A higher-rank logarithmic Kazhdan–Lusztig correspondence

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Outline

History / Motivation

In a rather famous series of papers [KL93–94], Kazhdan and Lusztig proved that two categories are braided-tensor equivalent:

- The category of finite-dimensional $\mathsf{U}_q(\mathfrak{g})$-modules for $q = \mathsf{e}^{\pi \mathsf{i}/\ell(\mathsf{k}+\mathsf{h}^\vee)}$ (ℓ is the lacing number).
- The parabolic category $\mathscr{O}_\mathsf{k}$ for $\widehat{\mathfrak{g}}$ ($\mathfrak{g}$ simple) of level k ($\mathsf{k} + \mathsf{h}^\vee \notin \mathbb{Q}_{\geqslant 0}$) in which D has finite-dimensional eigenspaces.

The latter is naturally interpreted in terms of the (simple) affine vertex operator algebra $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ as:

- The category of ordinary $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules for $\mathsf{k} + \mathsf{h}^\vee \notin \mathbb{Q}_{\geqslant 0}$.

The interpretation is warranted by the observation that the tensor product constructed by Kazhdan and Lusztig on this category is the fusion product of conformal field theory.

But, the nicest (and best understood) examples of $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-module categories are for $\mathsf{k} \in \mathbb{Z}_{\geqslant 0}$ [W84, GW86, V88, MS88–89, FZ92].

[F96] extended this Kazhdan–Lusztig correspondence to these levels (modulo a few cases that don't work — see [H13]) but with an additional semisimplification on the quantum group side.

This modification is necessary because quantum group categories are usually non-semisimple for q a root of unity, while $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ is rational.

This leads to two natural questions:

1. Why do such equivalences exist?
2. What's so natural about semisimplifications?

We won't try to answer the first and our view of the second is that the answer is "not a lot".

Logarithmic Kazhdan–Lusztig correspondences

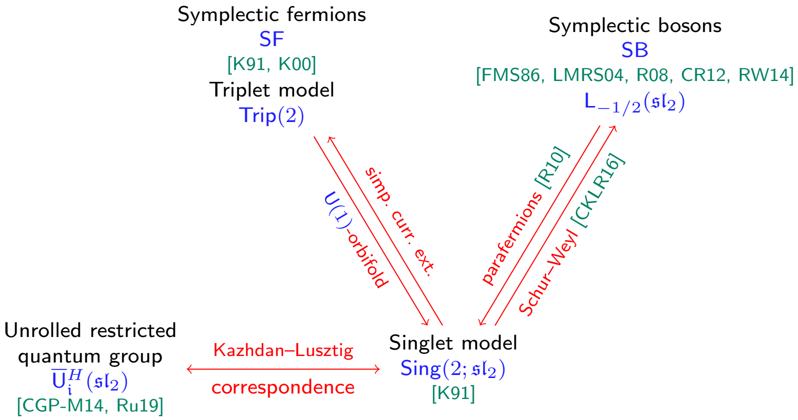
Suppose we reject the idea that we must semisimplify...

Perhaps there are more natural equivalences involving nonsemisimple categories for quantum groups and vertex algebras. These are sometimes called logarithmic Kazhdan–Lusztig correspondences.

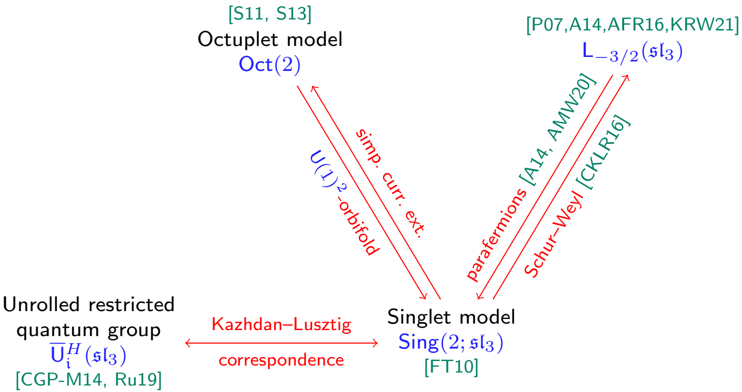
The first was proposed in [FGST06–07] for $\mathsf{U}_q(\mathfrak{sl}_2)$, $q = \mathsf{e}^{\pi \mathsf{i}/p}$, and the triplet algebras $\mathsf{Trip}(p)$ of [K91].
(The precise identification of the quantum group actually took many years, see eg. [CGR20].)

As representation theory is often harder for vertex algebras than for quantum groups, we have at least one obvious motivation for establishing (logarithmic) Kazhdan–Lusztig correspondences.

The logKL poster child:



Today's focus:



(The correspondence here is still conjectural. Our view is that it can still be used to get interesting conjectures for the related VOA-categories.)

The quantum group $\overline{\mathsf{U}}^{H}_{\mathfrak{i}}(\mathfrak{sl}_3)$

The unrolled restricted quantum group $\overline{\mathsf{U}}^{H}_{\mathfrak{i}}(\mathfrak{sl}_3)$ has generators

$$X_{\pm j}, H_j \quad (j = 1, 2) \quad \text{and} \quad K_\gamma \quad (\gamma \in \mathsf{Q})$$

and relations

$$K_0 = 1, \quad K_{\gamma_1} K_{\gamma_2} = K_{\gamma_1+\gamma_2}, \quad K_\gamma X_{\pm j} K_{-\gamma} = \mathfrak{i}^{\pm\langle \gamma | \alpha_j \rangle} X_{\pm j},$$
$$[H_j, X_{\pm j'}] = \pm A_{j,j'} X_{\pm j'}, \quad [H_j, H_{j'}] = [H_j, K_\gamma] = 0,$$
$$[X_j, X_{-j'}] = \delta_{j,j'} \frac{K_{\alpha_j} - K_{-\alpha_j}}{2\mathfrak{i}}, \quad X_{\pm j}^2 = 0, \quad (X_{\pm 1} X_{\pm 2})^2 = (X_{\pm 2} X_{\pm 1})^2,$$

as well as coproducts, counits and antipodes...

(There would also be q-Serre relations, but these are identically satisfied when $q = \mathfrak{i}$.)

We study the category $\mathscr{C}$ of finite-dimensional weight $\overline{\mathsf{U}}^H_{\mathfrak{i}}(\mathfrak{sl}_3)$-modules. $\overline{\mathsf{U}}^H_{\mathfrak{i}}(\mathfrak{sl}_3)$ is quasitriangular, hence $\mathscr{C}$ is braided [GP-M11,R19].

Here, "weight" means that H_j acts semisimply and K_γ acts as $\mathfrak{i}^{\langle \gamma | - \rangle}$ on simultaneous H_j-eigenvectors.

The irreducibles $\mathcal{L}_\lambda$ in $\mathscr{C}$ are highest-weight quotients of 8-dimensional Verma modules $\mathcal{M}_\lambda$.

Partition the highest weights $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ as follows:

	$\lambda_1 \in 2\mathbb{Z}$?	$\lambda_2 \in 2\mathbb{Z}$?	$\lambda_1 + \lambda_2 \in 2\mathbb{Z}+1$?
Type 0	✗	✗	✗
Type 1	✓	✗	✗
	✗	✓	✗
	✗	✗	✓
Type 2	✓	✗	✓
	✗	✓	✓
Type 3	✓	✓	✗

The structure of the Verma module $\mathcal{M}_\lambda$ depends on the type. Let $\mathcal{L}_\lambda^d$ indicate that $\dim \mathcal{L}_\lambda = d$. Then, the Loewy diagram of $\mathcal{M}_\lambda$ is

Type 0:

$$\mathcal{L}_\lambda^8$$

Type 1:

$$\begin{tikzcd} \mathcal{L}_\lambda^4 \arrow[d] \\ \mathcal{L}_{\lambda-\alpha_i}^4 \end{tikzcd}$$

Type 2:

$$\begin{tikzcd} & \mathcal{L}_\lambda^3 \arrow[dl] \arrow[dr] & \\ \mathcal{L}_{\lambda-\alpha_i}^1 \arrow[dr] & & \mathcal{L}_{\lambda+\alpha_i-2\alpha_3}^1 \arrow[dl] \\ & \mathcal{L}_{\lambda-\alpha_3}^3 & \end{tikzcd}$$

Type 3:

$$\begin{tikzcd} & \mathcal{L}_\lambda^1 \arrow[dl] \arrow[dr] & \\ \mathcal{L}_{\lambda-\alpha_1}^3 \arrow[dr] & & \mathcal{L}_{\lambda-\alpha_2}^3 \arrow[dl] \\ & \mathcal{L}_{\lambda-2\alpha_3}^1 & \end{tikzcd}$$

The projective cover $\mathcal{P}_\lambda$ of $\mathcal{L}_\lambda$ exists and its Loewy diagram is

Type 0:

$$\mathcal{L}_\lambda^8$$

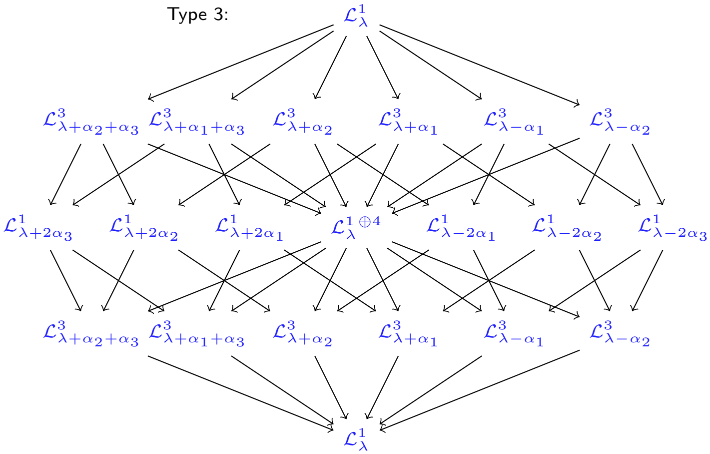
Type 1:

$$\begin{tikzcd}
 & \mathcal{L}_\lambda^4 \arrow[dl] \arrow[dr] & \\
\mathcal{L}_{\lambda+\alpha_i}^4 \arrow[dr] & & \mathcal{L}_{\lambda-\alpha_i}^4 \arrow[dl] \\
 & \mathcal{L}_\lambda^4 &
\end{tikzcd}$$

Type 2:

$$\begin{tikzcd}
 & & & \mathcal{L}_\lambda^3 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \mathcal{L}_{\lambda+\alpha_i}^1 \arrow[dl] \arrow[dr] \arrow[drrr] & & \mathcal{L}_{\lambda-\alpha_i}^1 \arrow[dlll] \arrow[dr] \arrow[drrr] & & \mathcal{L}_{\lambda+\alpha_i-2\alpha_3}^1 \arrow[dlll] \arrow[dl] \arrow[dr] & \\
\mathcal{L}_{\lambda+\alpha_3}^3 \arrow[dr] \arrow[drrr] & & \mathcal{L}_{\lambda+2\alpha_i-\alpha_3}^3 \arrow[dl] \arrow[drrr] & & \mathcal{L}_\lambda^3 \arrow[dlll] \arrow[dl] \arrow[dr] & & \mathcal{L}_{\lambda-\alpha_3}^3 \arrow[dlll] \arrow[dl] \\
 & \mathcal{L}_{\lambda+\alpha_i}^1 \arrow[drr] & & \mathcal{L}_{\lambda-\alpha_i}^1 \arrow[d] & & \mathcal{L}_{\lambda+\alpha_i-2\alpha_3}^1 \arrow[dll] & \\
 & & & \mathcal{L}_\lambda^3 & & &
\end{tikzcd}$$

Type 3:

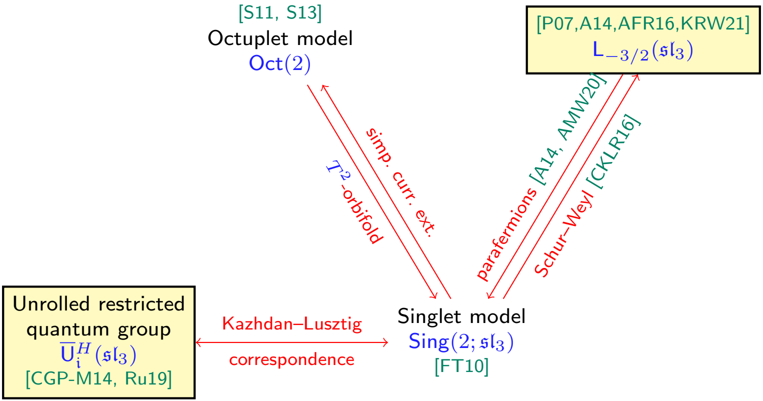


Note that the $\mathcal{P}_\lambda$ are tilting and BGG reciprocity holds [GP-M11,R19]. Their dimensions are 8, 16, 24 and 48, respectively.

The vertex operator algebra $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$

Although our (conjectured) logarithmic Kazhdan–Lusztig correspondence connects the representation theories of $\overline{\mathsf{U}}_{\mathfrak{i}}^{H}(\mathfrak{sl}_3)$ and $\mathsf{Sing}(2;\mathfrak{sl}_3)$, the latter is not easy to analyse directly.

Instead, we tackle the related vertex operator algebra $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$.



Relaxed highest-weight modules

We are interested in the category $\mathscr{W}$ of weight $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$-modules with finite-dimensional weight spaces.

- The irreducible highest-weight modules were classified in [P07].
- The irreducible relaxed highest-weight modules were classified in [AFR16] using Gelfand–Tsetlin combinatorics.
- This also follows directly from the highest-weight classification using Mathieu's coherent families [M00, KR19].
- The irreducibles in $\mathscr{W}$ are then spectral flows, indexed by elements of $\mathsf{P}^\vee$, of relaxed highest-weight modules.

We recall that a relaxed highest-weight vector is a weight vector that is annihilated by the x_n with $x \in \mathfrak{sl}_3$ and $n > 0$.

A relaxed highest-weight module is then a module that is generated by a single relaxed highest-weight vector.

There are four classes of irreducibles in $\mathscr{W}$:

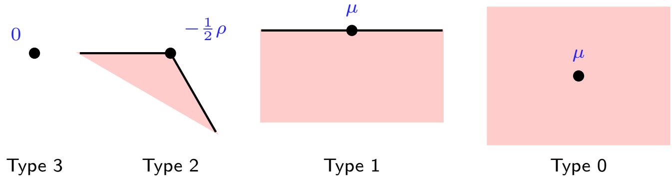
Type 3: Spectral flows of the vacuum module $\mathcal{V} = \mathsf{L}_{-3/2}(\mathfrak{sl}_3)$.

Type 2: Spectral flows of D_6-twists of the highest-weight module $\mathcal{H}$ of highest weight $-\frac{1}{2}\widehat{\rho}$.

Type 1: Spectral flows of D_6-twists of a single 1-parameter family of "semi-relaxed" highest-weight modules $\mathcal{S}_{[\mu]}$ (their top spaces are dense in one root direction).

Type 0: Spectral flows of a single 2-parameter family of "fully relaxed" highest-weight modules $\mathcal{R}_{[\mu]}$ (their top spaces are dense in all root directions).

They are qualitatively distinguished by the "shapes" of their top spaces:



The type of an irreducible $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$-module should match that of the corresponding $\overline{\mathsf{U}}_{\mathfrak{i}}^{H}(\mathfrak{sl}_3)$-module. To make this explicit, reduce to the coset

$$\mathsf{Sing}(2;\mathfrak{sl}_3) \cong \mathrm{Com}\big(\mathsf{H}_2 \hookrightarrow \mathsf{L}_{-3/2}(\mathfrak{sl}_3)\big).$$

Decomposing $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$-modules thus results in $\mathsf{Sing}(2;\mathfrak{sl}_3)$-modules, *eg.*

$$\mathcal{R}_{[\mu]} = \bigoplus_{\lambda\in[\mu]} \mathcal{E}^8_\lambda \otimes \mathcal{F}_\lambda, \qquad \text{(Type 0)}$$

and thus $\mathscr{W}$ yields a category $\mathscr{E}$ of $\mathsf{Sing}(2;\mathfrak{sl}_3)$-modules.

In fact, $\mathscr{E} \cong \mathscr{W}/\mathsf{P}^\vee$ (spectral flow). The irreducibles in $\mathscr{E}$ are therefore:

$$\begin{aligned}
\mathcal{R}_{[\mu]} &\rightsquigarrow \mathcal{E}^8_\lambda, & \lambda &\in \mu + \mathsf{Q}, & &\text{(Type 0)}\\
\mathcal{S}_{[\mu]},\ w_2(\mathcal{S}_{[\mu]}),\ w_1w_2(\mathcal{S}_{[\mu]}) &\rightsquigarrow \mathcal{E}^4_\lambda, & \lambda &\in \mu + \mathsf{Q}, & &\text{(Type 1)}\\
\mathcal{H},\ c(\mathcal{H}) &\rightsquigarrow \mathcal{E}^3_\lambda, & \lambda &\in -\tfrac{1}{2}\rho + \mathsf{Q}, & &\text{(Type 2)}\\
\mathcal{V} &\rightsquigarrow \mathcal{E}^1_\lambda, & \lambda &\in \mathsf{Q}. & &\text{(Type 2)}
\end{aligned}$$

Analysing these types leads to an explicit form for our correspondence.

Conjecture: [A logarithmic Kazhdan–Lusztig correspondence]

An equivalence between the $\overline{\mathsf{U}}_{\mathfrak{i}}^{H}(\mathfrak{sl}_3)$-module category $\mathscr{C}$ and the $\mathsf{Sing}(2;\mathfrak{sl}_3)$-module category $\mathscr{E}$ is given by

$$\mathcal{L}^8_\lambda \longleftrightarrow \mathcal{E}^8_{\phi(\lambda-\rho)}, \qquad \text{(Type 0)}$$

$$\mathcal{L}^4_\lambda \longleftrightarrow \mathcal{E}^4_{\phi(\lambda)} \quad \text{or} \quad \mathcal{E}^4_{\phi(\lambda-\rho)}, \qquad \text{(Type 1)}$$

$$\mathcal{L}^3_\lambda \longleftrightarrow \mathcal{E}^3_{\phi(\lambda)} \quad \text{or} \quad \mathcal{E}^3_{\phi(\lambda-\rho)}, \qquad \text{(Type 2)}$$

$$\mathcal{L}^1_\lambda \longleftrightarrow \mathcal{E}^1_{\phi(\lambda)}, \qquad \text{(Type 3)}$$

where ϕ is the linear map satisfying $\phi(\omega_1) = \frac{1}{2}\alpha_1$ and $\phi(\omega_2) = \frac{1}{2}\alpha_3$.

I do not know the provenance of ϕ, nor why we sometimes shift by ρ...

Comparing tensor products in $\mathscr{C}$ (brute force calculation) and conjectural Grothendieck fusion products in $\mathscr{W}$ (standard Verlinde formula [CR13,RW14]) suggests that this correspondence is braided-tensor.

Assuming this, we can transfer information about $\mathscr{C}$ to obtain highly non-trivial conjectures about $\mathscr{E}$, $\mathscr{W}$ and the module category of the (presumably C_2-cofinite) "octuplet algebra" $\mathrm{Oct}(2)$.

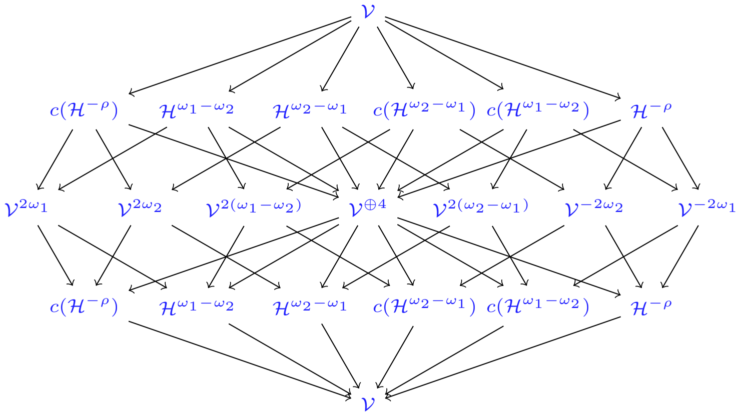
For each vertex operator algebra, we get conjectural:

- Loewy diagrams for the projective covers of each irreducible.
- Fusion rules for all irreducibles.

It is difficult to imagine how to otherwise imagine these conjectures.

The correspondence also works the other way: $\mathscr{W}$ is modular, hence so are $\mathscr{E}$ and $\mathscr{C}$. The provenance of the modularity of $\mathscr{C}$ is not yet clear...

As an example, we conjecture that the projective cover of the vacuum module $\mathcal{V} = \mathsf{L}_{-3/2}(\mathfrak{sl}_3)$ in $\mathscr{W}$ has the following Loewy diagram.



Here, the superscripts indicate the spectral flow parameter in $\mathsf{P}^\vee \cong \mathsf{P}$.

Outlook

So... where to now?

- Obviously, our correspondence is still conjectural and so requires proving...
- This requires more work on the side of vertex operator algebras.
- There are promising directions for affine cases and W-algebras (vertex tensor categories [HLZ10–11, CKM17], jet schemes and arc spaces [AM15–], inverse quantum hamiltonian reduction [A17], coherent families [KR19]).
- For example, logarithmic $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$-modules were recently constructed in [ACG21]. These feature rank-2 Jordan blocks for L_0 but our conjectural projectives suggest that this rank should be 3.
- Either way, to better understanding logarithmic Kazhdan–Lusztig correspondences, we must study higher-rank singlet algebras.
- And our first question still remains: why do Kazhdan–Lusztig correspondences exist?

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno

Hope to see you soon in Montréal!