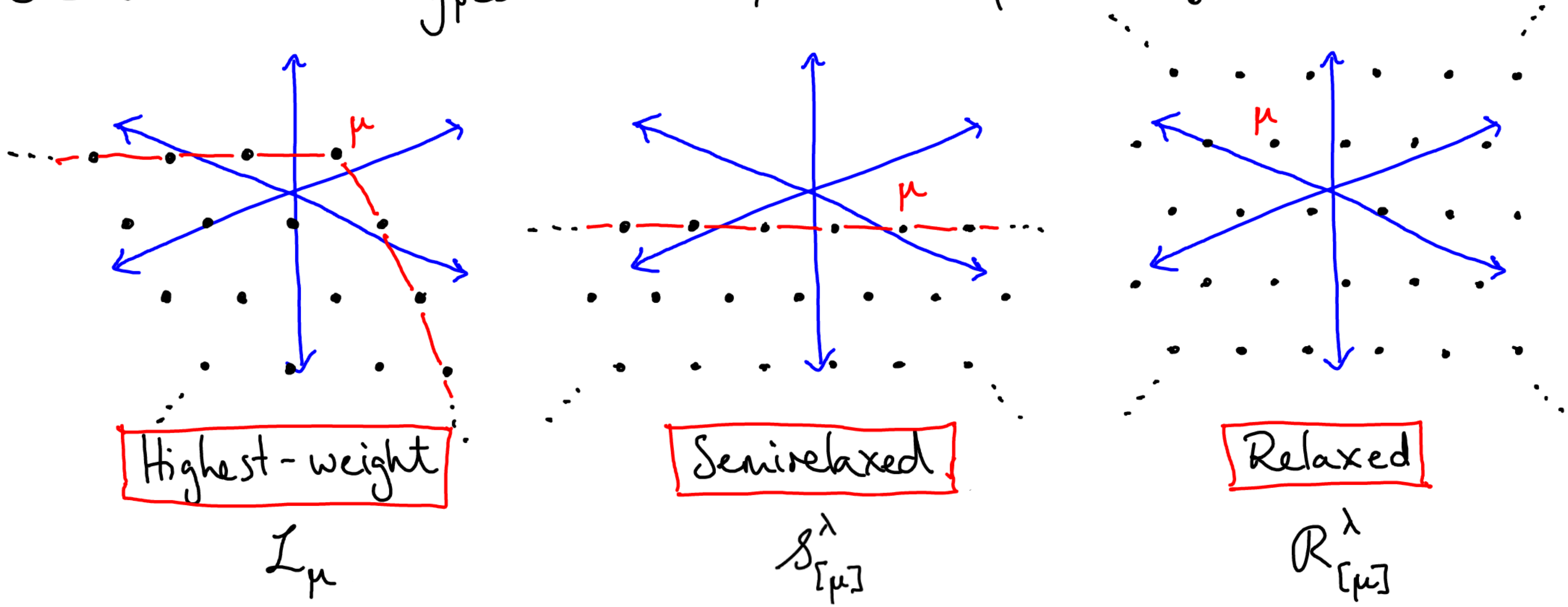


- Simple roots α_1, α_2.
- Highest root $\alpha_3 = \alpha_1 + \alpha_2$.
- Fundamental weights Λ_1, Λ_2.
- Weight lattice P. Root lattice Q.
- Weyl group $W = \langle w_1, w_2 : w_1^2 = w_2^2 = 1,\ w_3 = w_1 w_2 w_1 = w_2 w_1 w_2 \rangle \simeq S_3$.
- Dynkin symmetry d $(\alpha_1 \leftrightarrow \alpha_2)$. Conjugation $c = d w_3$ $(\alpha \leftrightarrow -\alpha)$.

$\therefore$ The automorphisms preserving the Cartan subalgebra generate D_6.

Irreducible positive-energy $A_2(u,v)$-modules are determined by their top spaces, ie by irreducible sl_3-modules. In $\hat{W}_{u,v}$, these are weight modules with finite multiplicities.

There are three types (Mathieu, 2000) up to D_6-twists:



Every $A_2(u,v)$-module is a level-k $\widehat{sl}_3$-module. If M is an sl_3-module, affinising then taking the almost-irreducible quotient gives an $\widehat{sl}_3$-module $\hat{M}$ with top space M.

Let: $\hat{P}^{\ell}_{\geq} = \{ \hat{\lambda} = (\lambda_0, \lambda_1, \lambda_2) : \lambda_i \in \mathbb{Z}_{\geq 0} \ \& \ \lambda_0 + \lambda_1 + \lambda_2 = \ell \}$,

$$A_{u,v} = \left\{ \hat{\lambda}^I - \frac{u}{v} \hat{\lambda}^F : \hat{\lambda}^I \in \hat{P}^{u-3}_{\geq} , \ \hat{\lambda}^F \in \hat{P}^{v-1}_{\geq} \right\}$$

and $A'_{u,v} = \left\{ w_1 \cdot \left(\hat{\lambda}^I - \frac{u}{v} \hat{\lambda}^F \right) : \hat{\lambda}^I \in P^{u-3}_{\geq} , \ \hat{\lambda}^F \in \hat{P}^{v-1}_{\geq} \ \& \ \lambda^F_1 \neq 0 \right\}$.

<u>Theorem (Arakawa 2012)</u>: Every highest-weight $A_2(u,v)$-module is isomorphic to <u>exactly one</u> $\hat{\mathcal{L}}_{\mu}$ with $\hat{\mu} \in A_{u,v} \sqcup A'_{u,v}$.

Given the highest-weight classification, there is an <u>algorithm</u> that produces the irreducible positive-energy classification (Kawasetsu-DR 2019).

Let: $B_{u,v} = \{ \hat{\lambda}^I - \frac{u}{v}\hat{\lambda}^F : \hat{\lambda}^I \in \hat{P}^{u-3}_{\geqslant}, \hat{\lambda}^F \in \hat{P}^{v-1}_{\geqslant} \ \& \ \lambda^F_1 \neq 0 \} \subseteq A_{u,v}$,

$C_{u,v} = \{ \hat{\lambda}^I - \frac{u}{v}\hat{\lambda}^F : \hat{\lambda}^I \in \hat{P}^{u-3}_{\geqslant}, \hat{\lambda}^F \in \hat{P}^{v-1}_{\geqslant}, \lambda^F_1 \neq 0 \ \& \ \lambda^F_2 = 0 \} \subseteq B_{u,v}$.

<u>Theorem</u>: Every irreducible positive-energy $A_2(u,v)$-module is isomorphic to a <u>D_6-twist</u> of:

- An $\hat{\mathcal{L}}_\mu$ with $\hat{\mu} \in A_{u,v} \sqcup A'_{u,v}$.
- An $\hat{\mathcal{S}}^{\lambda}_{[\mu]}$ with $\hat{\lambda} \in B_{u,v}$ & $[\mu - \lambda] \in (\mathbb{C}/\mathbb{Z})\alpha_1 \setminus \{2 \text{ points}\}$.
- An $\hat{R}^{\lambda}_{[\mu]}$ with $\hat{\lambda} \in C_{u,v}$ & $[\mu] \in \mathfrak{h}^*/Q \setminus \{3 \text{ lines}\}$.

There are some coincidences among the D_6-twists of this list, eg.

$$w_1(\hat{\mathcal{S}}^{\lambda}_{[\mu]}) \simeq \hat{\mathcal{S}}^{\lambda}_{[w_1(\mu)]} \quad \& \quad d(\hat{\mathcal{R}}^{\lambda}_{[\mu]}) \simeq \hat{\mathcal{R}}^{cw_2\cdot\lambda}_{[d(\mu)]}.$$

Either way, (Futorny-Tsylke 2001) implies the $\hat{W}_{u,v}$-classification.

<u>Theorem</u>: Every irreducible module in $\hat{W}_{u,v}$ is isomorphic to a <u>spectral flow</u> of a D_6-twist of an $\hat{\mathcal{L}}_\mu$, $\hat{\mathcal{S}}^{\lambda}_{[\mu]}$ or $\hat{\mathcal{R}}^{\lambda}_{[\mu]}$.

Spectral flow is an automorphism of $\hat{sl}_3$ given by

$$\begin{aligned} &\sigma^\Lambda(E^\alpha_n) = E^\alpha_{n-(\alpha,\Lambda)}, && \sigma^\Lambda(H_n) = H_n - \Lambda(H)\delta_{n,0}K, \\ &\sigma^\Lambda(K) = K, && \sigma^\Lambda(L_0) = L_0 - \Lambda_0 + \tfrac{1}{2}\|\Lambda\|^2 K, \end{aligned} \qquad \Lambda \in P.$$

And of course there are some coincidences among spectral flows,

eg. $$\sigma^{-\Lambda_2}\left(\hat{\mathcal{S}}^{\lambda}_{[\mu]}\right) \simeq c\left(\hat{\mathcal{S}}^{k\Lambda_2 - w_1(\lambda)}_{[k\Lambda_2 - \mu]}\right).$$

Finally, there are <u>degenerations</u> in each family of semirelaxed or relaxed $A_2(u,v)$-modules where the family member is reducible, eg.

$$\hat{\mathcal{S}}^{\lambda}_{[\lambda]} \simeq \hat{\mathcal{L}}_{\lambda} \oplus w_1\left(\hat{\mathcal{L}}_{w_1 \cdot \lambda}\right)$$

$$\& \quad \mathcal{R}^{\lambda}_{[\mu]} \simeq \hat{\mathcal{S}}^{\lambda}_{[\mu]} \oplus c\left(\hat{\mathcal{S}}^{cw_2 \cdot \lambda}_{[c(\mu) + (\frac{k}{2} - 1)\alpha_2]}\right) \qquad (\mu - \lambda \in \mathbb{C}\alpha_1).$$

We need to find them <u>all</u> if we want to verify modularity!

The sl_3 minimal model $A_2(3,2) = L_{-3/2}(sl_3)$

We now restrict to the "simplest" non-rational sl_3 minimal model $A_2(3,2)$. So, $u=3$ & $v=2$ $\Rightarrow$ $k=-\frac{3}{2}$ & $c=-8$.

This is the only (u,v) with $v \neq 1$ and linearly independent characters!

- $\exists$ 4 highest-weight modules $\hat{\mathcal{L}}_0$, $\hat{\mathcal{L}}_{-3\Lambda_1/2}$, $\hat{\mathcal{L}}_{-3\Lambda_2/2}$ & $\hat{\mathcal{L}}_{-\rho/2}$.

drop $\lambda = -\frac{3}{2}\Lambda_1$ in notation!

- $\exists$ 1 family of semirelaxed modules $\hat{\mathcal{S}}^{-3\Lambda_1/2}_{[\mu]}$, where $[\mu + \frac{3}{2}\Lambda_1] \in (\mathbb{C}/\mathbb{Z})\alpha_1$, excluding $[\mu] = [-\frac{3}{2}\Lambda_1]$ & $[-\frac{1}{2}\rho]$. $\rightarrow$ degenerations!
- $\exists$ 1 family of relaxed modules $\hat{\mathcal{R}}^{-3\Lambda_1/2}_{[\mu]}$, where $\mu \in \mathfrak{h}^*/Q$ excluding $[\mu] \in [-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_1] \cup [-\frac{1}{2}\rho + \mathbb{C}\alpha_2] \cup [-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_3]$. $\rightarrow$ degenerations!

The relaxed characters of $A_n(u,v)$ were computed in (Kawasetsu 2020).

BP(3,2)- character!

For $A_2(3,2)$, $\quad \mathrm{ch}[\hat{R}_{[\mu]}] = \mathrm{tr}_{R_{[\mu]}} z^{h_0} q^{L_0 - c/24} = \dfrac{\boxed{1}}{\eta(q)^4} \sum_{\nu \in [\mu]} z^{\nu}.$

The <u>standard modules</u> are the spectrally flowed relaxed modules.

Their characters are modular:

$$\mathrm{ch}[\sigma^{\lambda}(\hat{R}_{[\mu]})] \xrightarrow{S} \sum_{\lambda \in P} \int_{\mathfrak{h}^*_{\mathbb{R}}/Q} S^{\lambda,\lambda'}_{[\mu],[\mu']} \, \mathrm{ch}[\sigma^{\lambda'}(\hat{R}_{[\mu']})] \, d[\mu'],$$

$$S^{\lambda,\lambda'}_{[\mu],[\mu']} = e^{-2\pi i (k(\lambda,\lambda') - (\lambda,\mu') - (\mu,\lambda'))}.$$

The "S-matrix" is symmetric & unitary; it squares to conjugation (c).

Because of the degenerations, the standard characters form a <u>basis</u> for the space of all characters! eg.

$$\mu \in -\tfrac{3}{2}\Lambda_1 + \mathbb{C}\alpha_1 \implies \hat{\mathcal{R}}_{[\mu]} \overset{\text{deg.}}{\cong} \hat{\mathcal{S}}_{[\mu]} + c\left(\hat{\mathcal{S}}_{[-\mu-\alpha_2]}\right) \overset{\text{twists}}{\cong} \hat{\mathcal{S}}_{[\mu]} + \sigma^{-\Lambda_2}\left(\hat{\mathcal{S}}_{[\mu-\alpha_1/2]}\right)$$

$$\implies \operatorname{ch}\left[\hat{\mathcal{S}}_{[\mu]}\right] = \sum_{n=0}^{\infty}\left(\underbrace{\operatorname{ch}\left[\hat{\mathcal{R}}^{(-2n\Lambda_2)}_{[\mu]}\right] - \operatorname{ch}\left[\hat{\mathcal{R}}^{(-(2n+1)\Lambda_2)}_{[\mu-\alpha_1/2]}\right]}_{\text{notation}:\ \hat{\mathcal{R}}^{(\Lambda)}_{[\mu]} \equiv \sigma^{\Lambda}(\hat{\mathcal{R}}_{[\mu]})}\right).$$

(Similar formulae for the $\hat{\mathcal{L}}_\mu$!)

This gives S-transforms for all characters (into the standard basis),

eg. $\hat{\mathcal{L}}_0$: $$\overline{\overline{S}}{}^{0,\Lambda'}_{0,[\mu']} = \frac{1}{2\left[1 + \cos(2\pi(\Lambda_1,\mu')) + \cos(2\pi(\Lambda_2,\mu')) + \cos(2\pi(\Lambda_3,\mu'))\right]}.$$

Everything so far has been 100% rigorous. But we can push further if we conjecture that

- The category of weight $A_2(u,v)$-modules with finite multiplicities is rigid braided tensor. $\longrightarrow$ well defined Grothendieck fusion ring.

- The Grothendieck fusion rules are given by

$$[\hat{M}] \boxtimes [\hat{N}] = \sum_{\Lambda'' \in P} \int_{\mathfrak{h}^*_{\mathbb{R}}/Q} \left\langle \begin{matrix} & \hat{\mathcal{R}}^{(\Lambda'')}_{[\mu'']} & \\ \hat{M} & & \hat{N} \end{matrix} \right\rangle [\hat{\mathcal{R}}^{(\Lambda'')}_{[\mu'']}] \, d[\mu''],$$

where the Grothendieck fusion coefficients are given by

$$\left\langle \begin{matrix} & \hat{\mathcal{R}}^{(\Lambda'')}_{[\mu'']} & \\ \hat{M} & & \hat{N} \end{matrix} \right\rangle = \sum_{\Lambda''' \in P} \int_{\mathfrak{h}^*_{\mathbb{R}}/Q} \frac{S(\hat{M} \to \hat{\mathcal{R}}^{(\Lambda''')}_{[\mu''']}) \, S(\hat{N} \to \hat{\mathcal{R}}^{(\Lambda''')}_{[\mu''']}) \left(S^{\Lambda'', \Lambda'''}_{[\mu''],[\mu''']}\right)^*}{\overline{\overline{S}}^{0,\Lambda'''}_{0,[\mu''']}} \, d[\mu'''].$$

These conjectures allow us to compute Grothendieck fusion rules, eg.

$$\left[\hat{R}^{(\Lambda_2)}_{[\mu+\mu'+3\Lambda_2/2]}\right]$$

$$\left[\hat{R}_{[\mu]}\right] \boxtimes \left[\hat{R}_{[\mu']}\right] = \quad \left[\hat{R}^{(\Lambda_3)}_{[\mu+\mu'+3\Lambda_3/2]}\right] \quad 2\left[\hat{R}_{[\mu+\mu']}\right] \quad \left[\hat{R}^{(\Lambda_1)}_{[\mu+\mu'+3\Lambda_1/2]}\right]$$

$$\left[\hat{R}^{(-\Lambda_1)}_{[\mu+\mu'+3\Lambda_1/2]}\right] \quad \left[\hat{R}^{(-\Lambda_3)}_{[\mu+\mu'+3\Lambda_3/2]}\right],$$

$$\left[\hat{R}^{(-\Lambda_2)}_{[\mu+\mu'+3\Lambda_2/2]}\right]$$

$$\left[\hat{R}^{(\Lambda_2)}_{[\mu+\mu'+3\Lambda_2/2]}\right]$$

$$\left[\hat{S}_{[\mu]}\right] \boxtimes \left[\hat{R}_{[\mu']}\right] = \left[\hat{R}^{(\Lambda_3)}_{[\mu+\mu'+3\Lambda_3/2]}\right] \quad \left[\hat{R}^{(\Lambda_1)}_{[\mu+\mu'+3\Lambda_1/2]}\right],$$

$$\left[\hat{R}_{[\mu+\mu']}\right]$$

$$[\hat{\mathcal{L}}_{-\rho/2}] \boxtimes [\hat{\mathcal{R}}_{[\mu']}] = \begin{array}{ccc} [\hat{\mathcal{R}}^{(\Lambda_2)}_{[\mu'+\alpha_2/2]}] & & \\ & & [\hat{\mathcal{R}}^{(\Lambda_1)}_{[\mu'+\alpha_1/2]}] \\ [\hat{\mathcal{R}}_{[\mu'+\alpha_3/2]}] & & \end{array} ,$$

$$[\hat{\mathcal{S}}_{[\mu]}] \boxtimes [\mathcal{S}_{[\mu']}] = \begin{array}{ccc} & [\hat{\mathcal{R}}^{(\Lambda_2)}_{[\mu+\mu'+3\Lambda_2/2]}] & \\ [\hat{\mathcal{S}}^{(\Lambda_3)}_{[\mu+\mu'+3\Lambda_3/2]}] & & [\hat{\mathcal{S}}^{(\Lambda_1)}_{[\mu+\mu'+3\Lambda_1/2]}] \end{array} ,$$

$$[\hat{\mathcal{L}}_{-\rho/2}] \boxtimes [\hat{\mathcal{S}}_{[\mu']}] = \begin{array}{cc} [\hat{\mathcal{R}}^{(\Lambda_2)}_{[\mu'+\alpha_2/2]}] & \\ & [\hat{\mathcal{S}}^{(\Lambda_1)}_{[\mu'+\alpha_1/2]}] \end{array} ,$$

$$[\hat{\mathcal{L}}_{-\rho/2}] \boxtimes [\hat{\mathcal{L}}_{-\rho/2}] = \begin{array}{cccc} [\hat{\mathcal{L}}_0^{(2\Lambda_2)}] & & \\ & 2[c(\hat{\mathcal{L}}_{-\rho/2}^{(-\Lambda_1-\Lambda_2)})] & \\ & & [\hat{\mathcal{L}}_0^{(2\Lambda_1)}] \\ [\hat{\mathcal{L}}_0] & & \end{array} .$$

Note that <u>all</u> the Grothendieck fusion rules have non-negative integer multiplicities. Non-trivial check of consistency!

Note also that the Grothendieck fusion ring has a 1D representation that is <u>constant</u> of D_6- and spectral flow orbits, taking values

$$[\hat{R}_{[\mu]}] \longmapsto 8, \qquad [\hat{S}_{[\mu]}] \longmapsto 4, \qquad [\hat{\mathcal{L}}_{-\rho/2}] \longmapsto 3, \qquad [\hat{\mathcal{L}}_0] \longmapsto 1.$$

This is explained by a Kazhdan-Lusztig-type correspondence...

Conclusions

- We've classified the irreducible weight $A_2(u,v)$-modules with finite multiplicities and computed their D_6 / spectral flow orbits.
- These fall into families and we've determined the explicit decomposition when a family member degenerates (becomes reducible).
- This data is necessary for verifying modularity. We performed this verification for $A_2(u,v)$ and computed the Grothendieck fusion.
- I believe this is the first such "rank-2" analysis!

Outlook

- An obvious step is to generalise at least some of these results to $\mathfrak{sl}_n$ (or $\mathfrak{sp}_4$, $\mathfrak{g}_2$, ...). Need to understand the link between weight module classifications and nilpotent orbits.
- Characters linearly independent $\longrightarrow$ 1-pt functions (cf. Arakawa-van Ekeren 2016)
- Extend to and use W-algebras $\longrightarrow$ inverse quantum hamiltonian reduction (Adamović 2017) starting with exceptional W-algebras (Arakawa-van Ekeren 2019).

Exceptional W-algebra (Rational!) $\longrightarrow \cdots \longrightarrow$ Minimal W-algebra $\longrightarrow$ Admissible-level affine algebra (zero W-algebra)